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SHORT COMMUNICATIONS

# Investigation of a Spectral Problem for the Helmholtz Operator on the Plane

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## INTRODUCTION

The present paper deals with the investigation of a certain spectral problem for the Helmholtz operator on a plane, which appears in the diffraction problem (problems on the eigenwaves of a weakly directing dielectric waveguide). Under specific assumptions (e.g., see [1, p. 525; 2, p. 48; 3; 4]), this problem can be reduced to finding the values of the parameter $\lambda$ for which the Helmholtz equation

$$\Delta u + (q - \lambda)u = 0, \qquad x \in \Omega_i \cup \Omega_e, \tag{1}$$

has nontrivial solutions $u(x)$ satisfying the conjugation conditions

$$[u] = 0, \qquad [\partial u/\partial \nu] = 0, \qquad x \in \Gamma. \tag{2}$$

Here

$$q(x) = \begin{cases} q_i(x) & \text{for} \quad x \in \Omega_i, \\ q_e & \text{for} \quad x \in \Omega_e, \end{cases}$$

$q_i(x) = k_0^2 n_i^2(x)$, $q_e = k_0^2 n_e^2$, $q_i(x) > q_e > 0$, $x \in \Omega_i$, $n_i(x)$ and $n_e = \text{const}$ are the refraction coefficients of the waveguide and the ambient medium, $\partial u/\partial \nu$ is the outward normal derivative on the boundary $\Gamma$ of the bounded domain $\Omega_i$ on the plane $R^2$, $\Omega_e = R^2 \backslash \bar{\Omega}_i$, $[u]$ is the jump of the function $u$ on the contour $\Gamma$, $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, $\omega > 0$ is a given frequency of electromagnetic oscillations, $\varepsilon_0$ is the dielectric constant, $\mu_0$ is the magnetic constant, $\lambda = \beta^2$, and $\beta$ is the longitudinal propagation constant.

Following [5] (see also [6; 7, p. 228; 8, p. 31; 9]), we assume that the function $u$ satisfies the partial condition at infinity, i.e., can be represented in the form

$$u(x) = \sum_{n=-\infty}^{\infty} \alpha_n H_n^{(1)} \left( \sqrt{q_e - \lambda}\, r \right) \exp(in\varphi) \tag{3}$$

for sufficiently large $|x|$, where $r$ and $\varphi$ are the polar coordinates of the point $x$ and $H_n^{(1)}$ is the first-kind Hankel function of order $n$. Following [7, p. 228; 8, p. 31; 9], we can readily show that the eigenvalues of problem (1)–(3) must be sought on the Riemann surface $\Lambda$ of the function $\ln \sqrt{q_e - \lambda}$.

The exact solution of this problem is well known (e.g., see [1, p. 258]) in the case of a homogeneous waveguide with circular cross-section. In the case of some particular waveguide structures, approximate solutions were constructed, their asymptotic properties were analyzed, algorithms were developed for the computation of spectral characteristics of waveguides with cross-sections of special form and with a specific distribution of the refraction coefficient (e.g., see [1, p. 258; 2, p. 68; 3; 4]). Karchevskii [10, 11] suggested and investigated a numerical method for the case of a piecewise constant refraction coefficient.

In the present paper, we establish the localization domain of eigenvalues of problem (1)–(3) on the Riemann surface $\Lambda$ and reduce problem (1)–(3) to the spectral problem for a Fredholm operator function holomorphic in $\Lambda$. Using the results of [12, p. 39], we show that the characteristic set of this operator function can consist only of isolated points, which are its characteristic values. On the basis of the method similar to [13], we show that problem (1)–(3) has at least one simple real positive eigenvalue corresponding to a positive eigenfunction.

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## 1. LOCALIZATION OF EIGENVALUES

We assume that $q_i$ is a continuous function continuously differentiable in $\Omega_i$ and $\Gamma$ is a Lipschitz contour. By $U$ we denote the the space of continuous complex-valued functions continuously differentiable in $\bar{\Omega}_i$ and $\bar{\Omega}_e$ and twice continuously differentiable in $R^2\backslash\Gamma$. A nonzero function $u \in U$ is referred to as an *eigenfunction* of problem (1)–(3) corresponding to an eigenvalue $\lambda \in \Lambda$ if the relations of problem (1)–(3) are valid.

By $\Lambda_0$ we denote the main (physical) leaf of the surface $\Lambda$, which is determined by the conditions $-\pi < \arg\sqrt{q_e - \lambda} < 2\pi$ and $\operatorname{Im}\sqrt{q_e - \lambda} \geq 0$. Let $G$ be an interval of the real axis on the leaf $\Lambda_0$, namely, $G = \{\lambda \in \Lambda_0 : \ \operatorname{Im}\lambda = 0, \ q_e < \lambda < q_m\}$, and let $q_m = \max_{x\in\Omega_i} q_i(x)$.

**Theorem 1.** *On $\Lambda_0$, the eigenvalues of problem (1)–(3) can lie only in the interval $G$.*

**Proof.** If $\lambda \in \Lambda_0$ lies on the real axis in the domain $\lambda < q_e$, then the coefficient of Eq. (1) in $\Omega_e$ is a positive real number. In this case, problem (1)–(3) has only the trivial solution (e.g., see [14, p. 72]). For the remaining values of $\lambda \in \Lambda_0$, from the relations of problem (1)–(3) and from the asymptotic formula (e.g., see [15, p. 222 of the Russian translation])

$$H_n^{(1)}(z) = \sqrt{2/(\pi z)}\exp(i(z - n\pi/2 - \pi/4))(1 + O(1/z)), \qquad z \to \infty,$$

valid if $-\pi < \arg z < 2\pi$, we can readily obtain

$$\int\limits_{\Omega_i\cup\Omega_e} |\nabla u|^2 dx + \int\limits_{\Omega_i\cup\Omega_e} (\lambda - q)|u|^2 dx = 0. \tag{4}$$

To this end, one must use the Green formula in $\Omega_i$ and $\Omega_R = \{x \in \Omega_e : \ |x| < R\}$ and let $R$ tend to infinity. Only the zero function $u$ satisfies inequality (4) for real values of $\lambda$ lying in the domain $\lambda \geq q_m$. Taking the imaginary part of (4), we obtain

$$\operatorname{Im}\lambda \int\limits_{\Omega_i\cup\Omega_e} |u|^2 dx = 0.$$

Consequently, the eigenvalues of problem (1)–(3) on $\Lambda_0$ cannot have a nonzero imaginary part, which completes the proof of the theorem.

If $\lambda \in G$, then condition (3) is valid for functions exponentially decaying as $|x| \to \infty$ (surface waves); if $\lambda \in \Lambda\backslash\Lambda_0$, then the condition holds for exponentially growing functions (leaking waves).

## 2. A NONLINEAR SPECTRAL PROBLEM FOR A FREDHOLM HOLOMORPHIC OPERATOR FUNCTION

**Lemma.** *Let $u$ be the eigenfunction of problem (1)–(3) corresponding to an eigenvalue $\lambda \in \Lambda$. Then $u(x) = (B(\lambda)u)(x)$, $x \in R^2$, where*

$$(B(\lambda)u)(x) = \int\limits_{\Omega_i} \Phi(\lambda; x, y)p(y)u(y)dy, \qquad p(y) = q_i(y) - q_e,$$

$$\Phi(\lambda; x, y) = (i/4)H_0^{(1)}\left(\sqrt{q_e - \lambda}\,|x - y|\right), \qquad x \in R^2, \qquad y \in \Omega_i.$$

**Proof.** The desired assertion can be proved with the use of the Green formula and the relation (e.g., see [8, p. 35; 6; 16])

$$\int\limits_{\Gamma_R} \left(\frac{\partial u(y)}{\partial|y|}\Phi(\lambda; x, y) - \frac{\partial\Phi(\lambda; x, y)}{\partial|y|}u(y)dy\right) = 0,$$

which is valid for any $\lambda \in \Lambda$, an arbitrary function $u \in U$ satisfying condition (3), and a circle $\Gamma_R$ entirely lying in $\Omega_e$.

For a given $\lambda \in \Lambda$, we set

$$(K(\lambda)v)(x) = \int_{\Omega_i} \Phi(\lambda; x, y) p^{1/2}(x) p^{1/2}(y) v(y) dy. \tag{5}$$

We consider the operator $K(\lambda)$ treated as an operator acting in the space $L_2(\Omega_i)$ of complex-valued functions. We set $A(\lambda) = I - K(\lambda)$, where $I$ is the identity operator in $L_2(\Omega_i)$. For any $\lambda \in \Lambda$, $A(\lambda)$ is a Fredholm operator. Following [8, p. 71], we can readily show that the operator function $A(\lambda)$ is holomorphic in $\Lambda$. A nonzero function $v \in L_2(\Omega_i)$ is called an *eigenfunction* of the operator function $A(\lambda)$ corresponding to a characteristic value $\lambda_0 \in \Lambda$ if $A(\lambda_0)v = 0$. The characteristic set of the operator function $A(\lambda)$ is defined as the set of numbers $\lambda \in \Lambda$ such that the operator $A(\lambda)$ does not have a bounded inverse in $L_2(\Omega_i)$.

**Theorem 2.** *If $u \in U$ is an eigenfunction of problem* (1)–(3) *corresponding to an eigenvalue $\lambda_0 \in \Lambda$, then $v = p^{1/2}u \in L_2(\Omega_i)$ is an eigenfunction of the operator function $A(\lambda)$ corresponding to the characteristic value $\lambda_0$. If $v \in L_2(\Omega_i)$ is an eigenfunction of the operator function $A(\lambda)$ corresponding to a characteristic value $\lambda_0 \in \Lambda$, then $u = B(\lambda_0)(p^{-1/2}v) \in U$ is an eigenfunction of problem* (1)–(3) *corresponding to the eigenvalue $\lambda_0$.*

**Proof.** The first assertion of the theorem readily follows from the lemma. Let $v \in L_2(\Omega_i)$ be an eigenfunction of the operator function $A(\lambda)$ corresponding to the characteristic value $\lambda_0 \in \Lambda$. The kernel $\Phi(\lambda; x, y)$ is weakly polar for any $\lambda \in \Lambda$. Consequently, the function $u = B(\lambda_0)(p^{-1/2}v)$ is continuous on $\bar{\Omega}_i$ (e.g., see [17, p, 327]). By virtue of the well-known properties of the area potential [17, p. 463], the function $u$ is continuous and continuously differentiable in $R^2$ and twice continuously differentiable in $\Omega_i$ and $\Omega_e$. Furthermore, the number $\lambda_0$ and the function $u$ satisfy Eq. (1). Using the Graf addition theorem [18, p. 201], we can readily prove that the number $\lambda_0$ and the function $u$ satisfy condition (3). The proof of the theorem is complete.

Theorems 1 and 2, together with the results of [12, p. 39], imply the following assertion.

**Theorem 3.** *The characteristic set of the operator function $A(\lambda)$ consists only of isolated points, which are its characteristic values.*

## 3. THE EXISTENCE OF AN EIGENVALUE

Let the operator $K(\lambda)$ with given $\lambda \in G$ be determined by (5) and act in the space $L_2(\Omega_i)$ of real functions. For given $\lambda \in G$, we introduce the problem $v = \gamma K(\lambda)v$. The solutions $\gamma = \gamma(\lambda)$ and $v \neq 0$ of this problem are referred to as the *characteristic value* and the *eigenfunction* of the operator $K(\lambda)$. Note that $K(\lambda)$ with any $\lambda \in G$ is an integral operator with a symmetric weakly polar positive kernel [17, p. 327].

If for some $\lambda_0 \in G$, the function $v$ is an eigenfunction of the operator $K(\lambda_0)$ corresponding to the characteristic number $\gamma = 1$, then $v$ is an eigenfunction of the operator function $A(\lambda)$ corresponding to the characteristic value $\lambda_0$.

**Theorem 4.** *Problem* (1)–(3) *has at least one simple eigenvalue lying in the interval $G$ and corresponding to a positive eigenfunction.*

**Proof.** For given $\lambda \in G$, the operator $K(\lambda)$ has countably many positive characteristic values. The minimal of them satisfies the relation [17, p. 326]

$$\gamma_1(\lambda) = \inf_{f \in L_2(\Omega_i)} (f, f)/(K(\lambda)f, f), \tag{6}$$

where $(\cdot, \cdot)$ is the inner product in $L_2(\Omega_i)$. Let us now show that there exists a $\lambda \in G$ such that $\gamma_1(\lambda) = 1$. Since $\Phi(\lambda; x, y)$ continuously depends on $\lambda \in \Lambda$, it follows that $\gamma_1 = \gamma_1(\lambda)$ is a continuous function. This, together with the limit relation $\Phi(\lambda; x, y) \to \infty$ as $\lambda \to q_e$, implies that $\gamma_1(\lambda) \to 0$ as $\lambda \to q_e$.

Let us show that $\gamma_1(q_m) > 1$. Let $v \in L_2(\Omega_i)$ be an eigenfunction of the operator $K(\lambda)$ corresponding to the characteristic value $\gamma_1$ for a given $\lambda = q_m$. Following the proof of Theorems 1 and 2, for the function $u = \gamma_1 B(q_m)(p^{-1/2}v)$, we obtain the relation

$$\int\limits_{\Omega_i \cup \Omega_e} |\nabla u|^2 dx + (q_m - q_e) \int\limits_{\Omega_i \cup \Omega_e} |u|^2 dx - \gamma_1 \int\limits_{\Omega_i} p|u|^2 dx = 0.$$

Obviously, if $\gamma_1 \leq 1$, then the function $u$ must be zero identically. Hence $\gamma_1(q_m) > 1$.

By $\lambda_1$ we denote the unique solution of the equation $\gamma_1(\lambda) = 1$. By the Entsch theorem [17, p. 329], $\gamma_1(\lambda_1)$ is a simple characteristic value and corresponds to a positive eigenfunction $v_1$. Consequently, $\lambda_1$ is a simple eigenvalue of problem (1)–(3) corresponding to the positive eigenfunction $u_1 = B(\lambda_1)(p^{-1/2}v_1)$. The proof of the theorem is complete.

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