On Analytical Continuation in the Determinantal Method (*).

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1. – In a recent paper ([1]), NEWTON has studied the generalization of the determinantal method ([2-7]) to the continuous channels. To define $D^-_{\alpha\beta\dots}(E)$ (*i.e.* the value of the Fredholm determinant $D(E)$ of the Lippmann-Schwinger equation on one of the unphysical sheets of its Riemann surface support), he was led to exhibit two procedures:

a) The well-known method of analytic continuation ([1,2,5]).

b) The «substitution rule» which consists in using the definition of $D(E)$ as the Fredholm determinant of the Lippman-Schwinger equation:

$$D(E) = \mathrm{Det}\,(1 - G\cdot V)\,, \tag{1}$$

V being the Hermitian potential matrix and G the Green function channel matrix. To get from $D(E)$ to its analytic continuation $D^-(E)$, we must replace the G-elements G^+_α, G^+_β ... by G^-_α, G^-_β ....

NEWTON shows that, if the equivalence of the two procedures is *frequent*, not only the analytic continuation becomes meaningless in the continuous-channel case, but the equivalence is not even assured in every case of discrete thresholds.

He justifies this second assertion by remarking that, in the usual D/D formalism —applied to the overlapping threshold case—although we can define $D^-_{\alpha\beta\dots}(E)$ through the substitution rule, we cannot get, for instance, from $D(k_1, k_2)$ to $D(k_1, -k_2)$ through analytic continuation when $k_1^2 = k_2^2 = E = s$.

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([2]) R. G. NEWTON: *Journ. Math. Phys.*, **2**, 188 (1961).
([3]) R. G. NEWTON: *Scattering Theory of Waves and Particles* (New York, 1966).
([4]) R. BLANKENBECLER: *1963 Scottish Summer School* (Edinburgh, 1964).
([5]) M. BENAYOUN, J. LANCIEN and PH. LERUSTE: P.A.M. 68-06 and P.A.M. 68-07 (Paris, 1968); see also the paper submitted for publication to *Nuovo Cimento* (*Relativistic extension of the Le Couteur and Newton formalism*).
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([7]) K. J. LE COUTEUR: *Proc. Roy. Soc.*, A **256**, 115 (1960).

But the substitution rule, formulated in this way, depends entirely on the existence of a Schrödinger equation to describe the process (*i.e.* on a potential matrix).

We define here relations analogous to eq. (1), but independent of such an hypothesis; and which can be extended to the relativistic case, where the Schrödinger equation is no longer valid.

2. – We are going to express D through the determinants we can construct on the multichannel reduced-reactance K-matrix.

a) We shall respectively denote by K and K' the multichannel reduced-reactance and reactance matrices (*).

The Fredholm determinant of the Lippmann-Schwinger equation has been written

$$D(E) = \operatorname{Det}(1 - GV) . \tag{1}$$

The G-matrix of the Green functions $G_{ij} = G_i \delta_{ij}$ of the channels can be decomposed into two matrices, one of them (PG) depending on the momenta k_i, only through k_i^2:

$$G^{\pm} = PG \mp i\delta(E - H) = PG \mp ik\theta_N , \tag{2}$$

P standing for the Cauchy principal value ([3]), θ_N being the projector on the manifold N of all the channels; and k standing for the matrix of all the $k_{ij} = k_i \delta_{ij}$.

We obtain after a simple calculation (**)

$$D(E) = \operatorname{Det}(1 - GV) \det\left(1 + k^{\frac{1}{2}} V(1 - GV)^{-1} k^{\frac{1}{2}}\right) . \tag{3}$$

The K- and K'-matrices, which have all the desired properties of Hermiticity and meromorphy since G is Hermitian, are

$$K = - V(1 - GV)^{-1} , \tag{4a}$$

$$K' = - k^{\frac{1}{2}} V(1 - GV)^{-1} k^{\frac{1}{2}} . \tag{4b}$$

Then $D(s)$ can be written

$$D(s) = \operatorname{Det}(1 - GV) \det(1 - ikK) . \tag{5}$$

If we state

$$B_0(s) = \operatorname{Det}(1 - GV)$$

and

$$K_{ij}(s) = - \langle i | V(1 - GV)^{-1} | j \rangle , \tag{6}$$

the eq. (5) becomes

$$D(s) = B_0(s) \cdot \det(1 - ikK) . \tag{7}$$

(*) The kernel of the Lippmann-Schwinger equation shall be denoted by GV and not by K (see ref. ([1])); it must be understood as the operator product $G \cdot V$.

(**) We denote by « det », the determinants of matrices acting on the channel space; and by « Det », those of matrices acting also in the state space.

b) We shall denote by a capital Latin letter any submanifold of channels; the complete manifold of channels being denoted N; then θ_R stands for the projector upon the channel submanifold $R \subseteq N$; D_R is the value of D on the sheet which is reached by moving around the thresholds of the submanifold R, when it is meaningful.

The label of the B_L-functions is given by their definition:

$$B_R(s) = B_0(s) \det K_R = B_0(s) \det (\theta_R K \theta_R) \,, \qquad \forall\, R \subseteq N \tag{8}$$

We find, using (7) and (8),

$$D(s) \equiv D_0(s) = \sum_{\varnothing \subseteq L \subseteq N} (-\,i)^{\mathrm{Card}(L)} k_L B_L(s) \,, \tag{9}$$

where

$$k_L = \prod_{i \in L} k_i = \prod_{i \in L} (s - s_i)^{\frac{1}{2}} \qquad \text{and} \qquad k_\varnothing = 1 \,.$$

The expression (9) gives the dependence of D with respect to the determinants we can build with the K-matrix. These determinants, as meromorphic single-valued functions of s, can be continued analytically without any theoretical difficulty.

c) D_R is the value of D obtained through the substitution rule applied to the case of the R-channel submanifold; it can obviously be written

$$D_R = B_0(s) \det [1 - i(1 - 2\theta_R) kK] \,, \qquad \forall\, R \subseteq N \,. \tag{10}$$

Another expression of D_R can be obtained by analytic continuation—the channel momenta being equal or not—if we use the relation (9):

$$D_R = \sum_{\varnothing \subseteq L \subseteq N} (-)^{\mathrm{Card}(R \cap L)} (-\,i)^{\mathrm{Card}(L)} k_L B_L(s) \,. \tag{11}$$

d) On the other hand, we can easily deduce for every discrete threshold case

$$\det (S_R) = \det (\theta_R S \theta_R) = \det{}_R\, (1 + iK')(1 - iK')^{-1}] = \frac{D_R}{D_0} \,. \tag{12}$$

3. – All we have stated above leads us to the following conclusions:

a) The factor $(-)^{\mathrm{Card}(R \cap L)}$ in rel. (11) is equivalent to the change of the signs of the channel momenta contained in R. This change in signs, which defines the analytic continuation, is associated here to the B_L-functions of relation (8). Therefore there is no ambiguity any longer, whether the channel momenta are equal or not.

b) If there is no difficulty, with respect to eq. (11), in doing the necessary analytical continuations, it is because we use the K-matrix elements as basic quantities; moreover, the reduced-reactance matrix is always well defined, even in the relativistic case.

We can remark that, in the overlapping threshold case, the D_R we have defined are not the values of a *single* multivalued meromorphic function D [6].

c) The relations (7) and (10) which contain determinants of finite-rank matrices are simpler than eq. (1) — the relation deduced from it by the use of the substitution rule—where a functional determinant appears.

Hence eqs. (7) and (10) define a new formulation of Newton's substitution rule, since $(1 - 2\theta_R)$ is equivalent to the change

$$G_\alpha^+ \to G_\alpha^- , \qquad \forall\, \alpha \in R . \tag{13}$$

d) Equation (1) is of practical use for a small number of channels. When that number becomes greater than two, there appear constraint relations on the B_R-functions; those relations are the origin of the well-known constraints on the D-function [2,5].

e) In the case of an infinite number of channels (*i.e.* H is a manifold of infinite cardinals), the B_L-functions are in an infinite number of higher order: If H is our initial manifold, and $\mathscr{P}(H)$ the manifold of all the subsets of H, the number of the B_L-functions is equal to

$$\text{Card}\,[\mathscr{P}(H)] > \text{Card}\,(H) .$$

Then eq. (11) becomes difficult to use. On the other hand, eqs. (7) and (10) have a convenient form in order to be extended to this case.

f) As the relations (10) and (11) do not depend on the existence of a Schrödinger equation for the process involved, it should be possible to extend them to the relativistic case; and this does happen [5].

It follows from *a*) and *c*) above that our formulation gives back, in every case of discrete thresholds, a complete equivalence between the analytic continuation procedure and the substitution rule, in the form we have expressed them.