N-Dimensional Anisotropic Oscillator in a Uniform Time-Dependent Electromagnetic Field.

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Recently the method of time-dependent invariants (1) has been used by MALKIN and MAN'KO (2,3) to study the evolution of coherent states (4) in explicit time-dependent problems. This method is applied to the problem of an n-dimensional anisotropic oscillator in a uniform time-dependent electromagnetic field. Exact solutions to the corresponding time-independent oscillator have been obtained recently (5).

The Hamiltonian is assumed to have the form

$$H = \tfrac{1}{2}\boldsymbol{p}\boldsymbol{p} + \tfrac{1}{2}[\boldsymbol{q}, A(t)\boldsymbol{p}]_+ + \tfrac{1}{2}\boldsymbol{q}K(t)\,\boldsymbol{q} + \boldsymbol{f}(t)\boldsymbol{p} + \boldsymbol{g}(t)\,\boldsymbol{q}\,, \tag{1}$$

where $A(t), K(t)$ are real time-dependent $n\times n$ matrices and $\boldsymbol{f}(t)$, $\boldsymbol{g}(t)$ are real time-dependent n vectors. The dependence on the electromagnetic-field quantities in (1) is not given explicitly. $\boldsymbol{p}$ and $\boldsymbol{q}$ obey the commutation relation

$$[q_i, p_j]_- = i\hbar\delta_{ij}\,. \tag{2}$$

As in (3) we assume that the time-dependent invariants b_j (the range of j will be fixed later) have the form

$$b_j = \boldsymbol{v}_j(t)\,\boldsymbol{q} + iw_j(t)\boldsymbol{p} + \delta_j(t)\,, \tag{3}$$

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(1) H. R. LEWIS and W. B. RIESENFELD: *Journ. Math. Phys.*, **10**, 1458 (1969).
(2) I. A. MALKIN, V. I. MAN'KO and D. A. TRIFONOV: *Phys. Lett.*, **30** A, 414 (1969).
(3) I. A. MALKIN and V. I. MAN'KO: *Phys. Lett.*, **32** A, 243 (1970).
(4) R. J. GLAUBER: *Phys. Rev.*, **131**, 2766 (1963).
(5) E. E. BERGMANN and A. HOLZ: *Exact solutions of an n-dimensional anisotropic oscillator in a uniform magnetic field*, preprint Department of Physics, Lehigh University, Bethlehem, Pa.

where $\boldsymbol{v}_j(t), \boldsymbol{w}_j(t)$ are n vectors and $\delta_j(t)$ is a scalar. b_j will be subject to the condition

$$\frac{\mathrm{d}b_j}{\mathrm{d}t} = \frac{\partial b_j}{\partial t} + (i\hbar)^{-1}[b_j, H]_- = 0 . \tag{4}$$

For (4) to be satisfied identically for b_j of the form (3), the following set of equations has to hold:

$$\delta_j(t) = \int_0^t \big(i\boldsymbol{g}(t)\boldsymbol{w}_j - \boldsymbol{f}(t)\boldsymbol{v}_j\big)\,\mathrm{d}t , \tag{5}$$

$$\Gamma(t)\boldsymbol{\eta}_j(t) = -i\Xi\dot{\boldsymbol{\eta}}_j . \tag{6}$$

where $\Gamma(t)$ and Ξ are $2n \times 2n$ matrices given by

$$\Xi = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \qquad \Gamma(t) = \begin{pmatrix} I & \widetilde{A}(t) \\ A(t) & K(t) \end{pmatrix}, \tag{7}$$

and

$$\boldsymbol{\eta}_j(t) = \begin{pmatrix} -\boldsymbol{v}_j(t) \\ i\boldsymbol{w}_j(t) \end{pmatrix} \tag{8}$$

is a $2n$ vector. $\widetilde{A}$ is the transpose of A. I is a $n \times n$ unit matrix. We assume that, for $t < 0$, H is time independent and that

$$\boldsymbol{f}(t) = \boldsymbol{g}(t) = 0 \qquad \text{for } t \leqslant 0 . \tag{9}$$

The fundamental system of solutions of (6) can be found by means of the ansatz ([6])

$$\boldsymbol{\eta}_j = \boldsymbol{\xi}_j \exp[i\omega_j t] \qquad \text{for } t \leqslant 0 , \tag{10}$$

where $\boldsymbol{\xi}_j$ is a time-independent $2n$ vector determined by

$$\Gamma\boldsymbol{\xi}_j = \omega_j \Xi \boldsymbol{\xi}_j . \tag{11}$$

This equation, however, is studied in ([5]) where it is obtained for the creation and annihilation operators of the time-independent problem. The solutions of (11) will be put into the form of two $n \times n$ matrices

$$\widetilde{V}(t) = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n) , \qquad \widetilde{W}(t) = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n) . \tag{12}$$

([6]) E. A. CODDINGTON and N. LEVINSON: *The Theory of Ordinary Differential Equations* (New York, Toronto, London, 1955), p. 75.

The set of operators b_j and their Hermitian conjugates $b_j^\dagger$ (both satisfy (4)) can then be written in the form

$$\text{(13)} \qquad \boldsymbol{b} \equiv \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = V(t)\boldsymbol{q} + iW(t)\boldsymbol{p}\,, \qquad \boldsymbol{b}^\dagger \equiv \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = V^*(t)\boldsymbol{q} - iW^*(t)\boldsymbol{p}\,,$$

where V^* is the complex conjugate of V. It is easy to show that if (11) has only real eigenvalues which are different from zero all solutions can be normalized to satisfy the relations

$$\text{(14)} \qquad V^*\tilde{W} + W^*\tilde{V} = \hbar^{-1}I\,, \qquad V\tilde{W} - W\tilde{V} = 0\,.$$

A detailed derivation of (14) for positive definite Hamiltonians, which requires

$$\text{(15)} \qquad K - A\tilde{A}$$

to be positive definite, is given in (5). In the case where (15) holds all positive ω_j's will be associated with $\boldsymbol{b}$. From (14) it follows

$$\text{(16)} \qquad [b_i, b_j^\dagger]_- = \delta_{ij}\,, \qquad [b_i, b_j]_- = [b_i^\dagger, b_j^\dagger]_- = 0\,.$$

Denoting the matrices (12) for $t = 0$ by V_i, W_i we have as initial condition for the time-dependent problem

$$\text{(17)} \qquad \boldsymbol{b}(0) = V_i\boldsymbol{q} + iW_i\boldsymbol{p}\,, \qquad \boldsymbol{b}^\dagger(0) = V_i^*\boldsymbol{q} - iW_i^*\boldsymbol{p}\,.$$

If $\Gamma(t)$ is assumed to be continuous on the closed bounded t-interval $(0, t_f)$, then according to (7) the solutions of (6) are uniquely determined by the initial values. The solutions in general can only be given approximately. They satisfy, however, a number of important relations. It follows from (4) that

$$\text{(18}a\text{)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}[b_i, b_j^\dagger]_- = \hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}_j^\dagger(t)\boldsymbol{w}_i(t) + \boldsymbol{w}_j^\dagger(t)\boldsymbol{v}_i(t)\right) = 0\,,$$

$$\text{(18}b\text{)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}[b_i, b_j]_- = \hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(v_j(t)\,w_i(t) + w_j(t)v_i(t)\right) = 0$$

hold. Integration of (18a), (18b) and use of the initial conditions (17) shows that the relationships (14), (16) hold in the t-interval $(0, t_f)$ as well.

Let us consider next the evolution of the coherent states. From (14), (16) it follows that the coherent states can be constructed as the eigenstates of the b_j's as follows:

$$\text{(19)} \qquad b_j|\alpha\rangle = \alpha_j|\alpha\rangle \qquad \text{for } j = 1, \ldots, n\,,$$

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(7) E. A. Coddington and N. Levinson: *The Theory of Ordinary Differential Equations* (New York, Toronto, London, 1955), p. 20.

where α_j is an arbitrary complex number. For $t=0$ they assume the form

$$(20)\qquad |\alpha, i\rangle = \exp\left[-\frac{1}{2}\sum_{i=1}^{n}|\alpha_i|^2\right]\sum_{\{m_j\}}^{\infty}\prod_{j=1}^{n}\frac{\alpha_j^{m_j}}{(m_j!)^{\frac{1}{2}}}|\{m_j\}\rangle$$

in terms of the number eigenstates $|\{m_j\}\rangle$ of the initial Hamiltonian. In co-ordinate representation we have

$$(21)\qquad |\alpha, i\rangle = (2\pi h^2)^{-n/4}|\det W_i|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}|\alpha_i|^2\right]*\exp\left[-\boldsymbol{q}\frac{W_i^{-1}V_i}{2h}\boldsymbol{q}+h^{-1}\boldsymbol{\alpha}\widetilde{W}_i^{-1}\boldsymbol{q}\right],$$

where $\boldsymbol{\alpha}$ is a n vector with components α_i. The normalization of (21) follows by comparison of the Taylor expansion of (21) with (20) and the normalization of the ground state ([5]). By means of (14) it can be shown ([5]) that W and V are nonsingular and $\operatorname{Re}(WV^{-1})$ is positive definite.

The eigenstates of $b_j(t)$ for $t\geqslant 0$ can be chosen to satisfy Schrödinger's equation and the initial condition

$$(22)\qquad |\alpha, 0\rangle = |\alpha, i\rangle\,.$$

One obtains

$$(23)\qquad |\alpha, t\rangle = (2\pi h^2)^{-n/4}|\det W_i|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{j=1}^{n}|\alpha_j|^2\right]*$$
$$*\varphi(t,\alpha)\exp\left[-\boldsymbol{q}\frac{W^{-1}(t)V(t)}{2h}\boldsymbol{q}-h^{-1}(\boldsymbol{\alpha}-\boldsymbol{\delta}(t))\widetilde{W}^{-1}(t)\boldsymbol{q}\right],$$

where

$$(24)\qquad \varphi(t,\alpha)=\exp\left[-\int_0^t \mathrm{d}t\left(\frac{i}{4}\operatorname{Tr}\left(W^{-1}(t)V(t)\right)-h^{-1}\boldsymbol{f}(t)W^{-1}(t)(\boldsymbol{\alpha}-\boldsymbol{\delta})+\right.\right.$$
$$\left.\left.+\frac{1}{2}\left((\boldsymbol{\alpha}-\boldsymbol{\delta}(t))\widetilde{W}^{-1}(t)\right)^2\right)\right].$$

For $t\geqslant t_f$ the Hamiltonian is time independent again. We assume

$$(25)\qquad \boldsymbol{f}(t)=\boldsymbol{g}(t)=0\qquad\qquad \text{for } t\geqslant t_f\,.$$

The final coherent states for $t=t_f$ are given by

$$(26)\qquad |\beta, f\rangle = (2\pi h^2)^{-n/4}|\det W_f|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}|\beta_i|^2\right]*\exp\left[-\boldsymbol{q}\frac{W_f^{-1}V_f}{2h}\boldsymbol{q}+h^{-1}\boldsymbol{\beta}\widetilde{W}_f^{-1}\boldsymbol{q}\right],$$

where V_f and W_f are determined from (11) for $\Gamma(t_f)$. Note that W_f is note qual to $W(t_f)$.

By use of (23), 26) we obtain the generating function ([1]) for the transition amplitudes between the energy eigenstates of the initial and final Hamiltonian in the form

$$(27)\qquad \langle\beta, f|\alpha, t_f\rangle\exp\left[\tfrac{1}{2}\sum_{i=1}^{n}(|\alpha_i|^2-|\beta_i|^2)\right]=(\det R)^{-\frac{1}{2}}_{\text{p.v.}}|\det W_f|^{-\frac{1}{2}}|\det W_i|^{-\frac{1}{2}}\varphi(t_f,\alpha)*$$
$$*\exp\left[\tfrac{1}{2}\left(\boldsymbol{\beta}^*W_f^{\dagger-1}R^{-1}W_f^{*-1}\beta^*-2(\boldsymbol{\alpha}-\boldsymbol{\delta}(t_f))\widetilde{W}^{-1}(t_f)R^{-1}W_f^{*-1}\boldsymbol{\beta}^*+\right.\right.$$
$$\left.\left.+(\boldsymbol{\alpha}-\boldsymbol{\delta}(t_f))\widetilde{W}^{-1}(t_f)R^{-1}W^{-1}(t_f)(\boldsymbol{\alpha}-\boldsymbol{\delta}(t_f))\right)\right],$$

where

$$R = \hbar\big(W^{-1}(t_f)\,V(t_f) + W_f^{*-1}\,V_f^*\big)\,, \tag{28}$$

and the p.v. means principal value. By means of (14) it can be shown that the real part of R is positive definite hence $\det(R)$ will be nonsingular.

An alternative representation of (27) is

$$\langle\beta, f|\alpha, t_f\rangle \exp\left[\frac{1}{2}\sum_{i=1}^{n}\big(|\alpha_i|^2 + |\beta_i|^2\big)\right] = \sum_{\{m_i\}}\sum_{\{r_j\}}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{(\alpha_i)^{m_i}(\beta_j^*)^{r_j}}{(m_i!\,r_j!)^{\frac{1}{2}}}\langle\{r_j\}|\{m_i\}\rangle\,. \tag{29}$$

Here $|\{r_f\}\rangle$ are the number eigenstates of the final Hamiltonian and $|\{m_i\}\rangle$ are the states which have evolved out of the eigenstates of the Hamiltonian for $t=0$ via the interaction with the electromagnetic field. Comparing the Taylor series expansion of (27) with (29) determines the transition amplitudes [4]

$$\langle\{r_i\}|\{m_j\}\rangle\,. \tag{30}$$

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