IL NUOVO CIMENTO VOL. XXXVIII, N. 1 1° Luglio 1965

Singularities in the Physical Region (*).

S. COLEMAN (**)
Harvard University - Cambridge, Mass.

R. E. NORTON
University of California - Los Angeles, Cal.

(ricevuto il 9 Gennaio 1965)

Summary. — It is shown that a Feynman amplitude has singularities on the physical boundary if and only if the relevant Feynman diagram can be interpreted as a picture of an energy- and momentum-conserving process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time. As a by-product of the proof, the Feynman parameter associated with an internal line is identified (within a proportionality factor) with the time the particle exists between collisions, divided by its mass.

1. – Introduction.

In this note we present a simple criterion for the existence of singularities of a complex Feynman amplitude on the physical boundary. We define the physical boundary as that portion of the domain of the complex amplitude for which it is the Fourier transform of the vacuum expectation value of a time-ordered product of field operators. On the physical boundary all external momenta are real, but they need not be on the mass shell.

In Sect. 2 we show that the familiar Landau equations (1) (together with some readily-derived subsidiary conditions) form necessary and sufficient conditions for the existence of singularities on the physical boundary, and that these conditions are equivalent to the condition that the relevant Feynman diagram be interpretable as a picture of an energy- and momentum-conserv-

---

(*) Supported in part by the National Science Foundation.
(**) Alfred Sloan Research Fellow.
(1) L. LANDAU: *Nucl. Phys.*, **13**, 187 (1959); J. BJORKEN: unpublished; J. MATHES: *Phys. Rev.*, **113**, 381 (1959).

ing process occurring in space-time, with all internal particles real, on the mass shell, and moving forward in time.

This criterion is a generalization, applicable to an arbitrary Feynman amplitude, of an interpretation which has been employed by several authors ([2]) to illuminate features of singularities in specific diagrams, particularly the triangle graph as it relates to the Peierls mechanism ([3]).

2. – Argument.

The amplitudes corresponding to an arbitrary Feynman graph can be written as

$$I(p_i,\, m_i) = \int \nu(q_i)\, \delta(\Sigma\alpha_i - 1)\, D^{-N} \Pi_i\, \mathrm{d}\alpha_i \Pi_j\, \mathrm{d}^4 k_j \,, \tag{1}$$

where

$$D = \sum_i \alpha_i (q_i^2 - m_i^2) \,.$$

N is the number of internal lines, k_j is the independent four-momentum associated with the j-th closed loop, the p_j are the external momenta, $\nu(q_i)$ is a polynomial whose structure depends upon the spins of the internal particles, and α_i, q_i, and m_i are, respectively, the Feynman parameter, the propagating momentum, and the mass associated with the i-th internal line.

If we let the integration run over real k_i and real positive α_i, then eq. (1) evidently defines an analytic function for real external momenta and for internal masses with negative imaginary parts. The value of this function when the imaginary parts of the internal masses go to zero is the value of the amplitude on the physical boundary. The amplitude is analytic on the boundary unless either a singularity of the integrand appears at an end point of the contour of integration of a coalescing pair of singularities pinch the contour; any other

---

([2]) For example, J. B. Bronzan: *Phys. Rev.*, **134**, B 687 (1964); R. E. Norton: *Phys. Rev.*, **135**, B 1381 (1964); C. Kacser: *Phys. Lett.*, **12**, 269 (1964); I. J. R. Aitchison: *Phys. Rev.*, **133**, B 1257 (1964).

([3]) R. F. Peierls: *Phys. Rev. Lett.*, **6**, 641 (1961); C. Goebel: *Phys. Rev. Lett.*, **13**, 143 (1964); R. C. Hwa: *Phys. Rev.*, **130**, 1580 (1963). Because the internal particles involved are necessarily unstable, the Peierls singularity itself is not on the physical boundary. However, as we let the widths of the internal particles go to zero, the Peierls singularity moves to a location at which all external momenta are real; it therefore moves to the physical boundary, if it was on a sheet close to the physical boundary at the beginning. Thus ou rcriterion may be applied to determine if the Peierls singularity directly causes a peaking in a mass distribution in the small width limit. It can readily be verified that it doesn't.

singularities of the integrand may be avoided by shifting the contour at the last moment.

The conditions that there be either a coalescing pair of singularities or an end point singularity in each variable are the Landau [1] equations:

(2a) $$q_i^2 = m_i^2\,,$$

or

(2b) $$\alpha_i = 0$$

for each internal line, and

(3) $$\sum \alpha_i q_i = 0$$

for each closed loop. The conditions that these lie on the unshifted contour of integration are

(4) $$\alpha_i \geqslant 0\,,$$

and

(5) $$q_i = q_i^*\,.$$

Of course, coalescing singularities do not necessarily pinch the contour (in the one-dimensional case, for example, two poles may approach the contour from the same side). To show that in our case a pinch in fact occurs, we perform the k integrations in eq. (1). The amplitude then assumes the form [4]

(6) $$I(p_i, m_i) = \lim_{\varepsilon\to 0^+} \lambda \int_0^1 \Pi_i \,\mathrm{d}\alpha_i\, \delta(1-\Sigma\alpha_i)[C(\alpha)]^{n-2l-1}[D(\alpha, p) + i\varepsilon]^{2l-N}\,,$$

where λ is a constant, l is the number of closed loops, $C(\alpha)$ is a homogeneous polynomial in the α_i of degree l, and $D(\alpha, p)$ is a homogeneous polynomial in the α_i of degree $l+1$. If $\bar{\alpha}_i$ is a point of coalescence, and if we define $\eta_i = \alpha_i - \bar{\alpha}_i$, then in the immediate vicinity of $\bar{\alpha}_i$

(7) $$D = \tfrac{1}{2}\sum_{ij} D_{ij}\eta_i\eta_j\,,$$

where D_{ij} is the matrix composed of the second derivatives of D evaluated at $\bar{\alpha}_i$. D_{ij} is a real, symmetric matrix and hence can be diagonalized by a real, orthogonal co-ordinate transformation. In terms of these new co-ordinates,

---

(4) R. J. Eden: *Phys. Rev.*, **119**, 1763 (1960).

it is easy to see that the zeros of $D(\alpha, p) + i\varepsilon$ in eq. (6) must approach $\bar{\alpha}_i$ from opposite sides of the contour as $\varepsilon \to 0$. We should remark, of course, that this argument would break down if the matrix D_{ij} had zero eigenvalues other than the one that belongs to $\eta_i \propto \bar{\alpha}_i$. Such additional zeros, if they occur would be located at very exceptional points corresponding to cusps in the Landau curve in the physical region ([5]) The singularity would then be on the verge of leaving the physical boundary.

Thus, except for these possible exceptional points, eqs. (2)-(5) are necessary and sufficient conditions for the appearance of a singularity on the physical boundary. We now show that these equations admit a direct physical interpretation. Precisely, we demonstrate that they are the necessary and sufficient conditions for the consistency of the following picture of the transition: each vertex interaction occurs as an instantaneous event in space-time, and the internal particles propagate on the mass shell with the momenta q_i, forward in time, for just the correct distances and times to « tie together » the entire graph and allow it to be visualized as an ordered sequence of successive interactions. It is clear that, once the consistency of this picture is verified for a choice of the space-time intervals between the vertices, the consistency remains if all of these intervals are scaled by the same factor. In particular, there is no limit to the time over which the entire interaction can extend. As we shall see, this arbitrariness of scale corresponds to the fact that the α_i in eqs. (2)-(5) are determined only to within a common multiplier.

To show that eqs. (2)-(5) imply the consistency of the above picture, let us define (to within a scale factor) a space-time separation Δ_i between the two vertices connected by the propagating momentum q_i,

$$\Delta_i \propto \alpha_i q_i \,. \tag{8}$$

If alternative (2b) holds, this definition tells us that the two points are in fact one; in this case, we will shrink the line connecting them to a point, and apply all our subsequent arguments to the diagram thus short-circuited ([6]).

We observe:

1) As a consequence of eqs. (4) and (5), the space-time separation between any two points is real.

2) As consequence of eqs. (2a) and (5), the momentum of every internal particle is real and on the mass shell.

3) The separation given by eq. (8) is consistent with the motion of a classical particle with the momentum q_i, if we interpret (to within a constant

---

([5]) R. J. Eden, P. V. Landshoff, J. C. Polkinghorne and J. C. Taylor: *Journ. Math. Phys.*, **2**, 656 (1961).

([6]) A familiar stratagem: but in this instance graced with a physical interpretation.

of proportionality) α_i as the proper time the particle exists between collisions divided by the particle mass. Equation (4) tells us that the particle is moving forward in time.

4) Since any two points inside a diagram may be connected by a chain of internal lines, we may extend in an obvious way the definition of the separation given by eq. (8) to any two points. This extension is consistent if

$$\sum \Delta_i = 0 \tag{9}$$

around each closed loop. This is eq. (3).

Since all of our arguments are clearly reversible, eqs. (2)–(5) are both necessary and sufficient conditions for

1) the occurrence of a singularity on the physical boundary;

2) the consistency of interpreting the reaction as actually proceeding as a sequence of successive interactions in the way in which we have described.

The identity of these two notions is thus proved.

* * *

One of us (S.C.) would like to thank the Physics Department of U.C.L.A. for its hospitality and Mrs. D. SAXON for an excellent dinner.

RIASSUNTO (*)

Si mostra che un'ampiezza di Feynman ha singolarità al contorno fisico, se, e solo se, il relativo diagramma di Feynman può essere interpretato come la descrizione di un processo di conservazione nello spazio-tempo dell'energia e dell'impulso, con tutte le particelle interne reali sullo strato della massa, e dotate di moto in avanti nel tempo. Come risultato accessorio di questa dimostrazione, il parametro di Feynman associato ad una linea interna si identifica (a meno di un fattore di proporzionalità) con il tempo in cui la particella esiste fra le collisioni, diviso per la massa della particella.

(*) *Traduzione a cura della Redazione.*