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Sequent Correlations in Evolutionary Stochastic Point Processes and Its Application to Cascade Theories.

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Summary. — Sequent correlations in multiplicative stochastic processes are defined with the aid of product densities associated with the process. Such sequent correlations describe the process in a more detailed manner than the conventional correlations especially with regard to the parametric values characterizing the process. The evolution of any particular member of the multiplicative entities in the parameter is brought out by the sequent correlations. It is found that such types of correlation functions are very useful in the description of electromagnetic and perhaps nucleon cascades. To demonstrate this possibility, correlation functions of the first few orders are evaluated for the simple case of electron-photon cascade.

1. – Introduction.

Let us consider a multiplicative stochastic process progressing with respect to a certain continuous parameter t, characterizing the process. The process may depend on another continuous parameter x which may or may not be independent of t. Many processes arise in Physics and Biology where we deal with entities distributed over a continuous infinity of states. The parameter x may stand for the energy of a cosmic-ray particle or the age of a member of a bacterial growth. Thus a natural way to describe the system is by the introduction of an evolving continuous set of random variables. However, we encounter a difficulty due to a situation where only probability densities can be attached to particular values of the parameter x and not nonzero probabilities. For this reason such processes have been known as point processes (see for example BARTLETT [1]). While integration of the above-mentioned probability density over x will yield only the mean number of entities distributed over the range of integration, this procedure does not yield any result pertaining to the fluctuations in the mean which is a very important criterion. To overcome this difficulty, higher-order density functions expressing correlations in x-space have been introduced by KENDALL [2], BHABHA [3] and RAMAKRISHNAN [4]. These are called cumulant functions by KENDALL (the reason being that these functions enable us to obtain the cumulants) and product densities by RAMAKRISHNAN for the obvious reason that it is a probability magnitude in the product space. In view of the elegance and simplicity in interpretation, we shall use the product density notation of Ramakrishnan.

2. – Product densities.

If $N(x, t)$ is the stochastic variable representing the number of entities at time t with parametric value $X \leqslant x$, the density relation [*]

$$f_1(x, t)\,\mathrm{d}x = \mathscr{E}\{\mathrm{d}N(x, t)\} \tag{2.1}$$

exists [**] such that the probability of one entity having a parametric value in $(x, x+\delta x)$ is $f_1(x)\,\delta x$, the total probability of more than one entity having a parametric value in the range being $O(\delta x)$. $f_1(x, t)$ is termed the product density of degree one. The product density of degree two is defined by

$$f_2(x_1, x_2, t)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \mathscr{E}\{\mathrm{d}N(x_1, t)\,\mathrm{d}N(x_2, t)\}\,, \qquad x_1 \neq x_2\,. \tag{2.2}$$

(2.2) may be interpreted as the simultaneous probability of finding the parametric values of two entities in the intervals $(x_1, x_1+\mathrm{d}x_1)$ and $(x_2, x_2+\mathrm{d}x_2)$, provided the two intervals do not overlap. However, when there is a overlap (*i.e.*, $x_1 = x_2$),

$$\mathscr{E}\{\mathrm{d}N(x_1, t)\,\mathrm{d}N(x_2, t)\}_{x_1=x_2} = \mathscr{E}\{[\mathrm{d}N(x, t)]^2\} = \mathscr{E}\{\mathrm{d}N(x, t)\} = f_1(x, t)\,\mathrm{d}x\,. \tag{2.3}$$

In a similar manner higher-order densities can be defined.

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(1) M. S. BARTLETT: *Stochastic Processes* (Cambridge, 1955).

(2) D. G. KENDALL: *Journ. Roy. Statistist. Soc.*, B **11**, 230 (1949).

(3) H. J. BHABHA: *Proc. Roy. Soc.*, A **202**, 301 (1950).

(4) A. RAMAKRISHNAN: *Proc. Camb. Phil. Soc.*, **46**, 595 (1950).

(*) Throughout this paper, the symbol $\mathscr{E}$ will be used to denote the expectation value of the quantity within the brackets.

(**) The term existence is used from a purely physical point of view in the sense that there are a number of physical situations wherein such conditions hold good.

To obtain the second moment of the number of entities with parametric values over a finite interval (a, b) we need integrate $\mathscr{E}\{\mathrm{d}N(x_1, t)\,\mathrm{d}N(x_2, t)\}$ and take into account the degeneracy introduced by the overlap of $\mathrm{d}x_1$ and $\mathrm{d}x_2$. Writing the number of entities in the interval (a, b) as $N(a, b, t)$, we obtain

$$\mathscr{E}\{[N(a, b; t)]^2\} = \int_a^b f_1(x, t)\,\mathrm{d}x + \int_a^b\int_a^b f_2(x_1, x_2; t)\,\mathrm{d}r_1\,\mathrm{d}x_2\,. \tag{2.4}$$

Higher-order moments have been arrived at in a similar way and the explicit expression for the r-th moment in terms of linear combinations of integrals of product densities of degree less than or equal to r have been obtained by RAMAKRISHNAN (4).

So far, we have not paid any attention to the nature of the parameter x. The parameter x may be called the intrinsic parameter for obvious reasons. If x denotes the age of an individual in a birth and death process, the age is a deterministic function in the sense that if the time t_0 of birth is known, the age of the individual can be specified at any later time $t > t_0$, the parameter t standing for time in this case. However, there are a number of physical processes where the time or position in t-axis of a creation of a particle and its then x-value do not determine its subsequent x value. A concrete example is provided by the energy state x of an electron in the electron-photon cascade. In this case, the multiplicative stochastic process under consideration evolves with respect to both t and x, the energy parameter. In fact it makes sense to talk of the parametric value at the point of the creation of the particle. Experimentally, it is convenient to make energy measurements at the point of production. As a matter of fact, the data on electromagnetic cascades observed in nuclear emulsions makes reference only to the energies of the pairs of electrons at the point of production. Thus we can talk of product density in the product space of t and x. Such product densities have been used in the formulation of cascade theories (see for example reference (5) and (6)).

To define such a product density in the product space of x and t, we define $\mathscr{M}(x, t)$ as the random variable representing the number of entities which are « born » at a parametric value $T \leqslant t$, the intrinsic parametric value X being less than or equal x. The $\mathscr{M}(x, t)$ is the stochastic variable representing the number of entities that are created between t and $t+\mathrm{d}t$ and have an intrinsic parametric value between x and $x+\mathrm{d}x$ at its inception. If $\pi(n)$ is the probability that n entities are born between t and $t+\mathrm{d}t$ with intrinsic parametric

(5) A. RAMAKRISHNAN and S. K. SRINIVASAN: *Proc. Ind. Acad. Sci.*, A **44**, 263 (1956).

(6) S. K. SRINIVASAN, J. C. BUTCHER, B. A. CHARTRES and H. MESSEL: *Nuovo Cimento*, **9**, 77 (1958).

values between x and $x+\mathrm{d}t$, then

$$(2.5)\qquad \begin{cases} \pi(1) = f_1(x, t)\,\mathrm{d}x\,\mathrm{d}t + o(\mathrm{d}x\,\mathrm{d}t) = \mathscr{E}\{\mathrm{d}\mathscr{M}(x, t)\}\,, \\ \pi(0) = 1 - f_1(x, t)\,\mathrm{d}x\,\mathrm{d}t + o(\mathrm{d}x\,\mathrm{d}t)\,, \\ \pi(n) = o(\mathrm{d}x\,\mathrm{d}t)\,, \qquad x > 1. \end{cases}$$

From this point onwards, it is clear that a product-density technique for the product space Ω of x and t can be carried in toto. The moment formula (2.5) can be written for the product space by replacing $\mathrm{d}x$ by $\mathrm{d}t$.

From the above, it is clear that once the product densities are known, the problem of fluctuations involves only integration over the parametric range. Thus the main task in any problem will be the explicit determination of the product densities. The method of obtaining product densities for the case of electron-photon cascades has been dealt with by RAMAKRISHNAN (4) and BHABHA and RAMAKRISHNAN (7). However a glance at the Mellin transform solution obtained by BHABHA and RAMAKRISHNAN will convince us of the gigantic magnitude of the task of anyone who attempts to obtain some numbers from those results. Thus calculation of higher moments becomes, though not impossible, formidably difficult. To offset this difficulty, there have been attempts to reformulate the problem in such a way that the evaluation of the cumulants (factorial moments) of the number distribution is somewhat less tedious (see for example references (8) and (9)). This method completely eliminates the use of product densities of any other type of correlation functions and will not be discussed any further. Alternatively, we can introduce new density functions which include more information. Thus in terms of those functions, it may not be necessary to go beyond the second order in any physical situation. For example in the case of electromagnetic cascades, we can introduce two-point correlations in E-space. Specifically we can deal with particles that have been produced between t and $t+\mathrm{d}t$ with energy lying between E and $E+\mathrm{d}E$ at the point of production and are found to have an energy between E' and $E'+\mathrm{d}E'$ at $t' > t$. Such a generalization, apart from the possibility of overcoming certain computational difficulties mentioned above, may be interesting from the point of view of comparison with experiments, particularly in cascades. Moreover, there are certain physical features characterizing the particles, an example being polarization of electrons or mesons which is directly related to the energy at the point of production rather than at a later point of observation. These new densities associated with more than one interval of

(7) H. J. BHABHA and A. RAMAKRISHNAN: *Proc. Ind. Acad. Sci.*, A **32** e, 141 (1950).
(8) L. JANOSSY: *Proc. Phys. Soc. London*, A **63**, 241 (1950).
(9) S. K. SRINIVASAN: *Zeit. Phys.*, **161**, 346 (1961).

parametric space may be called *evolutionary sequent correlations* for obvious reasons. In Sect. 3, the sequent correlation functions are defined and the moment formulae for the number distribution as well as other quantities of physical interest are deduced. Section 4 of the paper will deal with physical examples, particularly from cascade theory of cosmic radiation.

3. – Sequent correlations.

The concept of sequent product densities were first introduced by RAMAKRISHNAN and RADHA ([10]) who distinguished between « instant » and « sequent » correlations in point processes. The instant correlation relates to the study of correlations of the random variables corresponding to the same value of t while the sequent correlation relates to that between the random variable corresponding to different values of t. Though the arguments used by RAMAKRISHNAN and RADHA heavily depend on t being the time parameter, the results are applicable to any ordered parameter. In fact if $N(x, t)$ in the notation of Sect. 2 is the stochastic variable representing the number of entities having a parametric value $X \geqslant x$, then the sequent product densities can be obtained by considering the expectation value of the product $\mathrm{d}N(x_1, t)_1 \cdot \mathrm{d}N(x_2, t_2) \ldots \mathrm{d}N(x_m, t_m)$. The sequent product densities contain more information than the instant product densities in that it partly explains the dependence of $N(x_i, t_i)$ on $N(x_j, t_j)$. However, the second-order sequence density is always expressible only in terms of second-order instant density and as such there seems to be no advantage from a computational point of view. On the other hand, the evolutionary sequent correlation will be shown to have a decisive advantage over the instant or even sequent densities.

We now proceed to a proper definition of evolutionary sequent correlations. Towards this end, let us define the primitive parametric value of an entity as the parametric value at (its time of inception) the point of its production. We can now consider the random variable $M(x_1, t_1; x_2, t_2)$ representing the number of entities produced between 0 and t_1, the primitive parametric value of each of which is greater than or equal x_1, the entities having a parametric value greater than or equal x_2 at $t = t_2$. Then we can define an evolutionary sequent correlation density by considering $\mathrm{d}M(x_1, t_1; x_2, t_2)$ which denotes the stochastic variable representing the number of entities that are produced between t_1 and $t_1 + \mathrm{d}t$, with the primitive parametric value of each of which is between x_1 and $x_1 + \mathrm{d}x_1$, the parametric value of these entities lying between x_2 and $x_2 + \mathrm{d}x_2$ at $t = t_2$. We shall reserve the symbol $\mathscr{F}$ to denote such a product density. If $P(n)$ is the probability that the random variable $\mathrm{d}M$

([10]) A. RAMAKRISHNAN and T. K. RADHA: *Proc. Cam. Phil. Soc.*, **57**, 843 (1961).

takes the value n, it is reasonable to assume that $P(1)$ is of the order $\delta\Omega$ while $P(n)$ for $n \geqslant 2$ is of a smaller order of magnitude as compared to $\delta\Omega$, $\delta\Omega$ being an infinitesimal element in the space in which the density functions are defined (*). Thus we can define the sequent correlation density of degree one by

$$(3.1) \qquad \begin{cases} P(1) = \mathscr{F}_1(x_1, t_1; x_2, t_2)\,\delta\Omega + o(\delta\Omega)\,, \\ \qquad = \mathscr{E}\{\mathrm{d}M(x_1, t_1; x_2, t_2)\}\,, \\ P(0) = 1 - \mathscr{F}_1(x_1, t_1; x_2, t_2)\,\delta\Omega + o(\delta\Omega) \\ P(n) = o(\delta\Omega)\,, \qquad\qquad n > 1. \end{cases}$$

Higher-order sequent correlation densities are defined in a similar manner. Correlation density of degree n is defined by

$$(3.2) \qquad \mathscr{E}\{\mathrm{d}M(x_1, t_1; x', t)\,\mathrm{d}M(x_2, t_2; x_2', t)\,\ldots\,\mathrm{d}M(x_n, t_n; x_n', t)\} = \\ = \mathscr{F}_n(x_1, t_1; x_2, t_2; \ldots x_n, t_n; x_1', x_2', \ldots x_n', t)\,\delta\Omega_1\,\delta\Omega_2 \ldots \delta\Omega_n$$

provided the $\delta\Omega_i$ are disjoint. If all the $\delta\Omega_i$'s are not disjoint, then a degeneracy as in the case of usual product densities occur. We shall not any further discuss since all the steps leading to the moment formula (2.5) are applicable in the present case provided we replace $\mathrm{d}x$ by $\mathrm{d}\Omega$.

4. – Sequent correlations in cascades.

4·1. *Two point correlations: sequent correlation density of degree one.* – The sequent correlation density is very useful in the description of electromagnetic cascades. Let us consider an electron-photon shower initiated by an electron or a photon of energy E_0. We shall assume that the shower develops by pair creation by photons and bremsstrahlung by electrons. Defining $\mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E_0)$ as the sequent product density of degree one of electrons in a shower initiated by a primary of i-th type ($i = 1, 2$ denotes an electron and a photon primary, respectively) we can obtain differential equations by considering the various possible outcomes of events that may happen in the infinitesimal interval $(0, \Delta)$, of t-axis. The primary electron (photon) may radiate a photon and drop down to a lower energy (create an electron positron pair) in the interval $(0, \Delta)$, thus giving rise to two independent primaries. To incorporate the contribution from such an outcome of events, we use the

(*) Ω is the product space of x_1, x_2 and t_1.

invariant imbedding method [11]. To be precise, we imbed the process corresponding to a thickness t into a class of processes obtained by letting t take any positive value.

Thus we obtain

$$(4.1)\quad \mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E_0) = \left(1-\Delta\int R^i(E'\,|\,E_0)\,\mathrm{d}E'\right)\mathscr{F}_1^i(E_1, t-\Delta; E_2, t-\Delta\,|\,E_0)+ \\ +\Delta\int R^i(E'\,|\,E_0)\,\mathrm{d}E'\,\{\mathscr{F}_1^i(E_1, t_1-\Delta; E_2, t-\Delta\,|\,E')+ \\ +\mathscr{F}_1^{3-i}(E_1, t_1-\Delta;\; E_2, t-\Delta\,|\,E_0-E')\}\,.$$

By making $\Delta\to 0$, we derive the differential equation

$$(4.2)\quad \left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t}\right)\mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E_0) = -\int R^i(E'\,|\,E_0)\,\{\mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E_0) - \\ -\mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E')\}\,\mathrm{d}E' + \int R^i(E'\,|\,E_0)\,\mathscr{F}_1^{3-i}(E_1, t_1; E_2, t\,|\,E_0-E')\,\mathrm{d}E'\,.$$

To solve (4.2), we note that asymptotic forms of $R^i(E'\,|\,E_0)$ are given by [12]

$$(4.3)\quad R^1(E'\,|\,E) = \left\{\frac{E-E'}{E}-\left(\frac{4}{3}+\alpha\right)\left(1-\frac{E}{E-E'}\right)\right\}\frac{1}{E}\,,$$

$$(4.4)\quad R^2(E'\,|\,E) = \left\{1-\left(\frac{4}{3}+\alpha\right)\left(\frac{E'}{E}-\frac{E'^2}{E^2}\right)\right\}\frac{1}{E}\,.$$

In view of the cross-sections $R^i(E'/E)$ being homogeneous in E and E', we notice that

$$(4.5)\quad \mathscr{F}_1^i(E_1, t_1; E_2, t\,|\,E_0)\,\mathrm{d}E_1\,\mathrm{d}E_2 = \mathscr{F}_1^i(\varepsilon_1, t_1; \varepsilon_2, t_2)\,\mathrm{d}\varepsilon_1\,\mathrm{d}\varepsilon_2 \qquad \left(\varepsilon_1=\frac{E_1}{E_0}\,,\ \varepsilon_2=\frac{E_2}{E_0}\right).$$

Thus (4.2) can be written as

$$(4.6)\quad \left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t}\right)\mathscr{F}_1^i(\varepsilon_1, t_1; \varepsilon_2, t) = -\int R^i(\varepsilon')\left\{\mathscr{F}_1^i(\varepsilon_1, t_1; \varepsilon_2, t)-\mathscr{F}_1^i\left(\frac{\varepsilon_1}{\varepsilon'}, t_1; \frac{\varepsilon_2}{\varepsilon'}, t\right)\right\}\mathrm{d}\varepsilon' + \\ +\int R^i(\varepsilon')\,\mathscr{F}_1^{3-i}\left(\frac{\varepsilon_1}{1-\varepsilon'}, t_1; \frac{\varepsilon_2}{1-\varepsilon'}, t\right)\mathrm{d}\varepsilon'\,.$$

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(11) R. BELLMAN, R. KALABA and G. M. WING: *Journ. Math. Phys.*, **1**, 280 (1960).
(12) H. A. BETHE and W. HEITLER: *Proc. Roy Soc.*, **146**, 83 (1934).

Defining the Mellin transform of $\mathscr{F}^i(\varepsilon_1, t_1; \varepsilon_2, t)$ as

(4.7) $$\mathscr{P}_1^i(s_1, t_1; s_2, t) = \int_0^1\int_0^1 \varepsilon_1^{s_1-1}\cdot\varepsilon_2^{s_2-1}\cdot\mathscr{F}_1^i(\varepsilon_1, t_1; \varepsilon_2, t)\,\mathrm{d}\varepsilon_1\,\mathrm{d}\varepsilon_2\,,$$

we can write (4.6) in the matrix form as

(4.8) $$\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t}\right)\mathscr{P}_1(s_1, t_1; s_2, t) = L\mathscr{P}_1(s_1, t_1; s_2, t)\,,$$

where $\mathscr{P}_1$ is a vector whose components are $\mathscr{P}_1^i(s_1, t_1; s_2, t)$ and L is the two-by-two matrix given by

(4.9) $$L = \begin{pmatrix} -A(s_1+s_2-1) & C(s_1+s_2-1) \\ B(s_1+s_2-1) & -D \end{pmatrix},$$

(4.10a) $$A(s) = \int_0^1 (1-\varepsilon^{s-1})\,R'(\varepsilon')\,\mathrm{d}\varepsilon'\,,$$

(4.10b) $$B(s) = 2\int_0^1 \varepsilon^{s-1}\,R^2(\varepsilon')\,\mathrm{d}\varepsilon'\,,$$

(4.10c) $$C(s) = \int_0^1 (1-\varepsilon)^{s-1}\,,$$

(4.10b) $$D = \int_0^1 R^2(\varepsilon')\,\mathrm{d}\varepsilon'\,.$$

The initial conditions to be imposed on (4.2) are given by

(4.11a) $$\mathscr{F}_1^1(E_1, 0; E_2, t\,|\,E_0) = 0\,,$$

(4.11b) $$\mathscr{F}_1^2(E_1, t_1; E_2, t\,|\,E_0) = 2R^2(E_1|E_0)\,\pi(E_2|E_1, t)\,,$$

where $\pi(E_2|E_1, t)\,\mathrm{d}E_2$ denotes the probability that an electron of energy E_1 at $t=0$ drops to an energy between E_2 and $E_2+\mathrm{d}E_2$ after traversing a thickness t. Thus the initial conditions satisfied by the system of eqs. (4.8) are given by

(4.12a) $$\mathscr{P}_1^1(s_1, 0; s_2, t) = 0\,,$$

(4.12b) $$\mathscr{P}_1^2(s_1, 0; s_2, t) = \exp[-A(s_1)\cdot t]\,B(s_1+s_2-1)\,,$$

where we have made use of the well-known expression for the Mellin transform of $\pi(E_2|E_1;t)$ [13]:

$$\pi(s|E_1;t) = \int_0^1 E_2^{s-1}\cdot\pi(E_2|E_1;t)\,\mathrm{d}E_2 = E_1^{(s-1)}\,. \tag{4.13}$$

(4.8) can be solved either by taking a Laplace transformation with respect to t and t_1 or by diagonalizing the matrix L and solving the resulting partial differential equations. As the method is straightforward, we refrain from giving the intermediate steps. The final solution is given by

$$\mathscr{P}_1^1(s_1, t_1; s_2, t) = \frac{C(s_1+s_2-1)\,B(s_1+s_2-1)}{\mu(s_1+s_2-1)-\lambda(s_1+s_2-1)}\cdot \\ \cdot\{\exp[-\lambda(s_1+s_2-1)t_1]-\exp[-\mu(s_1+s_2-1)t_1]\}\exp[-A(s_2)(t-t_1)]\,, \tag{4.14a}$$

$$\mathscr{P}_1^2(s_1, t_1; s_2, t) = \frac{B(s_1+s_2-1)}{\mu(s_1+s_2-1)-\lambda(s_1+s_2-1)}\cdot \\ \cdot\{[\mu(s_1+s_2-1)-D]\exp[-\lambda(s_1+s_2-1)t_1]- \\ -[D-\lambda(s_1+s_2-1)]\exp[-\mu(s_1+s_2-1)t_1]\}\exp[-A(s_2)(t-t_1)]\,. \tag{4.14b}$$

The mean number of electrons that are produced with primitive energies greater than ε_1 times the energy of the primary between 0 and t and remain above a certain fraction of the primary energy at $t'>t$ can be calculated by inverting E_1 and t_1 over the appropriate range. The occurence of functions with arguments s_1+s_2-1 and s_2 will simplify the evaluation of the inversion integrals to a great extent. Thus, we can conveniently study the mean numbers for small values of t'. The method of evaluation of inversion integrals of the type encountered on the right-hand side of (4.14) has been discussed by us [14] in connection with the fluctuation problem of electromagnetic cascades and can be taken over in toto. A study of these mean numbers will give us a deeper insight into the problem since the manner in which the electrons drop down in energy as t increases is made transparent.

4.2 *Sequent correlation density of degree one: alternative approach.* – The densities $\mathscr{F}_1^i(E_1, t_1; E_2, t|E_0)$ can be arrived at by different arguments.

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[13] L. JANOSSY: *Cosmic Rays* (Oxford, 1950).

[14] S. K. SRINIVASAN, K. S. S. IYER and N. V. KOTESWARA RAO: *Zeit. Phys.* (1963) (in press).

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The method is very simple and consists in expressing $\mathscr{F}_1^i$, in terms of the conventional product densities of degree one of electrons and photons that exist at t. Since the electron is produced between t_1 and $t_1+\mathrm{d}t_1$, it is necessary that there should be a photon at t_1 and it should create a pair between t_1 and $t_1+\mathrm{d}t_1$, one of the electrons having an energy in the prescribed range. Thus, if $g^i(E|E_0;\,t)$ is the product density of degree one of photons that exist at t due to a shower excited by a primary of energy E_0 and type i, then

$$\mathscr{F}_1^i(E_1,\,t_1;\,E_2,\,t_2|\,E_0) = 2\int g_1^i(E\,|\,E_0;\,t_1)\,R^2(E_1|\,E)\,\pi(E_2|\,E_1;\,t-t_1)\,\mathrm{d}E\,. \tag{4.15}$$

The equivalence of (4.14) and (4.13) can be verified by taking a Mellin transform of both sides of (4.14) and using well-known solutions ([15]) of cascade theory for $g_1^i(E|E_0;\,t)$. However the method indicated in the earlier part of this Section does not make use of the conventional product densities and illustrates the method of building up cascade theory purely in terms of the sequent correlation densities. The advantage of this method will become transparent in the final part of this Section where the invariant imbedding method will be shown to yield a simple set of equations for the second-order densities.

4·3 *Higher-order sequent correlation density: invariant imbedding approach.* – We now deal with $\mathscr{F}_2^i(E_1,\,t_1;\,E_2,\,t_2;\,E_3,\,E_4,\,t|E_0)$ the sequent correlation density of degree two of electrons produced between t_1 and $t_1+\mathrm{d}t_1$ and t_2 and $t_2+\mathrm{d}t_2$ and observed at a later thickness $t>t_1,\,t_2$. Exactly as in the first part of this Section, we analyse the possible outcome of events in $(0,\,\Delta)$ of the t-axis and use the invariant imbedding technique. Thus we have

$$\begin{aligned}(4.16)\qquad \mathscr{F}_1^i(E_1,\,t_1;\,E_2,\,t_2;\,E_3,\,E_4,\,t\,|\,E_0) &= \\ =\left(1-\Delta\int R^i(E'\,|\,E_0)\,\mathrm{d}E'\right)&\mathscr{F}_2^i(E_1,\,t_1-\Delta;\,E_2,\,t_2-\Delta,\,E_3,\,E_4,\,t-\Delta\,|\,E_0)\,+\\ +\Delta\int R^i(E'\,|\,E_0)\,\{&\mathscr{F}_2^i(E_1,\,t_1-\Delta;\,E_2,\,t_2-\Delta;\,E_3,\,E_4,\,t-\Delta\,|\,E')\,+\\ +\,&\mathscr{F}_2^{(3-i)}(E_1,\,t_1-\Delta;\,E_2,\,t_2-\Delta;\,E_3,\,E_4,\,t-\Delta\,|\,E_0-E')\,+\\ +\,&\mathscr{F}_1^i(E_1,\,t_1-\Delta;\,E_3,\,t-\Delta\,|\,E')\,\mathscr{F}_1^{3-i}(E_2,\,t_2-\Delta;\,E_4,\,t-\Delta\,|\,E_0-E')\,+\\ +\,&\mathscr{F}_1^{3-i}(E_1,\,t_1-\Delta;\,E_3,\,t-\Delta\,|\,E_0-E')\,\mathscr{F}_1^i(E_2,\,t_2-\Delta;\,E_4,\,t-\Delta\,|\,E')\}\,.\end{aligned}$$

([15]) H. J. BHABHA and W. HEITLER: *Proc. Roy. Soc.*, A **159**, 432 (1937).

By letting $\Delta \to 0$, we obtain

$$(4.17)\quad \left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t}\right)\mathscr{F}_2^i(E_1, t_1; E_2, t_2; E_3, E_4, t\,|\,E_0) =$$
$$= -\int R^i(E'\,|\,E_0)\,\{\mathscr{F}_2^i(E_1, t_1; E_2, t_2; E_3, E_4, t\,|\,E_0) -$$
$$- \mathscr{F}_2^i(E_1, t_1; E_2, t_2; E_3, E_4, t\,|\,E')\}\,\mathrm{d}E' +$$
$$+\int R^i(E'\,|\,E_0)\,\{\mathscr{F}_2^{3-i}(E_1, t_1; E_2, t_2; E_3, E_4, t\,|\,E_0-E') +$$
$$+ \mathscr{F}_1^i(E_1, t_1; E_3, t\,|\,E')\,\mathscr{F}_1^{3-i}(E_2, t_2; E_4, t\,|\,E_0-E') +$$
$$+ \mathscr{F}_1^{3-i}(E_1, t_1; E_3, t\,|\,E_0-E')\,\mathscr{F}_1^i(E_2, t_2; E_4, t\,|\,E')\}\,\mathrm{d}E' .$$

The boundary conditions satisfied by $\mathscr{F}_2^i$ are given by

$$(4.18a)\quad \mathscr{F}_2^1(E_1, 0; E_2, 0; E_3, E_4, t\,|\,E_0) = 0 ,$$

$$(4.18b)\quad \mathscr{F}_2^2(E_1, 0; E_2, 0; E_3, E_4, t\,|\,E_0) =$$
$$= 2R^2(E_1\,|\,E_0)\,\delta(E_0-E_1-E_2)\,\pi(E_3\,|\,E_1, t)\,\pi(E_4\,|\,E_2, t) ,$$

$$(4.18c)\quad \mathscr{F}_2^1(E_1, 0; E_2, t_2; E_3, E_4, t\,|\,E_0) = 0 ,$$

$$(4.18d)\quad \mathscr{F}_2^1(E_1, t_1; E_2, 0; E_3, E_4, t\,|\,E_0) = 0 .$$

In addition to the above conditions, we also need the value of $\mathscr{F}_2^2(E_1, 0; E_2, t_2; E_3, E_4, t\,|\,E_0)$ and $\mathscr{F}_2^2(E_1, t_1; E_2, 0; E_3, E_4, t\,|\,E_0)$. These cannot be specified directly but can be obtained after some calculations which are straightforward. To this end, we observe that by definition of $\mathscr{F}_2^2$,

$$(4.19)\quad \mathscr{F}_2^2(E_1, 0; E_2, t_2; E_3, E_4, t\,|\,E_0) = R^2(E_1\,|\,E_0)\big[\mathscr{F}\pi_1^2(E_2, t_2; E_3, E_4, t\,|\,E_1) +$$
$$+ \pi(E_3\,|\,E_1; t)\,\mathscr{F}_1^2(E_2, t_2; E_3, t\,|\,E_0-E_1)\big] ,$$

where $\mathscr{F}\pi_1^2(E_2, t_2; E_3, E_4, t\,|\,E_1)\,\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}E_3\mathrm{d}E_4\mathrm{d}t_2$ denotes the joint probability that an electron of primitive energy between E_2 and $E_2+\mathrm{d}E_2$ is created between t_2 and $t_2+\mathrm{d}t_2$ in the shower excited by an electron having an energy E_1, the primary and the secondary electron so created dropping the energies lying in the intervals $(E_3, E_3+\mathrm{d}E_3)$ and $(E_4, E_4+\mathrm{d}E_4)$ respectively. $\mathscr{F}\pi_1^2$ is an unknown function; however this need not trouble us since it is easy to write down integral equation satisfied by $\mathscr{F}\pi_1^2$, by the use of invariant imbedding techniques. The equation can be solved by simple transform technique and will not be discussed any further in this paper. $\mathscr{F}\pi_1^2$ by itself is important in the study of bursts produced by μ-mesons. The relevance of the techniques developed in this paper and the properties of the function $\mathscr{F}\pi_1^2$ are discussed by us in the following paper ([16]) dealing with the fluctuations in size of bursts produced by electrons and μ-mesons.

Just as in the case of two-point correlations, it is quite possible to relate the three-point correlation product density in terms of the conventional single-point product density as well as the product densities defined in the product space of E and t. We shall not go into the details since the object of the present contribution is just to point out the utility of sequent correlation densities to multiple processes and in particular to cascade theory of cosmic-ray showers.

Finally we observe that eq. (4.17) can be generalized to n-point sequent correlation densities easily and the equations do not lose the simplicity of the structure present in (4.17). This is particularly interesting if we compare the equations with those satisfied by the conventional product densities of n-th order (see for example MESSEL and POTTS ([17])) where we encounter matrix equations, the matrices being of order 2^n. The simple structure is essentially due to the formulation of the problem based on invariant imbedding techniques. The utility of such a formulation in stochastic multiplicative process has been discussed in detail by RSV ([18]).

5. – Concluding remarks.

In conclusion we wish to make a few general remarks. The functions $\mathscr{F}_1(x_1, t_1; x_2, t_2)$ introduced in Sect. 2 are defined in Ω, the product space of x_1, x_2 and t_1. On the other hand, it may be worthwhile to introduce the correlation between $\mathscr{M}(x, t)$, the random variable representing the number of entities that are created between 0 and t with primitive parametric values not less than x, and the variable $N(x, t)$ representing the number of entities that are found at t with parametric values not less than x. Thus, we can deal with the function $F(x_1, t_1; x_2, t_2)$ defined by

$$F(x_1, t_1; x_2, t_2)\,\mathrm{d}\Omega = \mathscr{E}\{\mathrm{d}\mathscr{M}(x, t_1)\,\mathrm{d}N(x_2, t_2)\}\,. \tag{5.1}$$

Two interesting cases arise according as $t_1 > t_2$ or $t_2 > t_1$ and both the cases are of great importance in the interpretation of data on cosmic ray showers.

([16]) S. K. SRINIVASAN and K. S. S. IYER: *Nuovo Cimento*, to appear (1964).
([17]) H. MESSEL and R. B. POTTS: *Phys. Rev.*, **86**, 847 (1952).
([18]) A. RAMAKRISHNAN, S. K. SRINIVASAN and R. VASUDEVAN: (1964) (to be published).

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Apart from this, $F(x_1, t_1; x_2, t_2)$ has some interesting limiting properties very similar to those of sequent product densities introduced by RAMAKRISHNAN and RADHA (10). The relevance of $F(x_1, t_1; x_2, t_2)$ and higher-order correlations to evolutionary Markovian processes has been discussed in detail by RSV (19).

* * *

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(19) A. RAMAKRISHNAN, S. K. SRINIVASAN and R. VASUDEVAN: (1964) (to be published).

RIASSUNTO (*)

Si definiscono le correlazioni susseguenti nei processi stocastici moltiplicativi, con l'aiuto delle densità di prodotto associate al processo. Tali correlazioni susseguenti descrivono il processo in modo più dettagliato delle correlazioni convenzionali, specialmente per quanto riguarda i valori parametrici che caratterizzano il processo. L'evoluzione di ciascun membro delle entità moltiplicative del parametro è messa in evidenza dalle correlazioni susseguenti. Si trova che questi tipi di funzioni di correlazione sono molto utili nella descrizione delle cascate elettromagnetiche e forse di quelle nucleoniche. Per dimostrare questa possibilità, si valutano le funzioni di correlazione del primo ordine nel caso semplice della cascata elettrone-fotone.

(*) *Traduzione a cura della Redazione.*