Three-Dimensional Lorentz Group and Harmonic Analysis of the Scattering Amplitude.

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Summary. — The many-particle scattering amplitude is projected on the matrix elements of the unitary irreducible representations of the three-dimensional Lorentz group. The usefulness of this transformation in the treatment of a certain class of integral equations satisfied by the amplitude is pointed out. A generalization taking into account a set of nonunitary representations is shown to lead to a transformation which has many of the properties of the classical Laplace transformation and can be used to obtain asymptotic expansions similar to those obtained from the Watson-Sommerfeld formula.

1. – Introduction.

The aim of this work is to enlarge and develop the ideas of a preceding paper (1) about a possible generalization of the partial-wave analysis of the scattering amplitude.

The main progress consists in the fact that here we treat an amplitude which involves an arbitrary number of particles with arbitrary spin. This generalization of the method developed in (1) for the scattering of two spinless particles is useful not only for its wider applicability, but also because it permits a deeper insight into the structure of the transformation and a more direct connection with the work of mathematicians.

The mathematical background of both the expansion developed here and of the usual partial-wave analysis can be found in the theory of « harmonic analysis », that is, the generalization of the Fourier analysis from functions

(1) L. SERTORIO and M. TOLLER: *Nuovo Cimento*, **33**, 413 (1964).

defined on a straight line or on a circle to functions defined on an arbitrary (locally compact) group ([2,3]). The usual partial-wave analysis can be regarded as the harmonic analysis applied to the scattering amplitude considered as a function defined on the rotation group. This is not very clear in the case of two-body scattering, because in this case it is simpler to consider the amplitude as a function defined on a spherical surface, but, when three or more particles are involved, this point of view leads to the most clear definition of the partial-wave amplitudes ([4-7]).

In order to understand the usefulness of our expansion, let us analyse which are the advantages of the usual partial-wave representation. The most interesting are the following:

a) it permits the description of the low-energy scattering by means of a small number of parameters;

b) it permits a simplification of some relations satisfied by the amplitude (*e.g.* integral equations of the Lippmann-Schwinger or of the Bethe-Salpeter type or unitarity relations).

Another method to represent the scattering amplitude is its analysis in terms of the complex angular momentum in the crossed channel. The analogous of the advantage *a*) is that the complex angular momentum representation can give rise to a description of very-high-energy scattering in terms of a small number of « Regge-pole parameters ». An advantage similar to *b*) is not very apparent because the usual theory does not indicate any direct connection between the complex angular momentum and the symmetry group of the equation to be simplified. This is connected with the fact that the usual complex angular momentum theory cannot be inserted directly into the framework of harmonic analysis.

Two attempts have been made to connect complex angular momentum with the group theory; the former uses some « local » representations of the rotation group ([7,8]), the latter uses the representations of the three-dimensional Lorentz group ([1]). Here we shall deal with the second one.

([2]) G. MACKEY: *Bull. Amer. Math. Soc.*, **56**, 385 (1950).

([3]) L. H. LOOMIS: *Ann Introduction to Abstract Harmonic Analysis* (Princeton, 1953).

([4]) M. JACOB and G. C. WICK: *Ann. Phys.*, **7**, 404 (1959); G. C. WICK: *Ann. Phys.*, **18**, 65 (1962).

([5]) R. L. OMNÈS: *On the Three-Body Scattering Amplitude*, I, II and III, U.C.R.L. reports and *Phys. Rev.*, **134**, B 1358 (1964).

([6]) J. B. HARTLE: *Phys. Rev.*, **134**, B 610, B 620 (1964).

([7]) M. ANDREWS and J. GUNSON: *Complex Angular Momentum in Many-Particle States*, I and II, preprint (University of Birmingham, 1963). This paper gives also the connection between the matrix elements of the local representations of the rotation group and of the representations of the three-dimensional Lorentz group.

([8]) E. G. BELTRAMETTI and G. LUZZATTO: *Nuovo Cimento*, **29**, 1003 (1963).

In order to apply harmonic analysis to the scattering amplitude, the first thing to do is to represent it as a function of the elements of a group (and of other parameters which are not affected by the following transformation). A general method to reach this object is described in Sect. **2** and **3**. This procedure gives rise to various possibilities, besides the one which leads to the usual partial waves. Each case corresponds to a certain subgroup of the Lorentz group and, as we show in Sect. **4** and **5**, in each case it is possible to project the amplitude on the matrix elements of the unitary irreducible representations of the corresponding group. Only one of these possibilities, connected with the three-dimensional Lorentz group, is investigated in detail in this paper, but also some of the others could lead to physically interesting concepts and deserve a detailed study.

The reason which has prevented an earlier investigation of the properties of these new representations of the scattering amplitude is, besides their somewhat hidden physical meaning, the difficulty inherent in the harmonic analysis of functions defined on noncompact noncommutative groups. In fact the unitary representations of these groups are all infinite-dimensional and generally depend on continuous parameters; these facts lead to the necessity of using infinite matrices and integrals instead of sums over the contributions of the various representations.

In Sect. **6** we show how the transformations defined in the preceding Sections can be used to diagonalize certain relations between amplitudes.

In Sect. **7** we generalize the harmonic analysis on the three-dimensional Lorentz group, obtaining a transformation which has many of the properties of the classical Laplace transformation. In particular, if certain conditions are satisfied, one can obtain from this transformation asymptotic expansions similar to those obtained from the Watson-Sommerfeld-Mandelstam formula [9] extended to the many-particle case [5-7] (Sect. **8**).

A disadvantage of our method with respect to the Watson-Sommerfeld-Mandelstam formula is due to the fact that it can be applied only to amplitudes which, roughly speaking, decrease more rapidly than $s^{-\frac{1}{2}}$ when the square of the center-of-mass energy s goes to infinity. However we remark that this is a natural difficulty for a transformation which diagonalizes a fixed momentum transfer integral equation. In fact, if the amplitude which forms the kernel of the equation has not a sufficiently «good» asymptotic behaviour, the kernel cannot be iterated, the Neumann series cannot be written for any value of the coupling constant and the equation does not make any sense. It may happen that the physically significant solution of the equation has a «bad» asymptotic behaviour, but, at least, if the projected equation is of

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(9) S. Mandelstam: *Ann. Phys.*, **19**, 254 (1962).

the Fredholm type, it can be obtained from a well-behaved solution adding to it Regge-pole contributions in the way explained in [1].

At last, we remark that the subject of this paper can be put in relation with the theory of the representations of the inhomogeneous Lorentz group [10-12]. An outgoing particle can be considered as an incoming particle with a negative energy and therefore we can associate to it a representation of the inhomogeneous (orthocronous) Lorentz group with real mass and negative energy. When one considers the direct product of a representation of this kind with a positive energy, real mass representation, corresponding to an incoming particle, it can easily be seen that this product contains representations with imaginary mass. The «little group» associated to these representations is the three-dimensional Lorentz group considered in this paper.

2. – The scattering amplitude as a function of the group elements.

We consider the off-shell scattering amplitude for an arbitrary number of incoming and outgoing particles with arbitrary spins. This amplitude depends on N four-vectors and N discrete indices which describe the four-momenta and the spins of the involved particles.

We adopt the usual Feynman convention, *i.e.*, when the four-momentum of an incoming (outgoing) particle has negative energy component, it has to be interpreted as the four-momentum of a corresponding outgoing (incoming) antiparticle with the sign changed.

The Lorentz invariance of the amplitude can be written in the form [13]

$$\sum_{s'_1 \dots s'_N} D^{(1)}_{s_1 s'_1}(a) \dots D^{(N)}_{s_N s'_N}(a)\, \boldsymbol{M}_{s'_1 \dots s'_N}(L(a^{-1})\, P_1 \dots L(a^{-1})\, P_N) = \boldsymbol{M}_{s_1 \dots, s_N}(P_1 \dots P_N)\,, \tag{1}$$

where a is an element of the Lorentz spinor group $\mathscr{L}$ (proper or with reflections included), $L(a)$ is the matrix of the corresponding transformation acting on four-vectors and $D^{(i)}_{ss'}(a)$ are the matrices of the corresponding spinor representations corresponding to the various particles involved.

The partial-wave analysis can be considered as a description of the behaviour of the amplitude when the parameters of the incoming particles are fixed and the parameters of the outgoing particles are contemporaneously rotated in the center-of-mass system. In order to generalize the partial-wave formalism, we divide the particles involved in the amplitude into two sets,

(10) E. P. Wigner: *Ann. Math.*, **40**, 149 (1939).
(11) Iu. M. Shirokov: *Sov. Phys., JETP.*, **6**, 919 (1958).
(12) H. Joos: *Forts. d. Phys.*, **10**, 65 (1962).
(13) H. P. Stapp: *Phys. Rev.*, **125**, 2139 (1962).

A and B, which do not necessarily coincide with the sets of the incoming and of the outgoing particles.

We use the indices $1A \dots nA$ to label the n particles of the set A and $1B \dots mB$ to label the m particles of the set B. In order to obtain more compact formulas, we use some shorthand notations: with P_A we indicate the set of vectors $P_{1A} \dots P_{nA}$, with s_A we indicate the set of spin indices $s_{1A} \dots s_{nA}$ and in a similar way we define P_B and s_B. Furthermore we call $D^{(A)}_{s_A s'_A}(a)$ the direct product of the matrices $D^{(1A)}_{s_{1A} s'_{1A}}(a) \dots D^{(nA)}_{s_{nA} s'_{nA}}(a)$, and in a similar way we define $D^{(B)}_{s_B s'_B}(a)$.

With these shorthand notations the Lorentz invariance (1) can be written as

$$(2) \qquad \sum_{s'_A s'_B} D^{(A)}_{s_A s'_A}(a)\, D^{(B)}_{s_B s'_B}(a)\, \boldsymbol{M}_{s'_A s'_B}\big(L(a^{-1})\, P_A,\; L(a^{-1})\, P_B\big) = \boldsymbol{M}_{s_A s_B}(P_A, P_B)\,.$$

With the sign convention we want to use, the four-vectors P_A indicate outgoing momenta and the four-vectors P_B indicate incoming momenta. This does not mean ingoing and outgoing particles because the energies can be negative. Then the energy-momentum conservation can be written in the form

$$(3) \qquad \sum_{i=1}^{n} P_{iA} = \sum_{i=1}^{m} P_{iB} = Q\,.$$

The meaning of Q is total energy-momentum if B is the set of the incoming particles, otherwise it is a four-momentum transfer.

Now we divide the $4n$-dimensional space of P_A into sets such that if P_A and P'_A belong to the same set they can be connected by a transformation of the group $\mathscr{L}$. By means of certain well-defined conditions, we choose for each set a representative element P^0_A. Then any P_A can be represented in the form

$$(4) \qquad P_A = L(a_A) P^0_A\,.$$

The same procedure can be applied to the vectors P_B and we obtain

$$(5) \qquad P_B = L(a_B) P^0_B\,.$$

It is clear that if $P'_A = L(a)\, P_A$ $(a \in \mathscr{L})$, P_A and P'_A correspond to the same P^0_A; this means that the independent components of P^0_A can be expressed by means of the invariants (with respect to $\mathscr{L}$) which can be built with the vectors P_A.

The conditions which determine the representative vectors P^0_A and P^0_B can be chosen in many ways (a possible set of conditions is given in the Appendix A); the only assumption that we need for a general discussion is that these conditions imply that

$$\sum_{i=1}^{n} P^0_{iA} = \sum_{i=1}^{m} P^0_{iB} = Q^0 . \tag{6}$$

Then we have

$$Q = L(a_A)Q^0 = L(a_B)Q^0$$

and

$$Q^0 = L(a_A^{-1}a_B)Q^0 . \tag{7}$$

This means that $g = a_A^{-1}a_B$ belongs to the subgroup G of $\mathscr{L}$ which contains all the transformations which leave the four-vector Q^0 unchanged.

From eq. (2) it follows that

$$\boldsymbol{M}_{s_A s_B}(L(a_A)P^0_A,\, L(a_B)P^0_B) = \sum_{s'_A s'_B} D^{(A)}_{s_A s'_A}(a_A)\, D^{(B)}_{s_B s'_B}(a_B)\, M_{s'_A s'_B}(P^0_A,\, a_A^{-1}a_B,\, P^0_B)\, , \tag{8}$$

where we have defined

$$M_{s_A s_B}(P^0_A, g, P^0_B) = \sum_{s'_B} \boldsymbol{M}_{s_A s'_B}(P^0_A,\, L(g)P^0_B)\, D^{(B)}_{s_B s'_B}(g^{-1}) \qquad (g \in G). \tag{9}$$

The function (9) is a different way of expressing the scattering amplitude. The advantage of expressing it as a function of the group element g is due to the existence of the expansion theorems which we shall use in Sect. **5**.

3. – Properties of the group G.

Now we assume that $\mathscr{L}$ is the proper Lorentz spinor group and investigate the properties of the group G. The structure of groups of this kind has been investigated by WIGNER in his classical work on the unhomogeneous Lorentz group (10). He distinguishes four cases (14):

i) $t = Q^2 > 0$; then we can chose $Q^0 = (\sqrt{t}, 0, 0, 0)$ and the group G is the group of spatial rotations (more exactly, the corresponding spinor group).

ii) $t = Q^2 < 0$; then we can chose $Q^0 = (0, 0, 0\, \sqrt{-t})$ and G is the group of the Lorentz transformations in the three-dimensional pseudo-

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(14) We use the metric $Q^2 = Q_t^2 - Q_x^2 - Q_y^2 - Q_z^2$.

euclidean space with one timelike dimension and two spacelike dimensions.

iii) $t = Q^2 = 0$; $Q \neq 0$.

iv) $Q = 0$.

In the first case our general treatment becomes the usual partial-wave analysis and the case iii) is a limiting case of the first two, not very interesting from the physical point of view.

The cases ii) and iv) have a great physical interest in connection, respectively, with nonforward and forward scattering. In this paper we begin the study of case ii) and leave case iv) to a future work.

A rather complete investigation of the mathematical properties of the three-dimensional Lorentz group has been given by BARGMANN ([15]) and we follow essentially his work with slightly modified notations.

The spinor group corresponding to the 3-dimensional Lorentz group can be identified with the group of the matrices

$$W(g) = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \tag{10}$$

with

$$\alpha\bar{\alpha} - \beta\bar{\beta} = 1\,. \tag{11}$$

The group elements can be identified by means of the parameters μ, ζ, ν through the relation

$$W(g) = \begin{bmatrix} \exp[-i\mu] & 0 \\ 0 & \exp[i\mu] \end{bmatrix} \cdot \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix} \cdot \begin{bmatrix} \exp[-i\nu] & 0 \\ 0 & \exp[i\nu] \end{bmatrix}. \tag{12}$$

The range of the parameters is

$$0 \leqslant \zeta < \infty\,, \qquad 0 \leqslant \mu < 2\pi\,, \qquad 0 \leqslant \nu < 2\pi\,,$$

but the parameters μ, ζ, ν and $\mu \pm \pi$, ζ, $\nu \pm \pi$ correspond to the same group element. The invariant measure on the group is given by

$$dg = (2\pi)^{-2} \sinh(2\zeta)\, d\mu\, d\nu\, d\zeta\,. \tag{13}$$

The whole measure of the group is infinite as the group is not compact.

([15]) V. BARGMANN: *Ann. Math.*, **48**, 568 (1947).

The transformation

$$X' = L(g)X$$

on the four-vector $X = (t, x, y, z)$ is given by

$$\begin{bmatrix} t'+z' & x'-iy' \\ x'+iy' & t'-z' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \cdot \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} \cdot \begin{bmatrix} \bar{\alpha} & \beta \\ \bar{\beta} & \alpha \end{bmatrix}$$

and therefore

$$(14)\qquad L(g) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\mu & -\sin 2\mu & 0 \\ 0 & \sin 2\mu & \cos 2\mu & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cosh 2\zeta & \sinh 2\zeta & 0 & 0 \\ \sinh 2\zeta & \cosh 2\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\nu & -\sin 2\nu & 0 \\ 0 & \sin 2\nu & \cos 2\nu & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

As required, the co-ordinate z is not affected by the transformation.

Now we show the connection between g and the center-of-mass energy. We assume that there are only two incoming particles with four-momenta P_{1A} and P_{1B}. According to the Appendix A we choose

$$(15)\qquad \begin{cases} P^0_{1A} = (-\omega_A, 0, 0, -q_A) , \\ P^0_{1B} = (\omega_B, 0, 0, q_B) . \end{cases}$$

The square of the centre-of-mass energy is given by

$$(16)\qquad s = (L(a_A)P^0_{1A} - L(a_B)P^0_{1B})^2 = (P^0_{1A} - L(g)P^0_{1B})^2$$

and from eq. (14) we have

$$(17)\qquad s = \omega_A^2 + \omega_B^2 + 2\omega_A\omega_B \cosh 2\zeta - (q_B + q_A)^2 .$$

Note that the high-energy behaviour of the scattering amplitude is given by the limit $\zeta \to \infty$.

4. – The generalized partial-wave expansion.

Let us remember some important properties of the unitary irreducible representations of a compact group G ([16]):

$$\sum_r \mathscr{D}^\sigma_{mr}(g)\,\mathscr{D}^\sigma_{rn}(g') = \mathscr{D}^\sigma_{mn}(gg') \tag{18}$$

(representation property),

$$\sum_r \mathscr{D}^\sigma_{mr}(g)\,\overline{\mathscr{D}^\sigma_{nr}(g)} = \delta_{mn} \tag{19}$$

(unitarity),

$$\int_G \mathscr{D}^\sigma_{mn}(g)\,\overline{\mathscr{D}^{\sigma'}_{m'n'}(g)}\,\mathrm{d}g = (N_\sigma)^{-1}\delta_{\sigma\sigma'}\delta_{mm'}\delta_{nn'} \tag{20}$$

(orthogonality). We assume that $\int_G \mathrm{d}g = 1$ and indicate by N_σ the dimension of the representation σ.

If $f(g)$ is an L^1 function defined on G, we can perform the following transformation

$$\mathscr{F}^\sigma_{mn} = \int_G \mathscr{D}^\sigma_{mn}(g)\,f(g)\,\mathrm{d}g \tag{21}$$

and, under certain more restrictive conditions, we have the following inversion formula

$$f(g) = \sum_\sigma N_\sigma \sum_{mn} \mathscr{F}^\sigma_{mn}\,\overline{\mathscr{D}^\sigma_{mn}(g)} = \sum_\sigma N_\sigma \sum_{mn} \mathscr{F}^\sigma_{mn}\,\mathscr{D}^\sigma_{nm}(g^{-1})\,. \tag{22}$$

If $f(g)$ is also an L^2 function, the following Plancherel formula holds

$$\int_G |f(g)|^2\,\mathrm{d}g = \sum_G N_\sigma \sum_{mn} |\mathscr{F}^\sigma_{mn}|^2\,. \tag{23}$$

These properties can be extended to a certain class of noncompact groups which contains the groups we are considering.

The main difference is that the unitary representations are all infinite-dimensional and there is a continuous infinity of them. Therefore we have

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([16]) E. P. WIGNER: *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Chapt. IX and X (New York, 1959).

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to replace the operation $\sum_\sigma N_\sigma$ with the more general one $\int \mathrm{d}\sigma$ where $\mathrm{d}\sigma$ is a suitably chosen measure in the space of unitary representations (Plancherel measure). The explicit form of this measure will be given later for the case which we consider.

In the following we consider the quantities $\mathscr{D}^\sigma_{mn}(g)$ and $\mathscr{F}^\sigma_{mn}$ as the matrix elements of the operators $\mathscr{D}^\sigma(g)$ and $\mathscr{F}^\sigma$. With these modifications the formulae (18), (19), (21), (22) and (23) become

$$\mathscr{D}^\sigma(g)\,\mathscr{D}^\sigma(g') = \mathscr{D}^\sigma(gg')\,, \tag{24}$$

$$\mathscr{D}^\sigma(g)[\mathscr{D}^\sigma(g)]^\dagger = [\mathscr{D}^\sigma(g)]^\dagger\,\mathscr{D}^\sigma(g) = I\,, \tag{25}$$

$$\mathscr{F}^\sigma = \int_G \mathscr{D}^\sigma(g)\,f(g)\,\mathrm{d}g\,, \tag{26}$$

$$f(g) = \int \mathrm{Tr}\,[\mathscr{F}^\sigma \mathscr{D}^\sigma(g^{-1})]\,\mathrm{d}\sigma\,, \tag{27}$$

$$\int_G |f(g)|^2\,\mathrm{d}g = \int_G \|\mathscr{F}^\sigma\|^2\,\mathrm{d}\sigma\,, \tag{28}$$

where $\|\mathscr{F}^\sigma\|$ is the Hilbert-Schmidt norm [17] of the operator $\mathscr{F}^\sigma$ and Tr indicates the trace. The correspondence (26) can be extended to functions $f(g)$ which are L^2 but not L^1 and the Plancherel relation (28) is still valid for this extended correspondence.

Now we apply the transformation (26) to the scattering amplitude expressed in the form (9). In order to remain in the framework of the simplest mathematical theorems, we have to assume that the amplitude is an L^1 or an L^2 function of g, but this is by no means true for all the physical amplitudes. It can be shown, however, that for certain Feynman diagrams the amplitude is actually an L^1 and also an L^2 function of g for almost all the values of the other parameters. Therefore it is reasonable to develop the theory for this simpler case and then, to look for the extension to more general cases.

According to this program we define the following operator-valued function of P^0_A, P^0_B, s_A, s_B:

$$\mathscr{M}^\sigma_{s_A s_B}(P^0_A, P^0_B) = \int_G \mathscr{D}^\sigma(g)\,M_{s_A s_B}(P^0_A, g, P^0_B)\,\mathrm{d}g\,. \tag{29}$$

This is the generalized partial-wave amplitude.

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[17] If the vectors χ_n form an orthonormal basis, the trace and the Hilbert-Schmidt norm are defined by $\mathrm{Tr}\,(\mathscr{F}) = \sum_n (\mathscr{F}\chi_n, \chi_n)$, $\|\mathscr{F}\| = \left[\sum_{mn} |(\mathscr{F}\chi_n, \chi_m)|^2\right]^{\frac{1}{2}}$.

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5. – The unitary representation of the three-dimensional Lorentz group.

In order to give to the formulae of the preceding Section a more explicit form when G is the spinor three-dimensional Lorentz group, we have to classify the unitarity irreducible representations of G and to find explicitely the « Plancherel measure » $d\sigma$.

Following BARGMANN [15], we call H_1, H_2 and H_0 the infinitesimal (Hermitian) operators of a representation, corresponding respectively to the infinitesimal Lorentz transformations along the axes x and y and to the infinitesimal rotations around the axis z. The operator $H_1^2+H_2^2-H_0^2$ commutes with all the operators of the representation and therefore we can write

$$H_1^2+H_2^2-H_0^2=qI\,, \tag{30}$$

where I is the identity operator and q is a number which can be used to label the representation.

However q is not sufficient to determine univoquely the representation and we have to consider also the spectrum of the operator H_0. Its possible eigenvalues are, of course, $m=0,\ \pm\frac{1}{2}\pm 1,\ \pm\frac{3}{2}\ldots$

According to these principles, BARGMANN classifies the representations in the following four classes:

$$C_q^0:\quad \begin{cases} q>0\,, \\ m=0\,,\quad \pm 1\,,\quad \pm 2\ldots\,; \end{cases}$$

$$C_q^{\frac{1}{2}}:\quad \begin{cases} q>\frac{1}{4}\,, \\ m=\pm\frac{1}{2}\,,\quad \pm\frac{3}{2}\ldots\,; \end{cases}$$

$$D_k^-:\quad \begin{cases} q=k(1-k)\,, \\ m=-k\,,\quad -(k+1)\,,\quad -(k+2)\ldots\,, \\ k=\frac{1}{2}\,,\quad 1\,,\quad \frac{3}{2}\ldots\,; \end{cases}$$

$$D_k^+:\quad \begin{cases} q=k(1-k)\,, \\ m=k\,,\quad (k+1)\,,\quad (k+2)\ldots\,, \\ k=\frac{1}{2}\,,\quad 1\,,\quad \frac{3}{2}\ldots\,. \end{cases}$$

One can distinguish between representations of the integral type, when m takes only integral values, and representations of the half-integral type when

m takes only half-integral values. If we call $(-e)$ the group element corresponding to the parameters $\mu=\zeta=0$, $\nu=-\pi$ (*i.e.*, a complete rotation around the z axis), these representations have the following property

$$\mathscr{D}(-e) = I \qquad \text{(integral type)},$$

$$\mathscr{D}(-e) = -I \qquad \text{(half-integral type)}.$$

From eq. (9) and (2) it follows that

$$M_{s_A s_B}(P^0_A, (-e)g, P^0_B) = \pm M_{s_A s_B}(P^0_A, g, P^0_B)\,, \tag{31}$$

where the $+$ sign appears if the set A contains an even number of half-integral spin particles and the $-$sign appears if this number is odd. It follows that in the first case only representations of the integral type are needed in the expansion and in the second case only representations of the half-integral type give a contribution.

The normalization properties of the representation matrix elements can be written with our notations in the form ([18])

$$\begin{cases} \mathscr{B}(g) = \displaystyle\int \psi^\sigma \mathscr{D}^\sigma(g)\,\mathrm{d}\sigma\,, \\ \displaystyle\int_G |(\mathscr{B}(g)\chi_1, \chi_2)|^2\,\mathrm{d}g = \|\chi_1\|^2\,\|\chi_2\|^2 \int |\psi^\sigma|^2\,\mathrm{d}\sigma\,, \end{cases} \tag{32}$$

where ψ^σ is an arbitrary function defined on the set of the unitary irreducible representations. By comparison with the analogous formulae given by BARGMANN, we obtain the explicit form of the « Plancherel measure » $\mathrm{d}\sigma$, which is given by the formula

$$\int \psi^\sigma\,\mathrm{d}\sigma = \int_0^\infty \psi^0_q 2s \operatorname{tgh} \pi s\,\mathrm{d}s + \int_0^\infty \psi^{\frac12}_q 2s \operatorname{ctgh} \pi s\,\mathrm{d}s + \sum_k (2k-1)(\psi^+_k + \psi^-_k)\,, \tag{33}$$

where $q=\frac14+s^2$.

Note that some unitary irreducible representations do not appear in eq. (33); they form a set of vanishing Plancherel measure.

([18]) We always understand that the integrations are extended to the whole range of the group parameters, which cover twice the group manifold.

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6. – Diagonalization of integral relations between amplitudes.

Now we consider a process which can be decomposed into two virtual processes in the way described in Fig. 1 (the arrows do not indicate the incoming and the outgoing particles, but give only the sign convention for the energy-momentum four-vectors).

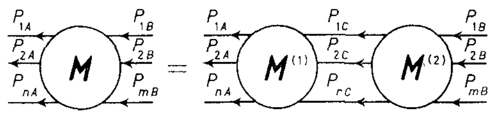

Fig. 1.

The connection between the corresponding amplitudes is of the form

$$(34)\qquad \boldsymbol{M}_{s_A s_B}(P_A, P_B) = \sum_{s_C s'_C} \int \boldsymbol{M}^{(1)}_{s_A s_C}(P_A, P_C)\, \varrho_{s_C s'_C}(P_C)\cdot \cdot \boldsymbol{M}^{(2)}_{s'_C s_B}(P_C, P_B)\, \delta^4\Big(Q - \sum_{i=1}^{r} P_{iC}\Big)\, \mathrm{d}^4P_{1C} \ldots \mathrm{d}^4P_{rC}\,,$$

where Q is defined by eq. (3). If the amplitudes M, $M^{(1)}$ and $M^{(2)}$ satisfy the Lorentz invariance conditions (2) and

$$(35)\qquad \begin{cases} \displaystyle\sum_{s'_A s'_C} D^{(A)}_{s_A s'_A}(a)\, D^{(C)}_{s_C s'_C}(a)\, \boldsymbol{M}^{(1)}_{s'_A s'_C}\big(L(a^{-1})\, P_A,\, L(a^{-1})\, P_C\big) = \boldsymbol{M}^{(1)}_{s_A s_C}(P_A, P_C)\,, \\ \displaystyle\sum_{s'_C s'_B} D^{(C')}_{s_C s'_C}(a)\, D^{(B)}_{s_B s'_B}(a)\, \boldsymbol{M}^{(2)}_{s'_C s'_B}\big(L(a^{-1})\, P_C,\, L(a^{-1})\, P_B\big) = \boldsymbol{M}^{(2)}_{s_C s_B}(P_C, P_B)\,, \end{cases}$$

the formula (34) is Lorentz-invariant if the weight-function $\varrho_{s_C s'_C}(P_C)$ satisfies the equation

$$(36)\qquad \sum_{s''_C s'''_C} D^{(C)}_{s''_C s_C}(a)\, D^{(C')}_{s'''_C s'_C}(a)\, \varrho_{s''_C s'''_C}\big(L(a) P_C\big) = \varrho_{s_C s'_C}(P_C)\,.$$

If we use eq. (9) and the similar equations

$$(37)\qquad \begin{cases} \displaystyle M^{(1)}_{s_A s_C}(P^0_A, g, P^0_C) = \sum_{s'_C} \boldsymbol{M}^{(1)}_{s_A s'_C}\big(P^0_A,\, L(g) P^0_C\big)\, D^{(C)}_{s_C s'_C}(g^{-1})\,, \\ \displaystyle M^{(2)}_{s_C s_B}(P^0_C, g, P^0_B) = \sum_{s'_B} \boldsymbol{M}^{(2)}_{s_C s'_B}\big(P^0_C,\, L(g) P^0_B\big)\, D^{(B)}_{s_B s'_B}(g^{-1})\,. \end{cases}$$

the eq. (34) becomes

$$(38)\qquad M_{s_A s_B}(P^0_A, g, P^0_B) = \sum_{s'_B s_C s'_C} \int \boldsymbol{M}^{(1)}_{s_A s_C}(P^0_A, P_C)\, \varrho_{s_C s'_C}(P_C)\cdot \cdot D^{(B)}_{s_B s'_B}(g^{-1})\, \boldsymbol{M}^{(2)}_{s'_C s_B}\big(P_C,\, L(g)\, P^0_B\big)\, \delta^4\Big(Q^0 - \sum_{i=1}^{r} P_{iC}\Big)\, \mathrm{d}^4P_{1C} \ldots \mathrm{d}^4P_{rC}\,.$$

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Now we take

$$P_C = L(a_C)\, P_C^0 \tag{39}$$

and note that the δ-function appearing in the integral (38) does not vanish only if a_C belongs to G.

Equation (39) can be used to perform a substitution of integration variables in eq. (38), *i.e.*, we transform an integral over P_C in an integral over the elements a_C of the group $\mathscr{L}$ and over the independent components of P_C^0. Takeing into account the δ-function, we can integrate over the group G instead of over the whole group $\mathscr{L}$. In the Appendix A we show that if the particles belonging to C are at least three [19] and $Q^2 < 0$, there is a measure $\mathrm{d}V_C$ in the space of the independent components of P_C^0 with the property

$$\int \varphi(P_C)\, \delta^4\Big(Q^0 - \sum_{i=1}^{r} P_{iC}\Big)\, \mathrm{d}^4P_{1C} \dots \mathrm{d}^4P_{rC} = \int_G \mathrm{d}g \int \mathrm{d}V_C\, \varphi\big(L(g)\, P_C^0\big)\,, \tag{40}$$

where $\varphi(P_C)$ is an arbitrary function and $\mathrm{d}g$ is the invariant measure on the group G. The explicit form of $\mathrm{d}V_C$ will be given in Appendix. If we take into account this formula, eq. (38) becomes

$$M_{s_A s_B}(P_A^0, g, P_B^0) = \sum_{s'_B s_C s'_C} \int \mathrm{d}V_C \int_G \mathrm{d}g'\, M^{(1)}_{s_A s_C}\big(P_A^0, L(g')\, P_C^0\big) \cdot$$
$$\cdot\, \varrho_{s_C s'_C}\big(L(g')\, P_C^0\big)\, M^{(2)}_{s'_C s'_B}\big(L(g')\, P_C^0, L(g)\, P_B^0\big)\, D^{(B)}_{s_B s'_B}(g^{-1})$$

and, after use of eqs. (35), (36) and (37), we obtain

$$M_{s_A s_B}(P_A^0, g, P_B^0) = \sum_{s_C s'_C} \int \mathrm{d}V_C \int_G \mathrm{d}g'\, M^{(1)}_{s_A s_C}(P_A^0, g', P_C^0) \cdot \tag{41}$$
$$\cdot\, \varrho_{s_C s'_C}(P_C^0)\, M^{(2)}_{s'_C s_B}(P_C^0, g'^{-1} g, P_B^0)\,.$$

If we consider only the dependence on the group elements g and g', this equa-

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[19] If only two spinless particles belong to one of the sets, some difficulty arises due to the fact that one of the group parameters is unnecessary. On the other hand the amplitude has the property $M(hg) = M(g)$ if h belongs to a subgroup H of G. If H is not compact, as it can happen if we consider the off-shell amplitude, $M(g)$ cannot be an L^1 or an L^2 function of g. These difficulties do not prevent the treatment of the two-particles case, as is shown in [1], but the application of the formalism of the present paper to this case needs a particular technique.

tion becomes

$$M(g) = \int_G \mathrm{d}g'\, M^{(1)}(g')\, M^{(2)}(g'^{-1}g)\,. \tag{42}$$

This means that M is the « convolution » of $M^{(1)}$ and $M^{(2)}$.

It is useful to give two sufficient conditions for the existence of an integral of the type (42) ([20]):

i) If $M^{(1)}(g)$ and $M^{(2)}(g)$ belong to the class L^1, $M(g)$ is defined almost everywhere in G and is an L^1 function.

ii) If $M^{(1)}(g)$ belongs to L^1 and $M^{(2)}(g)$ belongs to L^2, $M(g)$ is defined almost everywhere and belongs to L^2.

Now we show that a relation of the kind (41) can be considerably simplified by means of the transformation (29). We consider explicitly the simpler case (42) and we have

$$\begin{aligned}\mathscr{M}^\sigma &= \int_G M(g)\, \mathscr{D}^\sigma(g)\, \mathrm{d}g = \int_G \mathrm{d}g \int_G \mathrm{d}g'\, M^{(1)}(g')\, M^{(2)}(g'^{-1}g)\,\cdot \\ &\cdot \mathscr{D}^\sigma(g')\, \mathscr{D}^\sigma(g'^{-1}g) = \int_G \mathrm{d}g'\, M^{(1)}(g')\, \mathscr{D}^\sigma(g') \int_G \mathrm{d}g''\, M^{(2)}(g'')\, \mathscr{D}^\sigma(g'') = \mathscr{M}^{(1)\sigma} \mathscr{M}^{(2)\sigma}\,.\end{aligned} \tag{43}$$

The change of the order of integration is easily justified if the condition i) is satisfied. If the condition ii) is satisfied, the justification is less simple, but the result still holds.

In the general case (41) we obtain in the same way the relation between operator-valued functions

$$\mathscr{M}^\sigma_{s_A s_B}(P^0_A, P^0_B) = \sum_{s_C s'_C} \int \mathrm{d}V_C\, \mathscr{M}^{(1)\sigma}_{s_A s_C}(P^0_A, P^0_C)\, \varrho_{s_C s'_C}(P^0_C)\, \mathscr{M}^{(2)\sigma}_{s'_C s_B}(P^0_C, P^0_B)\,. \tag{44}$$

This is the desired « partially diagonalized » form of the formula (34). This procedure can be applied to integral equations of the Bethe-Salpeter type ([21]) which are useful to find the sum of infinite series of Feynman diagrams. Con-

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([20]) M. A. Naimark: *Normed Rings*, Sect. **28** (Groningen, 1964).

([21]) The possibility of writing down an integral equation for the relativistic three-particle amplitudes has been investigated by A. Tucciarone: *Thesis* (Roma, 1964).

sider for instance the equation

$$(45)\qquad \boldsymbol{M}(P_A, P_B) = \boldsymbol{V}(P_A, P_B) + \int \boldsymbol{K}(P_A, P'_A)\boldsymbol{M}(P'_A, P_B)\,\delta^4\Big(Q - \sum_{i=1}^{n} P'_{iA}\Big)\,\mathrm{d}^4P'_{1A}\ldots\mathrm{d}^4P'_{nA}$$

(all the particles are spinless).

If we apply to this equation the above explained procedure we obtain, using obvious notations

$$(46)\qquad M(P^0_A, g, P^0_B) = V(P^0_A, g, P^0_B) + \iint K(P^0_A, gg'^{-1}, P^{0\prime}_A)\, M(P^{0\prime}_A, g', P^0_B)\,\mathrm{d}g'\,\mathrm{d}V'_A$$

and finally

$$(47)\qquad \mathscr{M}^\sigma(P^0_A, P^0_B) = \mathscr{V}^\sigma(P^0_A, P^0_B) + \int \mathscr{K}^\sigma(P^0_A, P^{0\prime}_A)\mathscr{M}^\sigma(P^{0\prime}_A, P^0_B)\,\mathrm{d}V'_A\,.$$

The kernel of the eq. (47) can be considered an operator acting both on the representation space and on the space of the functions of P^0_A. Its complete Hilbert-Schmidt norm is given by

$$(48)\qquad \iint |\mathscr{K}^\sigma(P^0_A, P^{0\prime}_A)|^2\,\mathrm{d}V_A\,\mathrm{d}V'_A$$

and if this integral is finite the equation (47) can be solved by means of the Fredholm method.

Note that the original eqs. (45) or (46) can never be solved with the Fredholm method if the group G is not compact. In fact the integral

$$\int |K(P^0_A, gg'^{-1}, P^{0\prime}_A)|^2\,\mathrm{d}g\,\mathrm{d}g'\,\mathrm{d}V_A\,\mathrm{d}V'_A = \int |K(P^0_A, g, P^{0\prime}_A)|^2\,\mathrm{d}g\,\mathrm{d}V_A\,\mathrm{d}V'_A \cdot \int \mathrm{d}g'$$

is always divergent.

7. – The « Laplace-transform ».

The formulae given in Sect. **4** can be considered as the analogue of the Fourier transform; in fact the classical Fourier transform is given by the projection of a function $f(x)$ on the functions $\exp[ikx]$ which are the unitary representations of the translation group. In this Section we want to generalize the transformation (29) in the same way as the Laplace transform generalizes the Fourier transform. As the Laplace transform makes use of the functions $\exp[sx]$ (with complex s) which are in general nonunitary representations of the translation group, the generalization we are looking for should be connected to the nonunitary representations of the group G.

A general theory of the Laplace transform for functions defined on Abelian locally compact groups has been given by MACKEY ([22]), but we do not know a similar theory for noncommutative groups, which could provide a general mathematical framework for this investigation.

The advantage of the generalized Laplace transform with respect to the transformation given in Sect. 4 is twofold. First it permits the use of the theory of the operator-valued analytic functions which has been found very useful in the investigation of the properties of the complex angular momentum in potential and field-theoretical models ([23,24]). Secondly it generates high-energy asymptotic expansions of the scattering amplitude, as we shall see in the following Section.

The transform we are going to study is based on two families of representations which depend on a complex parameter l. They are given by the following linear operators acting on the L^2 functions defined in the interval $0 \leqslant \Phi < 2\pi$:

$$\mathscr{D}^l(g)\,\chi(\Phi) = |\alpha + \beta \exp[i\Phi']|^{2l+2}\,\chi(\Phi')\,, \tag{49}$$

$$\mathscr{D}'^l(g)\,\chi(\Phi) = (\alpha + \beta \exp[i\Phi'])\,|\alpha + \beta \exp[i\Phi']|^{2l+1}\,\chi(\Phi')\,, \tag{50}$$

where

$$\begin{cases} \exp[i\Phi] = \dfrac{\bar{\alpha}\exp[i\Phi'] + \bar{\beta}}{\alpha + \beta\exp[i\Phi']}\,, \\[2ex] \exp[i\Phi'] = \dfrac{\alpha\exp[i\Phi] - \bar{\beta}}{\bar{\alpha} - \beta\exp[i\Phi]}\,, \\[2ex] \dfrac{\mathrm{d}\Phi}{\mathrm{d}\Phi'} = |\alpha + \beta\exp[i\Phi']|^{-2}\,. \end{cases} \tag{51}$$

These representations have been investigated by BARGMANN ([15]); we give here for easier reference a summary of their properties.

The representations given by eq. (49) are of the integral type and those given by eq. (50) are of the half-integral type; in both cases they correspond

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([22]) G. MACKEY: *Proc. Nat. Acad. Sci. USA*, **34**, 156 (1948).
([23]) G. COSENZA, L. SERTORIO and M. TOLLER: *Nuovo Cimento*, **35**, 913 (1965).
([24]) G. TIKTOPOULOS: *Phys. Rev.*, **133**, B 1231 (1964).

to a value of q given by

(52) $$q = -l(1+l)\,.$$

If we define the scalar product in the usual way

$$(\chi_1, \chi_2) = \frac{1}{2\pi}\int_0^{2\pi} \chi_1(\Phi)\,\overline{\chi_2(\Phi)}\,\mathrm{d}\Phi\,,$$

the operators (49) and (50) are bounded and their norm is given by

(53) $$\|\mathscr{D}^l(g)\| = \|\mathscr{D}'^l(g)\| = \exp[|2\,\mathrm{Re}\,l + 1|\zeta]\,.$$

Moreover it can be shown that these operators are continuous functions of g in the strong operator topology and analytic operator-valued functions of l in the uniform topology. It can also be shown that the representations $\mathscr{D}'^l$ and $\mathscr{D}^l$ are « weakly equivalent » ([25]) to the representations $\mathscr{D}'^{-l-1}$ and $\mathscr{D}^{-l-1}$, respectively.

If $l = -\frac{1}{2} + is$ the representations $\mathscr{D}^l$ coincide with the unitary representations of the class C_q^0 with $q > \frac{1}{4}$ and the representations $\mathscr{D}'^l$ coincide with the unitary representations of the class $C_q^{\frac{1}{2}}$.

Now we are enabled to generalize eq. (29) and to define the following « Laplace transforms »

(54) $$\begin{cases} \mathscr{F}^l = \displaystyle\int_G \mathscr{D}^l(g)\, f(g)\,\mathrm{d}g\,, \\ \mathscr{F}'^l = \displaystyle\int_G \mathscr{D}'^l(g)\, f(g)\,\mathrm{d}g\,. \end{cases}$$

Of course if $l = -\frac{1}{2} + is$, $\mathscr{F}^l$ and $\mathscr{F}'^l$ coincide with some of the $\mathscr{F}^\sigma$ defined by (26).

We treat here in detail only the integral spin case; then of course $\mathscr{F}'^l = 0$ and we have to investigate only the properties of $\mathscr{F}^l$. The half-integral spin case can be treated in a very similar way.

If one requires the absolute convergence of the integral (54), the Laplace transform is defined as a bounded operator in the strip of the complex l-plane

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([25]) For a discussion of the concept of equivalence for nonunitary infinite-dimensional representations, see: M. A. NAIMARK: *Linear Representations of the Lorentz Group*, translations AMS (1957), p. 379; G. MACKEY: *Bull. Am. Math. Soc.*, **69**, 628 (1963).

where the integral

$$\int_G \exp\left[|2\,\mathrm{Re}\, l+1|\zeta\right]|f(g)|\,\mathrm{d}g \tag{55}$$

converges. However, the absolute convergence is a too restrictive condition four our pourposes and, on the other hand, it is very useful to know that $\mathscr{F}$ is an Hilbert-Schmidt operator. Therefore we interpret the integral (54) as

$$\mathscr{F}^l = \lim_{z\to\infty}\int_{\zeta<z}\mathscr{D}^l(g)\,f(g)\,\mathrm{d}g\,, \tag{56}$$

where we assume that the integral is an Hilbert-Schmidt operator and the limit has to be taken with respect to the Hilbert-Schmidt operator norm.

We do not investigate here all the properties of the classical Laplace transform which have their analogue in this case, but give only a sufficient condition for the existence of the limit (56).

We assume that $f(g)=f(\mu,\zeta,\nu)$ belongs to L^2; then the function

$$\hat{f}(\zeta)=\left[\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\,|f(\mu,\zeta,\nu)|^2\right]^{\frac{1}{2}} \tag{57}$$

and the operator-valued functions

$$\hat{\mathscr{F}}^l(\zeta)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\,\mathscr{D}^l(\mu,\zeta,\nu)\,f(\mu,\zeta,\nu) \tag{58}$$

are defined almost everywhere and moreover we can write

$$\mathscr{F}^l=\lim_{z\to\infty}\int_0^z\hat{\mathscr{F}}^l(\zeta)\sinh 2\zeta\,\mathrm{d}\zeta\,. \tag{59}$$

If we take

$$\chi_m(\Phi)=\exp[im\Phi]\,, \tag{60}$$

we have

$$\left(\mathscr{D}^l(g)\,\chi_n,\,\chi_m|\right)=\frac{1}{2\pi}\int_0^{2\pi}|\alpha+\beta\exp[i\Phi']|^{2l+2}\exp[in\Phi']\exp[-im\Phi]\,\mathrm{d}\Phi= \tag{61}$$

$$=\exp[-2im\mu]\,d^l_{mn}(\zeta)\exp[-2in\nu]\,,$$

where the function $d^l_{mn}(\zeta)$ can be expressed in terms of the hypergeometric function (see Appendix B). In the simplest case we have

$$d^l_{00}(\zeta) = P_l(\cosh 2\zeta)\,. \tag{62}$$

From the integral (61) it follows at once that

$$|d^l_{mn}(\zeta)| \leqslant d^{\operatorname{Re} l}_{00}(\zeta) = P_{\operatorname{Re} l}(\cosh 2\zeta)\,. \tag{63}$$

From (56) and (61) we have

$$(\hat{\mathscr{F}}^l(\zeta)\chi_n, \chi_m) = d^l_{mn}(\zeta)\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\, f(\mu, \zeta, \nu)\exp[-2im\mu]\exp[-2in\nu] \tag{64}$$

and using the inequality (63)

$$\|\hat{\mathscr{F}}^l(\zeta)\|^2 = \sum_{mn}|(\hat{\mathscr{F}}^l(\zeta)\chi_n, \chi_m)|^2 \leqslant$$
$$\leqslant [P_{\operatorname{Re} l}(\cosh 2\zeta)]^2\sum_{mn}\left|\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\, f(\mu, \zeta, \nu)\exp[-2im\mu]\exp[-2in\nu]\right|^2,$$

that is

$$\|\hat{\mathscr{F}}(\zeta)\| \leqslant P_{\operatorname{Re} l}(\cosh 2\zeta)\hat{f}(\zeta)\,. \tag{65}$$

Therefore we have that the limit of eq. (59) exists if l lays in the strip where the integral

$$\int_0^\infty P_{\operatorname{Re} l}(\cosh 2\zeta)\hat{f}(\zeta)\sinh 2\zeta\,\mathrm{d}\zeta \tag{66}$$

converges, and it is

$$\|\mathscr{F}^l\| \leqslant \int_0^\infty P_{\operatorname{Re} l}(\cosh 2\zeta)\hat{f}(\zeta)\sinh 2\zeta\,\mathrm{d}\zeta\,. \tag{67}$$

Note that the strip where the integral (66) converges can be wider than the strip where the integral (55) converges. If

$$\hat{f}(\zeta) = O(\exp[2L\zeta])\,, \qquad L < -\tfrac{1}{2} \tag{68}$$

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it can easily be deduced from the asymptotic properties of the Legendre function that (66) converges and therefore the Laplace transform (56) is defined for

$$L < \operatorname{Re} l < -L-1 \,. \tag{69}$$

From the analyticity of the operators $\mathscr{D}^l(g)$ it follows that the functions $d^l_{mn}(\zeta)$ are analytic in the whole l-plane and from eqs. (64) and (65) if follows that $\hat{\mathscr{F}}^l(\zeta)$ is an operator-valued analytic function of l whenever $\hat{f}(\zeta)$ is finite. From eq. (59) we deduce that the Laplace transform $\mathscr{F}^l$ is an analytic operator-valued function in the strip where the integral (66) converges.

Note that $\mathscr{F}^l$ does not determine uniquely $f(g)$; in fact if we add to $f(g)$ a matrix element of a representation of the discrete class, $\mathscr{F}^l$ does not change for $l=-\frac{1}{2}+is$, due to the orthogonality relations, and, from the analyticity of $\mathscr{F}^l$, it follows that it does not change anywhere. From Sect. **4** and **5** it is clear that, in the integral spin case, $f(g)$ is uniquely determined by $\mathscr{F}^l$ and by the projections $\mathscr{F}^{+k}$ and $\mathscr{F}^{-k}$ on the representations of the discrete classes.

8. – The asymptotic behaviour of the amplitude.

One of the merits of the classical Laplace transformation is given by the fact that asymptotic expansion of the original function can be obtained from the analytic properties of the transformed function [26]. A similar use can be made of the transformation defined in the preceding Section, but in this case theorems as powerful as in the classical case are not available, and further mathematical work in this field would be welcome.

An asymptotic expansion of the type

$$f(\mu, \zeta, \nu) = \sum_{i=1}^{N} f_i(\mu, \zeta, \nu) + R(\mu, \zeta, \nu) \tag{70}$$

can be interpreted in many ways, corresponding to the various possible conditions which can be imposed on the remainder $R(\mu, \zeta, \nu)$. For instance

i) conditions can be imposed on the asymptotic behaviour of $R(\mu, \zeta, \nu)$ for fixed values of μ and ν;

[26] G. Doetsch: *Theorie und Anwendung der Laplace-Transformation*, III Teil (New York, 1943).

ii) conditions can be imposed on the functions

$$R_{mn}(\zeta) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\, \exp[-2im\mu]\exp[-2in\nu]\, R(\mu, \zeta, \nu)$$

for m and n fixed;

iii) conditions can be imposed on the function

$$\hat{R}(\zeta) = \left[\frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\, |R(\mu, \zeta, \nu)|^2\right]^{\frac{1}{2}} = \left[\sum_{mn}|R_{mn}(\zeta)|^2\right]^{\frac{1}{2}}.$$

Of course the most useful expansions are those of the kind iii) and we shall investigate them in a further work. Here we give only some indication about the way to obtain expansions of the type ii), or in other words, to find asymptotic expansions for the functions

(71) $$f_{mn}(\zeta) = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\nu\, \exp[-2im\nu]\exp[-2in\nu]\, f(\mu, \zeta, \nu)\,.$$

From eq. (27) and (61) we have

(72) $$f_{mn}(\zeta) = \int \overline{d^{\sigma}_{mn}(\zeta)}\, \mathscr{F}^{\sigma}_{mn}\,\mathrm{d}\sigma\,,$$

where

(73) $$\mathscr{F}^{\sigma}_{mn} = (\mathscr{F}^{\sigma}\chi_n, \chi_m)\,.$$

The contribution to (72) of the representations of the discrete classes has already the form of an asymptotic expansion, the representations of the half-integral type do not contribute because we are considering the integral-spin case, therefore we have to consider only the contribution of the class C^0_q which can be written in the form

(74) $$f^{(c)}_{mn}(\zeta) = \frac{1}{2i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d^{-l-1}_{mn}(\zeta)\,\mathscr{F}^{l}_{mn}\,\frac{2l+1}{\operatorname{tg}\pi l}\,\mathrm{d}l\,;$$

using the eqs. (B.13), (B.14) and (B.16) of the Appendix B we can write eq. (74)

in the form

$$(75)\qquad f^{(c)}_{mn}(\zeta) = -i \int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} a^{-l-1}_{mn}(\zeta)\, \mathscr{F}^{l}_{mn} \frac{2l+1}{\operatorname{tg}\pi l}\, dl\,.$$

Now we assume that $\mathscr{F}^{l}_{mn}$ is a meromorphic function of l in the strip $-L-1 \leqslant \operatorname{Re} l \leqslant L$. This happens, *e.g.*, if the operator $\mathscr{F}^{l}$ is the resolvent of an integral equation of the Fredholm type, whose kernel is a meromorphic operator-valued function of l in this strip [23,24].

Then if certain conditions (which we do not investigate here) are satisfied, the asymptotic behaviour of $f_{mn}(\zeta)$ is dominated by the contribution of the poles encountered when the integration path is shifted towards the line $\operatorname{Re} l = -L-1$, *i.e.*, we can write

$$(76)\qquad f^{(c)}_{mn}(\zeta) \sim 2\pi \sum_i \frac{2l_i+1}{\operatorname{tg}\pi l_i}\, r^{i}_{mn} a^{-l_i-1}_{mn}(\zeta) + 2\sum_l \mathscr{F}^{l}_{mn} a^{-l-1}_{mn}(\zeta)(2l+1)\,,$$

where r^{i}_{mn} are the residues of the poles of $\mathscr{F}^{l}_{mn}$ at l_i; and the second sum has to be extended to the integral values of l in the interval $-L-1 < l < -\frac{1}{2}$.

That the last term of eq. (76) necessarily appears, can be understood as follows. Assume that $f_{mn}(\zeta) = 0$ for $\zeta > Z$; the contribution of the representations of the discrete class in general does not vanish. However they cannot give any contribution to the asymptotic behaviour of $f_{mn}(\zeta)$, therefore they have to be compensated by analogous terms in the asymptotic expansion of the contribution $f^{(c)}_{mn}(\zeta)$ of the continuous class. In fact, $a^{-l-1}_{mn}(\zeta)$ for integral l and $|m| \geqslant -l$, $|n| \geqslant -l$, $mn \geqslant 0$ coincides, apart a factor, with the corresponding matrix element of the representation of the discrete class with $k = -l$ $(l < 0)$.

Appendix A

We have seen in Sect. 2 that the conditions which define the vectors P^0 (we leave off the subscripts A or B which distinguish the set of particles) are somewhat arbitrary. Here we give a possible specification of the conditions when $Q^2 < 0$ and the particles belonging to the set considered are at least three. Moreover we assume that the group $\mathscr{L}$ does not contain the reflections.

Four of these conditions are given by

$$(\text{A.1})\qquad \begin{cases} Q^0_t = 0\,, \\ Q^0_x = 0\,, \\ Q^0_y = 0\,, \\ Q^0_z = \sqrt{-t} > 0\,. \end{cases}$$

In order to determine the other conditions we have to distinguish three cases, depending on the sign of the quantities

$$(\mathrm{A}.2)\qquad u=-\begin{vmatrix} Q^2 & (Q, P_1)\\ (P_1, Q) & P_1^2\end{vmatrix}, \qquad v=\begin{vmatrix} Q^2 & (Q, P_1) & (Q, P_2)\\ (P_1, Q) & P_1^2 & (P_1, P_2)\\ (P_2, Q) & (P_2, P_1) & P_2^2\end{vmatrix}.$$

These quantities are negative if the subspace spanned by the vectors involved is Euclidean, otherwise they are positive.

The three cases we have to consider are:

i) $u>0$, the conditions are

$$(\mathrm{A}.3)\qquad P^0_{1x}=0\,,\qquad P^0_{1y}=0\,,\qquad P^0_{2x}>0\,,\qquad P^0_{2y}=0\,.$$

ii) $u\leqslant 0$, $v>0$, the conditions are

$$(\mathrm{A}.4)\qquad P^0_{2x}=0\,,\qquad P^0_{2y}=0\,,\qquad P^0_{1x}>0\,,\qquad P^0_{1y}=0\,.$$

iii) $u<0$, $v<0$, the conditions are

$$(\mathrm{A}.5)\qquad P^0_{1t}=0\,,\qquad P^0_{1x}>0\,,\qquad P^0_{1y}=0\,,\qquad P^0_{2t}=0\,.$$

Note that when on-shell amplitudes are treated, we need to consider only the case i).

Now we have to calculate the explicit form of the measure $\mathrm{d}V$ in the space of the independent components of P^0. From eq. (40) we have

$$(\mathrm{A}.6)\qquad \int\varphi(P_1\ldots P_n)\,\delta^4\Big(Q_0-\sum_{i=1}^{n}P_i\Big)\,\mathrm{d}^4P_1\ldots d^4P_n=$$
$$=\int\varphi\Big(P_1\ldots P_{n-1},\,Q^0-\sum_{i=1}^{n-1}P_i\Big)\,\mathrm{d}^4P_1\ldots\mathrm{d}^4P_{n-1}=\int_G\mathrm{d}g\int\mathrm{d}V\,\varphi(L(g)P^0)\,.$$

The calculation proceeds in different ways in the three cases considered above. In the case i) we have [27]

$$(\mathrm{A}.7)\qquad 4\,\mathrm{d}g\,\mathrm{d}V=\mathrm{d}P_1\ldots\mathrm{d}P_{n-1}=|J|\,\mathrm{d}\mu\,\mathrm{d}\zeta\,\mathrm{d}\nu\,\mathrm{d}P^0_{1t}\,\mathrm{d}P^0_{1z}\,\mathrm{d}P^0_{2t}\,\mathrm{d}P^0_{2x}\,\mathrm{d}P^0_{2z}\,\mathrm{d}^4P^0_3\ldots\mathrm{d}P^0_{n-1}\,,$$

where

$$J=\frac{\partial(P_1,P_2,\ldots P_{n-1})}{\partial(\mu,\zeta,P^0_{1t},P^0_{1z},\nu,P^0_{2t},P^0_{2x},P^0_{2z},P^0_3\ldots P^0_{n-1})}\,.$$

([27]) The factor 4 is due to the fact that, when the group parameters vary in their range, the same vectors P_i are obtained four times.

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(when we do not indicate the relativistic index, we mean that all the four components should be written). If one writes down explicitly the determinant J one easily sees that it can be decomposed in the following way

$$J = \frac{\partial(P_{1t}, P_{1x}, P_{1y})}{\partial(\mu, \zeta, P^0_{1t})} \frac{\partial(P_{2'}, P_{2x}, P_{2y})}{\partial(\nu, P^0_{2t}, P^0_{2x})} \frac{\partial(P_3)}{\partial(P^0_3)} \cdots \frac{\partial(P_{n-1})}{\partial(P^0_{n-1})}.$$

Clearly only the first two terms are different from one and, if we evaluate them explicitly, we obtain

$$|J| = 8 \sinh 2\zeta \, (P^0_{1t})^2 P^0_{2x} , \tag{A.8}$$

From (A.7) and (A.8) taking into account the form (13) of the invariant measure dg, we obtain

$$\mathrm{d}V = 2(2\pi)^2 (P^0_{1t})^2 P^0_{2x} \, \mathrm{d}P^0_{1t} \mathrm{d}P^0_{1z} \mathrm{d}P^0_{2t} \mathrm{d}P^0_{2x} \mathrm{d}P^0_{2z} \mathrm{d}^4P^0_3 \ldots \mathrm{d}^4P^0_{n-1} \qquad (\text{case i})) . \tag{A.9}$$

In a similar way in the other two cases we get

$$\mathrm{d}V = 2(2\pi)^2 (P^0_{2t})^2 P^0_{1x} \, \mathrm{d}P^0_{2t} \mathrm{d}P^0_{2z} \mathrm{d}P^0_{1t} \mathrm{d}P^0_{2x} \mathrm{d}P^0_{2z} \mathrm{d}^4P^0_3 \ldots \mathrm{d}^4P^0_{n-1} \qquad (\text{case ii})) . \tag{A.10}$$

and

$$\mathrm{d}V = 2(2\pi)^2 (P^0_{1x})^2 P^0_{2y} \, \mathrm{d}P^0_{1x} \mathrm{d}P^0_{1z} \mathrm{d}P^0_{2x} \mathrm{d}P^0_{2y} \mathrm{d}P^0_{2z} \mathrm{d}^4P^0_3 \ldots \mathrm{d}^4P^0_{n-1} \qquad (\text{case iii})) . \tag{A.11}$$

APPENDIX B

In this Appendix we give the explicit form and some properties of the matrix elements of the representations used in the text.

If we use the orthogonal basis formed by the vectors χ_m with

$$H_0 \chi_m = m \chi_m , \tag{B.1}$$

we have that

$$(\mathscr{D}^\sigma(g) \chi_n, \chi_m) = \exp[-2im\mu] d^\sigma_{mn}(\zeta) \exp[-2in\nu] . \tag{B.2}$$

We report for easier reference the functions $d^{\pm k}_{mn}(\zeta)$ given by BARGMANN [15] for the unitary representations of the discrete classes. For $m \geqslant n$ it is

$$\begin{aligned} d^{+k}_{mn}(\zeta) &= \frac{1}{(m-n)!} \left(\frac{\Gamma(m+1-k)\Gamma(m+k)}{\Gamma(n+1-k)\Gamma(n+k)} \right)^{\frac12} (\cosh \zeta)^{-(m+n)} (\sinh \zeta)^{m-n} \cdot \\ &\quad \cdot F_{21}(k-n, 1-n-k; 1+m-n; -(\sinh \zeta)^2) = \\ &= \frac{(-1)^{n-k}}{\Gamma(2k)} \left(\frac{\Gamma(m+k)\Gamma(n+k)}{\Gamma(m+1-k)\Gamma(n+1-k)} \right)^{\frac12} (\sinh \zeta)^{-2k} (\mathrm{tgh}\, \zeta)^{m+n} \cdot \\ &\quad \cdot F_{21}(k-m, k-n; 2k; -(\sinh \zeta)^{-2}) , \end{aligned} \tag{B.3}$$

$$d^{-k}_{mn}(\zeta) = d^{+k}_{-n,-m}(\zeta) , \tag{B.4}$$

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and for $m \leqslant n$ we have

$$d^{\pm k}_{mn}(\zeta) = (-1)^{m-n} d^{\pm k}_{nm}(\zeta) \,. \tag{B.5}$$

The matrix elements of the representations $\mathscr{D}^l$ and $\mathscr{D}'^l$ are different from those given by BARGMANN because we use a different basis. Our matrix elements are less symmetric than BARGMANN's ones, but present the advantage of being analytic in the whole l-plane. From eq. (61) we have for the representations $\mathscr{D}^l$

$$d^l_{mn}(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh\zeta \exp[i\Phi] - \sinh\zeta)^{n-l-1} \cdot (\cosh\zeta - \sinh\zeta\exp[i\Phi])^{-l-n-1} (\exp[i\Phi])^{-m+l+1}\, \mathrm{d}\Phi \,. \tag{B.6}$$

For the representations $\mathscr{D}'^l$ if we introduce the basis

$$\chi'_m(\Phi) = \exp[i(m - \tfrac{1}{2})\,\Phi] \,, \tag{B.7}$$

we have in the same way

$$(\mathscr{D}'^l(g)\,\chi'_n, \chi'_m) = \exp[-2im\mu]\, d^l_{mn}(\zeta) \exp[-2in\nu] \,, \tag{B.8}$$

where $d^l_{mn}(\zeta)$ is still given by eq. (B.6) (note however that now m and n are half-integers).

The integral (B.6) can be transformed into an integral representation of the hypergeometric function [28] and after some calculations we obtain for $m \geqslant n$

$$d^l_{mn}(\zeta) = \frac{1}{(m-n)!}\,\frac{\Gamma(l+m+1)}{\Gamma(l+n+1)} (\cosh\zeta)^{m+n} (\sinh\zeta)^{m-n} \cdot F_{21}(m-l, m+l+1; m-n+1; -(\sinh\zeta)^2) \,. \tag{B.9}$$

For $m \leqslant n$ we have

$$d^l_{mn}(\zeta) = d^l_{-m,-n}(\zeta) \,. \tag{B.10}$$

Note also the following property which can be directly verified on eq. (B.9)

$$d^{-l-1}_{mn}(\zeta) = (-1)^{m-n} d^l_{nm}(\zeta) = \xi^{-l-1}_m d^l_{mn}(\zeta)\,\xi^l_n \,, \tag{B.11}$$

where

$$\xi^l_m = \frac{\Gamma(l+m+1)}{\Gamma(m-l)} \,, \qquad \xi^{-l-1}_m \xi^l_m = 1 \,. \tag{B.12}$$

([28]) A. ERDELYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI: *Higher Transcendental Functions*, vol. **1**, formula 2.1.3.13.

The last equations show that the representations corresponding to l and to $-l-1$ are weakly equivalent [25]. If $l=-\frac{1}{2}+is$, we have that $|\xi^l_m|=1$ and the representations are also unitary equivalent. From eq. (B.11), (61), (73) and (54) it follows that the Laplace transform defined in Sect. **6** has the property

$$\mathscr{F}^{-l-1}_{mn}=\xi^{-l-1}_m \mathscr{F}^l_{mn}\xi^l_n\,. \tag{B.13}$$

At last we express the matrix elements $d^l_{mn}(\zeta)$ in terms of functions which generalize the Legendre functions of the second kind and have a simple asymptotic behaviour for $\zeta\to\infty$. This decomposition can easily be obtained from a well-known property of the hypergeometric functions [29] and can be written as follows

$$d^l_{mn}(\zeta)=a^l_{mn}(\zeta)+b^l_{mn}(\zeta)\,, \tag{B.14}$$

where

$$a^l_{mn}(\zeta)=\frac{(-1)^{m-n}\Gamma(-2l-1)}{\Gamma(-l-m)\Gamma(-l+m)}(\sinh\zeta)^{-2l-2}(\operatorname{tgh}\zeta)^{m+n}\cdot F_{21}(l+1-n,\,l+1-m;\,2l+2;\,-(\sinh\zeta)^{-2}) \tag{B.15}$$

and

$$b^l_{mn}(\zeta)=\xi^l_m\,a^{-l-1}_{mn}(\zeta)\,\xi^{-l-1}_n\,. \tag{B.16}$$

Note that

$$a^l_{00}(\zeta)=\pi^{-1}\operatorname{tg}\pi l\,Q_l(\cosh 2\zeta)\,. \tag{B.17}$$

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(29) Loc. cit., formula 2.1.4.17.

RIASSUNTO

L'ampiezza di diffusione per un processo a più particelle viene proiettata sugli elementi di matrice delle rappresentazioni unitarie irriducibili del gruppo di Lorentz tridimensionale. Si mette in luce l'utilità di questa trasformazione nello studio di una certa classe di equazioni integrali a cui soddisfa l'ampiezza di diffusione. Si mostra che, utilizzando un certo insieme di rappresentazioni non unitarie, si ottiene una trasformazione più generale che ha molte delle proprietà della trasformazione classica di Laplace e può essere usata per ottenere sviluppi asintotici simili a quelli che si ottengono dalla formula di Watson-Sommerfeld.