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The Harmonic Oscillator in Stochastic Electrodynamics.

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Summary. — Classical electrodynamics with the hypothesis of a universal, Lorentz invariant, background radiation (stochastic electrodynamics) has been proposed as a possible alternative to quantum electrodynamics. The stochastic equations of motion of a charged particle are derived according to this theory, and they are compared with those of Brownian motion. A development of the equations in powers of the fine-structure constant α is considered. The harmonic oscillator is studied with the result that the oscillator performs a simple harmonic motion very stable in phase. The amplitude changes slowly and at random. The mean values of the kinetic and potential energy are calculated and agree quite well with the results of quantum electrodynamics up to first order in α. The existence of excited states is shown which prove to be very similar to the coherent states of the quantum oscillator. The calculated rate of spontaneous emission of radiation agrees with the result of quantum electrodynamics but the line width does not agree. Arguments are given which show that the quantum line width calculated according to the Weisskopf-Wigner theory cannot be correct in the case of the oscillator. A general expression for the evolution of the expectation value of any observable of the oscillator in quantum electrodynamics is also derived.

1. – Stochastic electrodynamics.

It has been suggested that classical electrodynamics may be sufficient to interpret the experimental facts of the atomic domain, that is, that classical electrodynamics, properly understood, may be an alternative to quantum electrodynamics (1). For this to be true, one must assume the existence of

(1) A review of the field is presented in the article by M. SURDIN: *Ann. Inst. H. Poincaré*, **15**, 203 (1971). New developments and further references are given by E. SANTOS: *Lett. Nuovo Cimento*, **4**, 497 (1972).

a random background radiation in the whole space. We emphasize that this is not an additional postulate to classical electrodynamics. On the contrary, a new postulate will be to assume that there is no background radiation. In fact, the general solution of Maxwell's equations is always the sum of a particular solution plus the general solution of the homogeneous equations; this last represents a background radiation. In order to develop classical electrodynamics consistently, the background radiation cannot be excluded from the beginning. Rather, the experiments must decide whether it exists or not. The point is that the quantum phenomena may be the experimental evidence for a background radiation.

The complexity of the world implies that the background radiation must be considered a random field. Then, in order to preserve the equivalence of all inertial frames, we must assume that the stochastic parameters of the radiation are Poincaré invariant. It can be shown [2] that this implies that the spectrum of such radiation is of the form

$$\varrho(\omega) = \text{const}\ \omega^3\,, \qquad \text{const} \equiv \hbar/2\pi^2 c^3\,, \tag{1.1}$$

where c is the speed of light (introduced here for later convenience) and $\hbar$ some constant which gives a measure of the intensity of the radiation. On empirical grounds, the constant $\hbar$ is identified with the reduced Planck constant. According to eq. (1.1), the total energy density of the radiation diverges as the fourth power of ω, so that a cut-off at high frequencies must be assumed and, accordingly, a loss of Poincaré invariance. However, this loss of invariance would be observable only in processes taking place in very short time intervals. Therefore, we will assume that the spectrum (1.1) is correct up to an angular frequency $\omega_{\max}$ and consider that the theory makes predictions only about those quantities which are finite in the limit $\omega_{\max} \to \infty$. Physical bases for the cut-off might be the gravitational interaction or the pair creation [1]. The theory based upon these hypotheses is called stochastic electrodynamics.

Stochastic electrodynamics has provided a classical interpretation for several phenomena usually considered purely quantal [1]. A consequence of the theory is that the motion of any charged particle has a random character. On the other hand, it has been speculated that the (complex) Schrödinger equation might be interpreted as a pair of real stochastic equations [3], which might give the general connection between stochastic electrodynamics and quantum mechanics. Nevertheless, there is no proof that the random motion

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[2] T. W. MARSHALL: *Proc. Cambridge Phil. Soc.*, **61**, 537 (1965); T. H. BOYER: *Phys. Rev.*, **182**, 1374 (1969).

[3] E. SANTOS: *Brownian motion and the stochastic theory of quantum mechanics*, in *Irreversibility in the Many-Body Problem*, edited by L. M. GARRIDO, J. BIEL and J. REA (New York, 1972). Standard references are: E. NELSON: *Phys. Rev.*, **150**, B 1079 (1966); L. DE LA PEÑA-AUERBACH: *Journ. Math. Phys.*, **10**, 1620 (1969).

due to the background radiation is just what is needed to derive the Schrödinger equation. (The derivation of the Schrödinger equation directly from stochastic electrodynamics, made by SURDIN [4], does not seem conclusive to the author of the present paper.)

At first sight, there is an essential difficulty with stochastic electrodynamics because it can apply only to charged particles. Consequently, it seems unsuitable as an alternative for quantum mechanics in dealing with uncharged particles. In other words, the electromagnetic background radiation might be the origin of the quantum behaviour of charged particles but apparently it cannot explain the quantum behaviour of the uncharged ones. The idea of a new, independent, explanation is very unsatisfactory, so that we are forced to assume that all particles are composed of charged parts (it is not necessary to assume that all bodies have a total charge different from zero). But, if this is so, the electromagnetic interaction is more fundamental that the strong (nuclear) interaction (and, indeed, more fundamental than the weak one). This conclusion is the opposite to the current opinion in elementary-particle physics. We see, then, that the acceptance of stochastic electrodynamics as the basic theory of the microworld means a very radical departure from the current opinions in theoretical physics. In fact, we would return to a conception of the physical world very similar to that of eighty years ago.

The purpose of this paper is to study the nonrelativistic harmonic oscillator according to stochastic electrodynamics. Several papers have been devoted to this subject [5], but many points remain unclear. In this paper we rederive some known results and develop the theory further. In this way, the analogies and differences between stochastic and quantum electrodynamics can be seen more clearly. The main interest of this research is to shed some light on the way in which we might deal with nonlinear systems, which have not been studied till now.

2. – Equations of motion.

If we neglect the force due to the magnetic radiation field, as is usual in many nonrelativistic calculations, the equation of motion of a particle in the presence of radiation is

$$m\ddot{\boldsymbol{r}} = e\boldsymbol{E} + \boldsymbol{f} + m\tau\dddot{\boldsymbol{r}}\,, \qquad \tau \equiv 2e^2/3mc^3\,, \tag{2.1}$$

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[4] M. SURDIN: *Intern. Journ. Theor. Phys.*, **4**, 117 (1971).

[5] T. W. MARSHALL: *Proc. Roy. Soc.*, A **276**, 475 (1963); P. BRAFFORT and C. TZARA: *Compt. Rend.*, **239**, 1779 (1954); P. BRAFFORT, M. SURDIN and A. TARONI: *Compt. Rend.*, **261**, 4339 (1965).

where m is the mass and e the charge of the particle, $\boldsymbol{f}=\boldsymbol{f}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)$ the external (given) force and $\boldsymbol{E}$ the electric field of the radiation. The last term of eq. (2.1) represents the damping due to the reaction on the particle of the radiation emitted by it. The time variation of the electric force due to the radiation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(e\boldsymbol{E}) = e\frac{\partial \boldsymbol{E}}{\partial t} + e(\dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla})\boldsymbol{E}\,.$$

In the nonrelativistic domain it is consistent to neglect the second term compared with the first one and, therefore, to assume that the electric field is a function of time but not of position. It can be shown that this is equivalent to the electric dipole approximation of quantum electrodynamics, familiar from atomic physics.

Equation (2.1) is a stochastic differential equation which can be solved, in principle, provided we know the properties of the stochastic process $\boldsymbol{E}(t)$. The information available about $\boldsymbol{E}(t)$ is the power spectrum, which can be derived from eq. (1.1). We define the power spectrum of any function, $x(t)$, in the form

$$(2.2)\qquad G_x(\omega) \equiv 2\operatorname*{limit}_{T\to\infty}|\tilde{x}(\omega, T)|^2\,, \qquad \tilde{x}(\omega, T) \equiv (4\pi T)^{-\frac{1}{2}}\int_{-T}^{T} x(t)\exp[i\omega t]\,\mathrm{d}t\,.$$

The autocorrelation function of any (possibly complex) variable is defined by

$$(2.3)\qquad \langle x^*(t)x(t+T)\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty} G_x(\omega)\exp[-i\omega T]\,\mathrm{d}\omega\,.$$

For real variables, where $G_x(\omega) = G_x(-\omega)$, this relation takes the more usual form (Wiener-Khintchine theorem)

$$(2.4)\qquad \langle x(t)x(t+T)\rangle = \int_{0}^{\infty} G_x(\omega)\cos\omega T\,\mathrm{d}\omega\,.$$

Now, the power spectrum of $\boldsymbol{E}(t)$ is derived from eq. (1.1) as follows. The average value of the electric field of the radiation is related to the radiant energy density u in the form

$$u = \int_{0}^{\omega_{\max}} \varrho(\omega)\,\mathrm{d}\omega = (1/8\pi)\langle \boldsymbol{E}^2 + \boldsymbol{H}^2\rangle = (1/4\pi)\langle \boldsymbol{E}^2\rangle = (3/4\pi)\langle E_x^2\rangle\,,$$

where we have taken into account that the mean-square values of any component of the electric and the magnetic fields of the radiation are equal. Now, by com-

parison with eq. (1.1) it follows that the power spectrum of a component of the electric field, say $E_x(t)$, is

$$G_E(\omega) = 2\hbar|\omega|^3/3\pi c^3 . \tag{2.5}$$

We write $|\omega|$ (not just ω) in order that, for negative frequencies, the power spectrum $G_E(\omega)$ be positive, as it should be in agreement with eq. (2.2).

Equation (2.1) is nonlinear in general and, therefore, very difficult to solve. In this paper we will deal only with the linear problems (free particle and harmonic oscillator) which are easy to handle. Nevertheless, we make in the following some general considerations which might be useful in the study of nonlinear problems. In the first place, it must be pointed out that eq. (2.1) is a third-order differential equation, but only two independent solutions are physically meaningful. For instance, if $\boldsymbol{E}$ and $\boldsymbol{f}$ were zero, the general solution of eq. (2.1) would be

$$\boldsymbol{r} = \boldsymbol{A} + \boldsymbol{B}t + \boldsymbol{C} \exp [t/\tau] .$$

The exponentially increasing function of time is absurd, and it is present due to the approximate nature of the damping term, $m\tau \dddot{\boldsymbol{r}}$ ([6]). A procedure to eliminate these undesirable solutions is to use, instead of eq. (2.1), the following integro-differential equation:

$$m\ddot{\boldsymbol{r}} = e\int_0^\infty \boldsymbol{E}(t+\tau s) \exp [-s]\,\mathrm{d}s + \int_0^\infty \boldsymbol{f}(t+\tau s) \exp [-s]\,\mathrm{d}s \equiv \boldsymbol{F}(t) + \boldsymbol{f}_{\mathrm{ef}}(t) . \tag{2.6}$$

It is easy to show that this equation has the same solutions as eqs. (2.1) except the undesirable ones. Equation (2.6) has the formal appearance of the Newton law for a particle which is subject to a random force $\boldsymbol{F}$ besides the external force $\boldsymbol{f}_{\mathrm{ef}}(t)$. The random force has a power spectrum which can be easily derived from that of $\boldsymbol{E}$. For one of the components of $\boldsymbol{F}$ it is

$$G_F(\omega) = m\hbar\tau|\omega|^3/[\pi(1+\tau^2\omega^2)] . \tag{2.7}$$

The comparison of this with eq. (2.2) shows that the damping term, $m\tau \dddot{\boldsymbol{r}}$, gives rise to some kind of cut-off in the power spectrum of the effective random force. Unfortunately, the simplicity of eq. (2.6) is misleading, because the effective external force $\boldsymbol{f}_{\mathrm{ef}}$ cannot be determined before the equation is solved.

([6]) See, for example, L. D. LANDAU and E. M. LIFSHITZ: *Classical Theory of Fields* (London, 1965).

In the search for a solution of eqs. (2.1) or (2.6) it is useful to ask for a small number which may allow some perturbative approach. By combining the parameters of the basic equations (1.1) and (2.1), which are m, c, e and $\hbar$, a single independent pure number can be obtained, which is convenient to write in the form $\alpha \equiv e^2/\hbar c$. For a particle with the elementary charge, this is just the fine-structure constant, which has so fundamental a role in quantum electrodynamics (QED). If stochastic electrodynamics (SED) is to be an alternative to QED, we must compare the predictions of both theories to every order in α. In particular, the zeroth-order approximation to SED should be a theory paralleling ordinary quantum mechanics (QM). For short, we will call SM (stochastic mechanics) this zeroth-order approximation to SED. The limit $\alpha \to 0$ means $\tau \to 0$ for given m, c and $\hbar$. However, it is important to point out that the limit $\tau \to 0$ may not be a good approximation (even for $\alpha \ll 1$) in some cases, as, for example, in studying the motion in time intervals of order τ ($\simeq \hbar\alpha/mc^2$). So, we expect a break down of SM for times of order τ or lengths of order $\hbar\alpha/mc$. This might be the case for quantum mechanics also, being here the origin of the difficulties in elementary particle theory. Taking the limit $\tau \to 0$ in eqs. (2.6) and (2.7), we expect to obtain the basic equations of SM, which we write

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}(t) + \boldsymbol{f}(t + \pi\varepsilon)\,, \tag{2.8}$$

$$G_F^0(\omega) = m\hbar\varepsilon(\omega)|\omega|^3\,, \qquad \varepsilon(\omega) \equiv \lim_{\tau\to 0} (\tau/\pi)/(1 + \tau^2\omega^2)\,. \tag{2.9}$$

The quantity ε appearing in eq. (2.8) is an infinitesimal equivalent to τ/π, as is shown in the following:

$$\int_0^\infty \boldsymbol{f}(t + \tau s)\exp[-s]\,\mathrm{d}s = \boldsymbol{f}(t) + \dot{\boldsymbol{f}}(t)\tau + o(\tau) = \boldsymbol{f}(t + \tau) + o(\tau)\,.$$

It is important to retain it in eq. (2.8) because the limit $\varepsilon \to 0$ is to be taken only at the end of the calculation. The symbolic function $\varepsilon(\omega)$ has a meaning only under an integral and can be defined as a distribution such that, for any bounded function $f(\omega)$,

$$\int_{-\infty}^{\infty} \varepsilon(\omega) f(\omega)\,\mathrm{d}\omega = \langle f\rangle \equiv \lim_{\Omega\to\infty} (2\Omega)^{-1}\int_{-\Omega}^{\Omega} f(\omega)\,\mathrm{d}\omega\,. \tag{2.10}$$

The function $\varepsilon(\omega)$ can be also considered the inverse of the Dirac δ for positive ω, that is

$$y(x) = \delta(x), x > 0 \Leftrightarrow x(y) = \varepsilon(y)\,, y > 0\,.$$

It must be noted that the infinitesimal ε of eq. (2.8) must be such that the quotient $\varepsilon(\omega)/\varepsilon$ is unity.

Equations (2.8) and (2.9), being the basic equations of SM, must correspond to the basic equations of QM, that is either the Schrödinger equation plus the probability interpretation of the square modulus of the wave function, or the Heisenberg equation plus the fundamental commutation relations. The connection between the basic equations of SM and QM is not trivial and it will be studied in subsequent papers.

3. – Momentum and energy.

A problem which we must solve is to define the momentum and the energy of a charged particle in the presence of radiation. This is not trivial because only the momentum and the energy of the whole system (particle plus radiation) are clearly defined. Indeed, the definitions cannot be arbitrary if we wish that the calculated quantities agree with the experimental ones. So, the momentum of the particle must be defined as a function of observable quantities and in such a way that the change in momentum with time is equivalent to the experimentally controllable forces acting on the particle (therefore excluding those due to the background radiation). A similar statement is true for the energy. Therefore, a particle subject only to the random forces—which we will call a free particle in the following—must have a constant linear momentum. It is obvious that the momentum of such a particle cannot be defined by means of the equation

$$\boldsymbol{p} = m\dot{\boldsymbol{r}}\,, \tag{3.1}$$

this being a rapidly fluctuating quantity. According to the above-stated principle, the rate of change in (observable) momentum must be equal to the external force, that is

$$\dot{\boldsymbol{p}} = \boldsymbol{f}\,. \tag{3.2}$$

From this and eq. (2.1) follows the definition

$$\boldsymbol{p} = m\dot{\boldsymbol{r}} + (e/c)\boldsymbol{A} - m\tau\ddot{\boldsymbol{r}} \qquad (\text{gauge } \varphi = 0) \tag{3.3}$$

where $\boldsymbol{A}$ is the vector potential (remember that $\boldsymbol{A}$ was assumed to depend only on time). It is seen that the mechanical momentum (eq. (3.1)) has two parts: the observable momentum $\boldsymbol{p}$ and a hidden, fluctuating momentum given by minus the two last terms of eq. (3.3). In order that the (observable) momentum of a free particle equal the mean mechanical momentum, the potential $\boldsymbol{A}$ must be so chosen that its time averag is also zero, and we will make this choice from now on.

It is not easy to define the kinetic energy in such a way that it is related to observable quantities. The rate of change of the kinetic energy should be equal to the work made by the external forces, *i.e.*

$$\mathrm{d}K/\mathrm{d}t = \boldsymbol{f}\cdot\dot{\boldsymbol{r}}\,. \tag{3.4}$$

However, the kinetic energy so defined is a rapidly fluctuating quantity. This is best seen if we write eq. (3.4) in terms of the momentum, taking into account eqs. (3.2) and (3.3). In fact, eq. (3.3) can be transformed into

$$m\dot{\boldsymbol{r}}(t) = \int_0^\infty \boldsymbol{p}(t+\tau s)\exp[-s]\,\mathrm{d}s - (e/c)\int_0^\infty \boldsymbol{A}(t+\tau s)\exp[-s]\,\mathrm{d}s\,. \tag{3.5}$$

Combining this with eqs. (3.2) and (3.4) we obtain

$$m\mathrm{d}K/\mathrm{d}t = \dot{\boldsymbol{p}}(t)\cdot\int_0^\infty \boldsymbol{p}(t+\tau s)\exp[-s]\,\mathrm{d}s + \dot{\boldsymbol{p}}\cdot\boldsymbol{q}\,, \tag{3.6}$$

where $\boldsymbol{q}(t)$ is the last term of eq. (3.5). This equation shows that the fluctuating part of the kinetic energy is due to the quantity $\boldsymbol{q}$, which is related only to the random forces produced by the radiation field. We have assumed that these are independent of the external forces, so that the mean value of the last term of eq. (3.6) is zero. This suggests to define the (averaged) kinetic energy so that the following relation holds:

$$m\mathrm{d}K/\mathrm{d}t = \dot{\boldsymbol{p}}(t)\cdot\int_0^\infty \boldsymbol{p}(t+\tau s)\exp[-s]\,\mathrm{d}s\,. \tag{3.7}$$

Indeed, with this definition, the kinetic energy of a free particle does not change with time. Nevertheless, this definition is not very useful because eq. (3.7) cannot be integrated. Physically, this means that the change in kinetic energy does not only depend on the change in momentum but also on the speed of this change. As we have dropped the last term of eq. (3.6) because it is a fluctuating quantity, eq. (3.7) seems reasonable only when the momentum changes slowly with time. In this case the constant τ can be neglected and eq. (3.7) leads to the following definition of kinetic energy:

$$K = (\boldsymbol{p}(t))^2/2m\,. \tag{3.8}$$

Nevertheless, other definitions of energy seem possible in SED and some ambiguity remains. In the limit $\tau \to 0$ (SM) the situation is much better because eq. (3.7) can be integrated directly to give eq. (3.8).

The passage from the mechanical definition of kinetic energy ($m\dot{\boldsymbol{r}}^2/2$) to eq. (3.8) corresponds to the mass renormalization of quantum electrodynamics. In fact, the mechanical definition leads to an energy which is divergent when $\omega_{\max}$ goes to infinity as we will show in the next Section. In the nonrelativistic theory which we are considering, although there is an energy renormalization there is no mass renormalization. In fact, if the observable mass is defined as the ratio between force and acceleration, we show in the following that the mass which appears in the equation of motion (2.1) is already the observable mass. In order to define an observable acceleration, independent of the time intervals used to measure it, we will consider a constant external force $\boldsymbol{f}$. As we are assuming that $\boldsymbol{E}$ is only a function of time and $\boldsymbol{f}$ a constant, there are solutions of eq. (2.6) such that

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{R},$$

where

$$m\ddot{\boldsymbol{R}} = \boldsymbol{f}, \qquad m\ddot{\boldsymbol{r}}_0 = \boldsymbol{F}.$$

If we remember that the mean value of $\boldsymbol{F}$ is zero due to Lorentz invariance, the mean acceleration is

$$\langle \ddot{\boldsymbol{r}} \rangle = \langle \ddot{\boldsymbol{r}}_0 \rangle + \langle \ddot{\boldsymbol{R}} \rangle = \langle \ddot{\boldsymbol{R}} \rangle = \ddot{\boldsymbol{R}} = \boldsymbol{f}/m,$$

which shows that the observable mass does not need renormalization. This fact contrasts with the situation in QED, where the mass must be renormalized even in the nonrelativistic approximation ([7]).

The above definitions of momentum and energy (eqs. (3.3) and (3.8)) were first proposed by BRAFFORT, SURDIN and TARONI ([5]). In our justification of these definitions we have shown that the first follows from very general principles, but the second is not without ambiguity.

4. – Comparison with Brownian motion.

The stochastic theory best known in physics is the theory of Brownian motion ([8]) and it is useful to compare this theory with stochastic electrodynamics. In its simplest form—due to EINSTEIN and SMOLUCHOWSKI—the theory of Bro nian motion states that the motion of a particle in the absence of external forces—subject only to the random forces produced by the interaction with the molecules of the liquid in which it moves—is characterized by a stochastic

([7]) See, for example, J. J. SAKURAI: *Advanced Quantum Mechanics* (New York, 1967), p. 70.

([8]) N. WAX (Editor): *Selected Papers on Noise and Stochastic Processes* (New York, 1954).

process $x(t)$, known as Wiener process. The Wiener process has the property that its mean-square fluctuation in a time T is proportional to T, *i.e.*

$$2DT = \Delta x \equiv \langle [x(t+T) - x(t)]^2 \rangle = 2\langle x(t)^2 \rangle - 2\langle x(t)x(t+T) \rangle , \tag{4.1}$$

where the brackets mean ensemble averages, and D is a constant, called the diffusion coefficient, which measures the intensity of the random motion. (For simplicity, we will work in one dimension throughout this Section.) From eq. (4.1) it can be shown that the probability density of $x(t) - x(0)$ is Gaussian or, equivalently, that this probability density obeys the diffusion equation

$$\frac{\partial \varrho(x, t)}{\partial t} = D \frac{\partial^2 \varrho(x, t)}{\partial x^2} . \tag{4.2}$$

If we assume now that eq. (4.1) holds also for time averages, it follows, from the Wiener-Khintchine theorem, that

$$\tfrac{1}{2}\Delta x = \langle x(t)^2 \rangle - \langle x(t)\, x(t+T) \rangle = \int_0^\infty G_x(\omega)(1 - \cos \omega T)\, \mathrm{d}\omega = DT . \tag{4.3}$$

The power spectrum that fulfils this equality is

$$G_x^0(\omega) = 2D/\pi\omega^2 , \tag{4.4}$$

and it fully characterizes the Wiener process. In this way we have, for Brownian motion, a connection between the power spectrum (4.4) of its characteristic stochastic process (Wiener process) and the evolution equation (4.2) of the associated probability density (the Fokker-Planck equation of the stochastic process). On the contrary, in zeroth-order stochastic electrodynamics (SM) although it is easy to obtain the power spectrum of the basic stochastic process, the associated Fokker-Planck equation is not known. (This equation should be equivalent to the Schrödinger equation if SM is to be an alternative to QM.) The power spectrum of the basic stochastic process of SM is

$$G_x^0(\omega) = \hbar\varepsilon(\omega)/m|\omega| . \tag{4.5}$$

This is obtained from eqs. (2.8) and (2.9) when $f = 0$. It is seen that eq. (4.5) has some analogy with eq. (4.4) if we identify

$$\hbar = 2mD .$$

This is just the equality which has been used in all stochastic theories of quantum mechanics ([3]). On the other hand, there are clear differences between eqs. (4.4)

and (4.5), which show that the basic stochastic process of SM is different from the one of Brownian motion (Wiener process). Indeed, it is now well established that the Wiener process is not suitable for a stochastic theory of quantum mechanics (9).

The analogy between the basic stochastic processes of SM and Brownian motion is summarized in that both are Markovian processes characterized by a single parameter of dimensions, lenght squared over time. The Markovian property can be roughly defined by the equality

$$\langle \dot{x}(t)\, \dot{x}(t+T)\rangle \equiv \int_0^\infty G_x(\omega)\, \omega^2 \cos \omega T \, \mathrm{d}\omega = 0\,, \qquad \text{if } T \neq 0\,; \tag{4.6}$$

this means that the velocities at two different times are uncorrelated. Strictly speaking, the velocity $\dot{x}$ is not defined because $x(t)$ is not differentiable, but eq. (4.6) can be stated rigorously as

$$\lim_{\tau \to 0} \mathrm{Re} \int_0^\infty G_x(\omega)\, \omega^2 \exp\left[i\omega T - \tau\omega\right] \mathrm{d}\omega = 0\,, \qquad \text{if } T \neq 0\,. \tag{4.7}$$

It is easy to show that this condition holds for both eqs. (4.4) and (4.5). An interesting question is whether there are other Markov processes characterized by a single parameter of dimensions lenght squared over time. The answer is important because a conclusion of the stochastic theories of quantum mechanics is that there are only two basic processes that fulfil the apparently related property eq. (4.1) (note that this equation refers to ensemble averages, not to time averages). It seems as if these two processes were those which we are considering here (whose power spectra are eqs. (4.4) and (4.5)). Indeed, it has been shown that one of them gives rise to a diffusion-type equation (which suggested to identify it with the Wiener process) and the other leads to the Schrödinger equation (3). If it can be shown that this second process is the same as that which appears in SM (whose spectrum is given by eq. (4.5)), we would have an indirect derivation of the Schrödinger equation from stochastic electrodynamics. However, there are many unclear points and a true derivation does not exist at present. For example, the spectrum given by eq. (4.5) does not fulfil eq. (4.3) although it might fulfil eq. (4.1) for suitably chosen ensemble averages. This problem will be dealt with in subsequent papers.

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(9) See E. SANTOS: *Brownian motion and the stochastic theory of quantum mechanics*, in *Irreversibility in the Many-Body Problem*, edited by L. M. GARRIDO, J. BIEL and J. RAE (New York, 1972).

Let us consider now the theory of Brownian motion due to ORNSTEIN and UHLENBECK. The basic equation of the theory is

$$m\ddot{x} = F - m\tau^{-1}\dot{x} + f, \tag{4.8}$$

where f is the external (systematic) force and F the stochastic force. The power spectrum of the stochastic force is white (roughly speaking, it is the derivative of a Wiener process). It can be written

$$G_F(\omega) = m^2 D/\pi\tau\,, \qquad \tau^{-1} \equiv 6\pi a\eta/m\,, \qquad D \equiv k\theta\tau/m\,, \tag{4.9}$$

where m is the mass of the Brownian particle and a its radius, η the viscosity of the medium and θ its absolute temperature, k being the Boltzmann constant. Actually, eq. (4.8) is incorrect because the acceleration $\ddot{x}$ and the force F are not defined (the Wiener process has no derivative), and it should be substituted by a suitable pair of equations (Langevin equations). Nevertheless, we write the basic equation as in eq. (4.8) because it is easier to understand physically (compare with eq. (2.1)). If there are no external forces ($f = 0$), eq. (4.8) can be written

$$\dot{x} = u(t)\,, \tag{4.10}$$

where the spectrum of the velocity $u(t)$ is

$$G_u(\omega) = (2D/\pi)/(1 + \tau^2\omega^2)\,. \tag{4.11}$$

It can be shown that this implies the following Gaussian distribution for the velocities of the Brownian particle (Maxwell distribution):

$$\varrho(u) = (2\tau/\pi D)^{\frac{1}{2}} \exp\left[-\tau u^2/2D\right], \tag{4.12}$$

a fact which will be used in the next Section.

The Ornstein-Uhlenbeck theory of Brownian motion (OU) is similar to SED in the same way that the Einstein-Smoluchowski theory (ES) is similar to SM. Indeed, ES is the limit of OU as SM is the limit of SED. In fact, if we search for a pure number by combining the parameters of OU—m, a, η, $k\theta$—we obtain

$$\gamma \equiv mk\theta/\eta^2 a^4\,.$$

This number is small for large enough Brownian particles (remember that a^4/m is proportional to the size for particles of constant density). For fixed conditions of the medium in which the Brownian particle moves (*i.e.* θ and η fixed) it is easy to see that the Brownian particles have characteristic times τ

proportional to γ if they have the same diffusion parameter D. Then, the limit $\tau \to 0$ usually becomes a good approximation if $\gamma \ll 1$. We emphasize that this may not always be the case, similarly to the situation encountered in SM (and, presumibly, in QM). In the limit $\tau \to 0$, eqs. (4.8) and (4.9) become

$$\dot{x} = u(t) + v(t)\,, \qquad u \equiv \operatorname*{limit}_{\tau \to 0} \tau F/m\,, \qquad v \equiv \operatorname*{limit}_{\tau \to 0} \tau f/m\,, \tag{4.13}$$

where the spectrum of the stochastic velocity u is

$$G_u^0(\omega) = 2D/\pi\,. \tag{4.14}$$

These are just the basic equations of ES. Indeed, from eqs. (4.13) and (4.14) eq. (4.4) follows easily in the absence of systematic forces.

Finally, it is useful to compare the mean-square velocity and the fluctuation of position in SED and OU. For simplicity, let us consider a free particle (*i.e.* $f = 0$ in eqs. (4.8) and (2.1)). In OU, the spectrum of the velocity $u(t)$ is given by eq. (4.11) and that of the co-ordinate $x(t)$ by

$$G_x(\omega) = G_u(\omega)/\omega^2 = (2D/\pi\omega^2)/(1 + \tau^2\omega^2)\,. \tag{4.15}$$

Hence, the mean-square velocity is

$$\langle u^2 \rangle_{\mathrm{OU}} = \int_0^\infty G_u(\omega)\,\mathrm{d}\omega = D/\tau\,, \tag{4.16}$$

and the mean fluctuation of position is

$$\Delta x_{\mathrm{OU}} = \int_0^\infty G_x(\omega)(1 - \cos\omega T)\,\mathrm{d}\omega \simeq \begin{cases} DT & \text{for large } T\,, \\ DT^2/\tau & \text{for small } T\,. \end{cases} \tag{4.17}$$

In SED, the spectrum of the velocity is eq. (4.9) and that of the co-ordinate

$$G_x(\omega) = (\hbar\tau/\pi m|\omega|)/(1 + \tau^2\omega^2)\,. \tag{4.18}$$

Hence, the mean-square velocity results logarithmically divergent and the fluctuation of the position is

$$\Delta x_{\mathrm{SED}} = \frac{\hbar\tau}{\pi m}\int_0^\infty \frac{(1 - \cos\omega T)\,\mathrm{d}\omega}{\omega + \tau^2\omega^3} \simeq \begin{cases} (\hbar\tau/\pi m)\log\dfrac{T}{\tau} & \text{for large } T\,, \\ (\hbar T^2/2\pi m\tau)\log\dfrac{\tau}{T} & \text{for small } T\,. \end{cases} \tag{4.19}$$

(See Appendix A for the calculation of this integral.) We see that the fluctuation in position is larger in SED that in OU for small time intervals, but smaller for large time intervals. This is a consequence of the fact that random motion has the character of a vibration in SED but in Brownian motion it is similar to a sequence of random steps. It is to be noted that in Brownian motion the fluctuation is a very simple function of T for large T and it is independent of τ. In SED, however, the dependence of the fluctuation in T is rather complex and depends on τ even in the limit $T \to \infty$. This is the origin of the difficulty in finding the Fokker-Planck equation of SM (presumably equivalent to the Schrödinger equation).

5. – Ground state of the oscillator.

The harmonic oscillator is characterized by an external force of the type

$$\boldsymbol{f} = -\text{const}\, \boldsymbol{r} \equiv -m\omega_0^2 \boldsymbol{r}\,. \tag{5.1}$$

It is useful to study the oscillator both starting with eqs. (2.1) and (2.2) (SED) and from eqs. (2.8) and (2.9) (SM). The results obtained from this last pair of equations must agree with the limit $\alpha \to 0$ of this obtained from the first one. This will give us a test of the usefulness of eqs. (2.8) and (2.9). In the problem of the oscillator a new parameter, ω_0, appears which is not present in the general theory. Hence, another pure number can be obtained besides $\alpha \equiv e^2/\hbar c$, which is conveniently written in one of the forms

$$\beta \equiv \hbar\omega_0/mc^2\,, \qquad \gamma \equiv 2e^2\omega_0/3mc^3 \equiv 2\alpha\beta/3 \equiv \tau\omega_0\,. \tag{5.2}$$

The number β gives a measure of the validity of the nonrelativistic approximation. Although it is usually much smaller than α (for instance, $\beta \simeq 10^{-10}$ in molecular vibrations), it is less significant for the theory. Most times, the numbers α and β will appear combined in such a way that γ is the best expansion parameter (*i.e.* we will obtain the results of SM by taking the limit $\gamma \to 0$ in the results of SED).

The equation of motion of the oscillator in SED is obtained from eqs. (2.1) and (3.1). In one dimension it is written

$$\ddot{x} = (e/m)E_x - \omega_0^2 x + \tau\dddot{x}\,. \tag{5.3}$$

The general solution of this equation is the sum of the general solution of the homogeneous part plus a particular solution of eq. (5.3) itself. The general solution will be considered later. Let us now find the particular solution such that

$$\langle x\rangle = \langle \dot{x}\rangle = \langle \ddot{x}\rangle = 0\,. \tag{5.4}$$

(The bracket means now time average, a convention which we will always use in the following.) This particular solution represents a motion which we will call ground state of the oscillator. From eqs. (2.2), (5.3) and (5.4), the power spectrum of $x(t)$ is easily found to be

$$G_x(\omega) = (\hbar\tau/\pi m)|\omega|^3/[(\omega^2-\omega_0^2)^2+\tau^2\omega^6]\,. \tag{5.5}$$

Hence, if we take eq. (3.2) into account, the power spectrum of the momentum $p(t)$ is obtained:

$$G_p(\omega) = (m\hbar\tau\omega_0^4/\pi)|\omega|/[(\omega^2-\omega_0^2)^2+\tau^2\omega^6]\,. \tag{5.6}$$

The peaks of these functions at $\omega \simeq \pm\omega_0$ indicate an approximately harmonic variation of $x(t)$ and $p(t)$.

In order to study the motion of the oscillator in more detail, it is useful to follow the path in phase space, which is a two-dimensional manifold for a system with one degree of freedom. It is convenient to represent the points of phase space by complex numbers a, such that

$$\mathrm{Re}\, a = m\omega_0 x/(2m\hbar\omega_2)^{\frac{1}{2}}, \qquad \mathrm{Im}\, a = (\omega_1 p/\omega_0)/(2m\hbar\omega_2)^{\frac{1}{2}}, \tag{5.7}$$

where the angular frequencies ω_1 and ω_2 will be defined later. In this way, the motion of the oscillator in phase space can be represented by a single complex function of time. We will see that the angular frequencies ω_1 and ω_2 become identical with ω_0 in the limit $\alpha \to 0$ (*i.e.* in SM). In this case the function $a(t)$ is written

$$a(t) = [m\omega_0 x(t) + ip(t)]/(2m\hbar\omega_0)^{\frac{1}{2}}, \tag{5.8}$$

which shows the parallelism between $a(t)$ and the usual ladder operator of quantum mechanics. The power spectrum of the function $a(t)$ can be obtained from eq. (5.5) (remember our definition of power spectrum for complex functions, eq. (2.3)). It is

$$G_a(\omega) = (\tau\omega_0^2/2\pi\omega_2)|\omega|(\omega_1+\omega)^2/[(\omega^2-\omega_0^2)^2+\tau^2\omega_6]\,. \tag{5.9}$$

At this moment, we define ω_2 by normalizing $a(t)$ in such a way that $\langle|a|^2\rangle$ is one-half (which is the value which it takes for $\omega_1=\omega_2=\omega_0$, *i.e.* in SM). This normalization is important in order to compare the fluctuations of $a(t)$ for different values of ω_1. Then, we have ω_2 in terms of ω_1 as follows:

$$\tfrac{1}{2} = \langle|a|^2\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty} G_a(\omega)\,\mathrm{d}\omega = (1/4\pi\omega_0\omega_2)[(\pi-\gamma)(\omega_1^2+\omega_0^2) - 2\gamma\log\gamma\omega_0^2] + o(\gamma). \tag{5.10}$$

(See Appendix A for the calculation of the integral.)

It is seen that the function $G_a(\omega)$ has a peak at $\omega \simeq \omega_0$, but, unlike eqs. (5.5) and (5.6), the peak at $\omega \simeq -\omega_0$ is not present (remember that $\omega_1 \simeq \omega_0$). This shows that the function $a(t)$ varies with time approximately as $\exp[-i\omega_0 t]$ and it suggests introducing the new function

$$b(t) = a(t) \exp[i\omega_3 t], \tag{5.11}$$

where the angular frequency ω_3 can be considered the mean frequency of the oscillator if it is chosen so that the function $b(t)$ varies with time as slowly as possible. This leads us to define ω_1 and ω_3 in such a way that the fluctuation of $b(t)$ is a minimum, that is

$$\begin{aligned} \Delta b \equiv \langle |b(t+T) - b(t)|^2 \rangle = \\ = 2\langle |b(t)|^2 \rangle - \langle b^*(t)\, b(t+T) \rangle - \langle b^*(t)\, b(t-T) \rangle = \text{minimum}. \end{aligned} \tag{5.12}$$

This fluctuation must be calculated from the power spectrum of $b(t)$ which, from eq. (5.11), can be found to be

$$G_b(\omega) = G_a(\omega + \omega_3). \tag{5.13}$$

Hence, if we take into account eqs. (5.12) and (2.4), the fluctuation Δb is given by

$$\begin{aligned} \Delta b = \int_{-\infty}^{\infty} G_b(\omega)(1 - \cos \omega T)\, \mathrm{d}\omega = \int_{-\infty}^{\infty} G_a(\omega)[1 - \cos(\omega - \omega_3) T]\, \mathrm{d}\omega = \\ = 1 - \int_{-\infty}^{\infty} G_a(\omega) \cos(\omega - \omega_3) T, \end{aligned} \tag{5.14}$$

where the last equality was written taking eq. (5.10) into account. The values of ω_1 and ω_3 which make eq. (5.14) a minimum depend on T, so that we must specify the value of T at which the minimum must be calculated. It seems that the best choice is to consider large values of T. Then, the contribution to the last integral in eq. (5.14) comes mainly from frequencies near ω_3, so that the minimum of Δb occurs at the maximum of $G_a(\omega_3)$. If we take into account eq. (5.10), this leads to the following values of ω_1, ω_2 and ω_3:

$$\omega_1 = \omega_0[1 - (2\gamma/\pi) \log \gamma], \quad \omega_2 = \omega_0[1 - \gamma/\pi - (2\gamma/\pi) \log \gamma], \quad \omega_3 = \omega_0, \tag{5.15}$$

calculated up to first order in γ.

The fluctuation Δb can be calculated now from eqs. (5.14) and (5.15). It can be shown that this fluctuation is very small (of order γ) for times of order ω_0^{-1} or less. This is best seen by evaluating the fluctuations of $x(t)$ and $p(t)$ for

$T = 2\pi n/\omega_0$, n being an integer. It follows that

$$\Delta x = 2\int_0^\infty G_x(1-\cos\omega T)\,\mathrm{d}\omega = \tag{5.16}$$
$$= (\hbar/\pi m\omega_0)[-\gamma\log\gamma + \gamma\cosh(2\pi n) + 2\pi n\gamma\sinh(2\pi n)] + o(\gamma)\,,$$

$$\Delta p = 2\int_0^\infty G_p(\omega)(1-\cos\omega T)\mathrm{d}\omega = m\hbar\omega_0 n\gamma\sinh(2\pi n) + o(\gamma)\,. \tag{5.17}$$

For $n \ll 1/\gamma$, these fluctuations are very small in comparison with the mean values of x^2 and p^2, respectively. These are

$$\langle x^2\rangle = \int_0^\infty G_x(\omega)\,\mathrm{d}\omega = (\hbar/2m\omega_0)[1-(2\gamma/\pi)\log\gamma - \gamma/\pi] + o(\gamma)\,, \tag{5.18}$$

$$\langle p^2\rangle = \int_0^\infty G_p(\omega)\,\mathrm{d}\omega = (m\hbar\omega_0/2)(1-\gamma/\pi) + o(\gamma)\,. \tag{5.19}$$

For times much larger than ω_0^{-1} it is convenient to write $G_b(\omega)$ in terms of the variable

$$u = 2\omega/\gamma\,, \tag{5.20}$$

and to retain only terms of zero order in γ. In this way we have

$$G_b(\omega)\,\mathrm{d}\omega \simeq (\omega_0/\pi)\,\mathrm{d}u/[u^2+\omega_0^2]\,. \tag{5.21}$$

Hence, the following value for the fluctuation Δb is obtained:

$$\Delta b = \int_{-\infty}^\infty G_b(\omega)(1-\cos\omega T)\,\mathrm{d}\omega \simeq 2\gamma\omega_0 T\,, \quad \text{if} \quad \omega_0^{-1} \ll T \ll \gamma^{-1}\omega_0^{-1}\,. \tag{5.22}$$

The limit of Δb for $T\to\infty$ is unity, as it should be because Δb approaches $2\langle|b|^2\rangle$ in this limit (see eq. (5.12)) and this equals $2\langle|a|^2\rangle$ (eq. (5.11)), which is unity (eq. (5.10)).

The linear dependence of the fluctuation Δb on T (eq. (5.22)) is typical of Brownian motion (compare with eq. (4.1)). Also, the power spectrum of $b(t)$ (eq. (5.21)) is identical with that of the velocity in the Ornstein-Uhlenbeck theory of Brownian motion (eq. (4.11)) with the replacement $\tau\to\omega_0^{-1}$, $2D/\tau\to 1$. Then, as eq. (4.12) is a consequence of eq. (4.11), the following probability distribution results for a from eq. (5.21):

$$\varrho(|a|) = 2\pi^{-1}\exp[-|a|^2]\,. \tag{5.23}$$

In the limit $\alpha \to 0$ this becomes (see eq. (5.7))

$$\varrho(x, p) = (1/2\pi\hbar) \exp\left[-m\omega_0 x^2/2\hbar - p^2/2m\hbar\omega_0\right]. \tag{5.24}$$

The picture which emerges from the results obtained up to now, is as follows. The oscillator performs a harmonic motion almost as if the background radiation and the damping were not present. Both the phase and the amplitude are extremely stable for many periods (about 10^{10} periods in molecular vibrations). Nevertheless, they change slowly, performing a kind of Brownian motion, with the result that the memory of the initial phase and amplitude is finally lost. The probability distributions of position and momentum are Gaussian and independent of each other (eq. (5.24) can be factorized).

In the limit $\alpha \to 0$, eqs. (5.5) and (5.6) become

$$G_x(x) = (\hbar/m)\,\delta(\omega^2 - \omega_0^2), \qquad G_p(\omega) = m\hbar\omega_0^2\,\delta(\omega^2 - \omega_0^2). \tag{5.25}$$

These power spectra can be obtained also from eqs. (2.8) and (2.9) (SM), if some care is used in taking the limit $\varepsilon \to 0$. Similarly, one can obtain

$$G_a(\omega) = \delta(\omega - \omega_0), \qquad G_b(\omega) = \delta(\omega). \tag{5.26}$$

Hence, the first eq. (5.10) can be also found, but the fluctuations Δb, Δx and Δp become zero for all T (compare this result with eqs. (5.16), (5.17) and (5.22)). The mean values of x^2 and p^2 can be obtained from eq. (5.25) and agree with the results eqs. (5.18) and (5.19) in the limit $\gamma \to 0$. Nevertheless, eq. (5.21), which is essential in order to find the probability distribution of x and p, cannot be obtained from eq. (5.26). This shows that the limit $\varepsilon \to 0$ cannot be taken at intermediate stages of the derivation or some information is lost.

6. – Comparison with quantum theory.

In the following we show that the ground state of the oscillator is very similar in stochastic and quantum electrodynamics. We start with the comparison in zeroth order of α, that is we first compare the ground state in SM and QM. After this, we will study the corrections due to the finite value of α both in SED and QED.

The ground state of the oscillator in QM is characterized by an energy

$$E_0 = \tfrac{1}{2}m\omega_0^2\langle \boldsymbol{r}^2\rangle + (1/2\,m)\langle \boldsymbol{p}^2\rangle = \tfrac{3}{2}\hbar\omega_0, \tag{6.1}$$

and the following probability distributions for the co-ordinate and the mo-

mentum:

$$\begin{cases} \varrho(\boldsymbol{r}) = (m\omega_0/2\pi\hbar)^{\frac{3}{2}} \exp\left[-m\omega_0 \boldsymbol{r}^2/2\hbar\right], \\ \varrho(\boldsymbol{p}) = (2\pi\hbar m\omega_0)^{-\frac{3}{2}} \exp\left[-\boldsymbol{p}^2/2m\hbar\omega_0\right]. \end{cases} \tag{6.2}$$

In SM, the mean energy is given also by eq. (6.1) (see eqs. (5.18) and (5.19)) and there is a probability distribution in phase space given by eq. (5.24).

It is seen that the predictions of both theories are similar, but there are some differences between them. In the first place, in SM it is assumed that the particle has a precise position and a precise momentum at any time; this allows one to define a probability distribution in phase space. In QM, there is no such probability distribution in phase space because it is assumed that the position and the momentum cannot be dispersionless simultaneously. Actually, this difference seems not very important in practice because eq. (5.24) is just the product of the two functions (6.2). A more dramatic difference exists in the predictions about the energy. In fact, QM predicts that the energy is dispersionless and that it takes the value $\frac{3}{2}\hbar\omega_0$. In contrast, in SM the oscillator can have any instantaneous energy, and the value $\frac{3}{2}\hbar\omega_0$ represents only the time average over an infinite time interval. The probability distribution for the energy is given by eq. (5.24), which can be written

$$\varrho(E) = (1/\hbar\omega_0) \exp\left[-E/\hbar\omega_0\right]. \tag{6.3}$$

Incidentally, we note that this probability distribution is just the Boltzmann distribution for a temperature

$$k_{\mathrm{B}}\theta = \hbar\omega_0 . \tag{6.4}$$

These differences between the predictions of SM and QM are summarized in the statement that the ground state is a pure state in QM, but it is a mixture (or statistical ensemble) in SM. In fact, in SM the ground state of the oscillator does not correspond to a single path $\boldsymbol{r}(t)$ but to a whole class of paths, *i.e.* all those fulfilling eqs. (5.3) and (5.4). (More precisely, the ground state in SM is defined by a probability distribution in the space of the functions $\boldsymbol{r}(t)$.) In order to see whether these differences are important in practice we should ask what quantities can be actually measured in the ground state of the oscillator. As any measurement is a time-dependent phenomenon, no definite conclusion can be obtained from the study of a strictly stationary state.

The finite value of α produces changes in the probability distributions of position, momentum and energy. Unfortunately, there is no procedure available for making calculations in QED other than perturbation theory, so that we must limit ourselves to making comparisons in the first orders in α. In principle, the whole probability distributions can be calculated, but we will con-

sider only the mean values of $\boldsymbol{r}^2$, $\boldsymbol{p}^2$ and E. The first-order correction to $\langle \boldsymbol{p}^2 \rangle$ is given, both in QED and SED, by

$$\Delta\langle \boldsymbol{p}^2 \rangle = -\hbar^2 \omega_0^2 \alpha / \pi c^2 . \tag{6.5}$$

The SED result is derived easily from eq. (5.19); for the QED calculation see Appendix B. The leading correction to $\langle \boldsymbol{r}^2 \rangle$ in SED is a logarithmic term, which shows that the quantity $\langle \boldsymbol{r}^2 \rangle$ is not analytic in α. Up to terms of order α, the correction is, from eq. (5.18)

$$\Delta\langle \boldsymbol{r}^2 \rangle = (2\hbar^2 \alpha / \pi m^2 c^2)[\log (3mc^3/2\hbar\omega_0 \alpha) - \tfrac{1}{2}] . \tag{6.6}$$

In QED, the first-order correction is divergent; this may be interpreted as indicating that the quantity $\langle \boldsymbol{r}^2 \rangle$ is also nonanalytic in α. If we introduce a cut-off $\omega_{\max}$ in the frequencies, the first-order correction is (see Appendix B for the calculations)

$$\Delta\langle \boldsymbol{r}^2 \rangle = (2\hbar^2 \alpha / \pi m^2 c^2)[\log (1 + \omega_{\max}/\omega_0) - \tfrac{1}{2}] . \tag{6.7}$$

It is remarkable that both the coefficient of the logarithmic term and the linear term are the same in SED and QED.

There is some uncertainty in the comparison of the corrections to the energy due to the fact that the definition of kinetic energy is ambiguous in SED. In fact, the definition eq. (3.8) is only correct in the limit $\alpha \to 0$ as we have indicated in Sect. **3**. If we accept this definition as valid up to first order of α, we obtain, from eqs. (6.5) and (6.6),

$$E = (\hbar^2 \omega_0^2 \alpha / \pi m c^2)[\log (3mc^3/2\hbar\omega_0 \alpha) - 1] . \tag{6.8}$$

This result was first derived by SOKOLOV and TUMANOV [10] from a theory in which the electron was assumed classical, but the radiation field was quantized. From a purely classical theory (*i.e.* from SED) this result was first obtained by BRAFFORT, SURDIN and TARONI [5].

In QED the energy is not just the sum of kinetic and potential energy because it is assumed that an additional interaction energy is present. Then, the correction to the energy is not obtained from eqs. (6.5) and (6.7) but must be calculated directly. The first-order correction is divergent and, introducing a cut-off, we have (see Appendix B)

$$\Delta E = (\hbar^2 \omega_0^2 \alpha / \pi m c^2) \log (1 + \omega_{\max}/\omega_0) . \tag{6.9}$$

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[10] A. A. SOKOLOV and V. M. TUMANOV: *Sov. Phys. JETP*, **30**, 802 (1956).

Although the agreement with eq. (6.8) is good, the linear term which appears in eq. (6.8) is not present in the QED result. It seems probable that this is because something equivalent to the interaction energy, which was considered in the derivation of eq. (6.9), is lacking in eq. (6.8).

7. – Excited states.

Up to now, we have obtained a particular solution of the complete equation (5.3) (although, of course, the explicit form of the function $\boldsymbol{r}(t)$ cannot be found). The homogeneous equation has the solution

$$\boldsymbol{r}(t) = (\boldsymbol{A} \sin bt + \boldsymbol{B} \cos bt) \exp[-at] + \boldsymbol{C} \exp[ct], \tag{7.1}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are arbitrary constants and $a + bi$, $a - bi$, c are the roots of the characteristic equation

$$c^2 + \omega_0^2 - \tau c^3 = 0\,. \tag{7.2}$$

One of these roots is real and positive and it gives rise to the last term of eq. (7.1). This term is absurd because it implies for the oscillator an always rising energy (see the comment just before eq. (2.6)), so that we take $\boldsymbol{C} = 0$. The complex roots, calculated up to first order in τ (or γ), are

$$a \pm ib = -\tau\omega_0^2/2 \pm i\omega_0 \equiv -\gamma\omega_0/2 \pm i\omega_0\,, \tag{7.3}$$

and eq. (7.1) becomes

$$\boldsymbol{r}(t) = (\boldsymbol{r}_1 \sin \omega_0 t + \boldsymbol{r}_2 \cos \omega_0 t) \exp[-\gamma\omega_0 t/2]\,. \tag{7.4}$$

Once we have obtained the general solution of the homogeneous part of eq. (5.3), we can determine the general solution of the complete equation by adding the particular solution corresponding to the ground state of the oscillator. The momentum associated with the motion represented by eq. (7.4) is

$$\boldsymbol{p} = m\dot{\boldsymbol{r}} \simeq m\omega_0(\boldsymbol{r}_1 \cos \omega_0 t - \boldsymbol{r}_2 \sin \omega_0 t) \exp[-\gamma\omega_0 t/2]\,, \tag{7.5}$$

and the energy

$$E = (m/2)(\omega_0^2 \boldsymbol{r}^2 + \dot{\boldsymbol{r}}^2) = (m\omega_0^2/2)(\boldsymbol{r}_1^2 + \boldsymbol{r}_2^2) \exp[-\gamma\omega_0 t]\,, \tag{7.6}$$

where we have neglected higher-order terms in γ. We see that the co-ordinate and momentum of the oscillator at any time is the sum of a systematic part (given by eqs. (7.5) and (7.6)) and a stochastic part (with a probability distri-

bution given by eq. (5.24)). Thus, although we cannot determine the actual path of the particle we are able to calculate the probability distribution of its co-ordinate and momentum at any time. This is obtained by combining eq. (5.24) with eqs. (7.4) and (7.5) and the following results:

$$\varrho(r, p, t) = (2\pi\hbar)^{-\frac{3}{2}} \exp\left[-(m\omega_0/2\hbar)(\boldsymbol{r} - \boldsymbol{r}_1(t)\cos\omega_0 t - \boldsymbol{r}_2(t)\sin\omega_0 t)^2\right] \times \exp\left[(\boldsymbol{p} + m\omega_0 \boldsymbol{r}_1(t)\sin\omega_0 t - m\omega_0 \boldsymbol{r}_2(t)\cos\omega_0 t)^2/(2\hbar m\omega_0)\right]. \tag{7.7}$$

In this expression the functions $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ depend on time in the form

$$\boldsymbol{r}(t) = \boldsymbol{r}(0)\exp\left[-\gamma\omega_0 t/2\right]. \tag{7.8}$$

In zeroth order of α (SM), the functions $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ become constant and eq. (7.7) gives the same probability distributions of the co-ordinate and the momentum which gives QM for the coherent states of the oscillator. These states are represented by time-dependent minimum uncertainty wave packets and they form a nonorthogonal overcomplete system in the Hilbert space of the states of the oscillator ([11]). The probability distribution eq. (7.7) shows the analogy betewen SM and QM. In both theories there are an infinity of different states characterized by the values of the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. These states can be represented by « packets » whose centre moves according to classical mechanics. In QM the packets are considered of wave character. In SM they are the obvious representation of our ignorance of the actual position and momentum of the particle, this ignorance being inherent to the stochastic nature of the theory.

We have studied only the states given by eq. (7.7), but it is clear that any probability distribution $\varrho(\boldsymbol{r}, \boldsymbol{p}, t)$ obtained by linear combination (with real and positive coefficients) of several distributions of the type (7.7) can represent a state of our knowledge of the oscillator. These probability distributions will contain less information than that of eq. (7.7), which is the maximum one compatible with the stochastic character of the theory. These states may correspond to the mixed states of quantum statistical mechanics, while eq. (7.7) corresponds to a pure state of the quantum theory.

The absence of truly stationary excited states of the oscillator in SM shows that there is a striking difference between SM and QM. In SM only states represented by eq. (7.7) (or mixtures with real positive coefficients of these states) are possible. In particular, there are no states similar to the quantum-mechanical excited eigenstates of the Hamiltonian. Stated differently, in quantum mechanics any wave function of the form

$$\psi(\boldsymbol{r}, t) = \sum c_n \varphi_n(\boldsymbol{r}) \exp\left[-iE_n t/\hbar\right],$$

([11]) See, for example, S. STENHOLM: *Phys. Lett.*, **60**, 1 (1973) for a study of coherent states and its applications.

where $\varphi_n(\boldsymbol{r})$ are eigenfunctions of the Hamiltonian, represents a possible state. On the other hand, the physically realizable states of SM correspond only to some particular states of QM (the coherent states), *i.e.* to some sets of values of the c_n. This difference is very important because it shows that SM does not fulfil one of the basic postulates of quantum mechanics, the superposition principle. It is obvious that the interpretation of the experiments according to stochastic electrodynamics, if it is possible, needs to be radically different from the interpretation given by the quantum theory.

8. – Emission and absorption of radiation.

We have seen that the predictions of QM and SM are similar both for the ground and for the excited states of the oscillator, and also that the corrections due to the finite value of α are similar in SED and QED for the ground state. Now, we must compare the consequences of the finite value of α for the excited states. These are the phenomena of spontaneous emission and line breadth of the spectrum. We proceed first to the study of spontaneous emission.

In SED, the state characterized by eq. (7.7) is a decaying state as eq. (7.8) shows. The rate of energy decrease of the oscillator is given by eq. (7.6), which can be written

$$\mathrm{d}E/\mathrm{d}t = -\gamma\omega_0 E, \tag{8.1}$$

where E is the energy above the ground state. According to QED, the probability of decay per unit time from a state with quantum numbers n_1, n_2, n_3, is (see Appendix C)

$$\Gamma = (2\alpha\hbar\omega_0^2/3mc^2)(n_1 + n_2 + n_3) = \gamma E/\hbar, \tag{8.2}$$

where E is also the energy above the ground state. Now, taking into account that the energy change in the transition between two states is $\hbar\omega_0$, we have

$$\mathrm{d}E/\mathrm{d}t = -\Gamma\hbar\omega_0; \tag{8.3}$$

this shows that eq. (8.1) is also true in QED, although the interpretation is different from the one of SED. In QED, the perturbation techniques used in the calculations are not suitable for the study of continuous evolution and there are no other techniques available. The calculation of eq. (8.2) is typical of the way in which quantum field theoretical calculations are made. Probability transitions are calculated between quasi-stationary states and, although the time evolution is assumed to be continuous in principle, we deal always with quasi-stationary states and discontinuous jumps between them, in practice.

So, in quantum theoretical language, we speak of a collection of oscillators, each one in a quasi-stationary state, making discontinuous transitions between these states from time to time, in such a way that eq. (8.1) holds in the mean. Nevertheless, this picture is not unavoidable in quantum theory, but a consequence of the use of perturbation techniques.

In stochastic electrodynamics we have a quite different picture. Any oscillator is in a state which can be represented by a probability distribution of the type of eq. (7.7). Each one radiates continuously in such a way that eq. (8.1) holds individually. In this theory there is no room for the « quanta » of light. We note that we speak here only about the coherent radiation emitted by the oscillator. Besides this, the oscillator is continuously emitting and absorbing radiation to or from the background field. This is almost unobservable because it is rapidly fluctuating. In other words, the presence of the oscillator changes the background field without altering its stochastic parameters. Actually, the random absorption and emission of radiation give rise to some observable effects. For instance, if there are two oscillators, with the same characteristic frequency, some distance apart, the absorption and emission give rise to some correlation in its motion with the result that there is an effective attraction between them. It has been shown that this is just similar to the long-range van der Waals force predicted by QED ([12]).

The result eq. (8.1) has been obtained by MARSHALL ([13]), with a technique different from the one used here. MARSHALL studied, according to SED, the evolution of excited states analogous to the stationary excited states of quantum mechanics. Nevertheless, it seems difficult to take seriously such states in a purely classical theory because they have « probability » distributions in phase space which are not positive definite. Instead of introducing unphysical states in SED, we must study the time evolution of the coherent states in QED. In order to study quantitatively the time evolution of the coherent states of the oscillator in QED, we have used time-dependent perturbation theory. The change of the state of the oscillator has been calculated in a time interval which is large in comparison with the period of the oscillator, but short enough for perturbation theory to be valid. It is shown then that the evolution of a coherent state is represented by a minimum uncertainty wave packet whose centre moves according to the classical laws eqs. (7.4) and (7.5). The details of the calculation can be seen in Appendix C. The similarity between the time evolution of the coherent states in QED and SED is remarkable. In fact, the only difference in the evolution of the probability distributions of the co-ordinate and the momentum is that in SED the distribution is defined in phase space, but in QED there are two different distributions, one in co-ordinate space and the other in momentum space.

([12]) T. H. BOYER: *Phys. Rev. A*, **6**, 314 (1972).
([13]) T. W. MARSHALL: *Izvestiya VUZ, Fizika*, **12**, 34 (1968).

Besides the intensity of the spontaneously emitted radiation, it is possible to measure its spectrum (and maybe other properties such as the angular distribution). For simplicity, let us consider an oscillator in one dimension and the radiation emitted in a jump between eigenstates of the unperturbed Hamiltonian. The line width, being directly observable, must be independent of the approach used in the calculation, *i.e.* we have no need to consider coherent states in this case. According to QED the emission of radiation from a state with quantum number n carries the oscillator to the state with quantum number $n-1$. (The emission to any other state is forbidden.) The mean frequency of the emitted radiation is obviously ω_0 and the line shape can be calculated by means of the Weisskopf-Wigner theory. According to this theory, the line width is the sum of the inverses of the lifetimes of the initial and the final states. The inverse of the lifetime of a state is just eq. (8.2), so that the line width in the case considered here is

$$\Gamma = \Gamma_1 + \Gamma_2 = (2n-1)\gamma\omega_0 . \tag{8.4}$$

Then, the line shape is given by

$$I(\omega) \propto (n-\tfrac{1}{2})\gamma\omega_0/[(\omega-\omega_0)^2 + (n-\tfrac{1}{2})^2\gamma^2\omega_0^2] = \\ = (E\gamma/\hbar)/[(\omega-\omega_0)^2 + E^2\gamma^2/\hbar^2], \tag{8.5}$$

where E is half the sum of the energies of the final and the initial states above the ground state.

In SED, the excited states have a time dependence which is given by eqs. (7.7) and (7.8). The line shape of the emitted radiation can be obtained from the Fourier transform of eq. (7.4), which gives

$$I(\omega) \propto (\gamma\omega_0/2)/[(\omega-\omega_0)^2 + \gamma^2\omega_0^2/4] . \tag{8.6}$$

It is seen that this result only agrees with the QED result (eq. (8.5)) for the quantum excited state with $n=1$. The line width calculated with SED is smaller than the QED prediction for all other excited states. The above calculation of the line width in SED may be too naive because only the systematic part of the motion (eq. (7.4)) has been considered, neglecting the random part, which may be incorrect. In order to see whether the discrepancy between QED and SED is real, let us calculate the line width in SED from another point of view. We consider, instead of the spontaneous emission, a situation of equilibrium between the oscillator and the radiation. We assume that the background field has an additional amount of radiation besides the random one. Then, the spectrum of the electric field will be of the form (compare with eq. (2.2))

$$G_E(\omega) = (2\hbar/3\pi c^3)(|\omega|^3 + f(\omega)) , \tag{8.7}$$

where $f(\omega)$ is some function which we consider slowly varying (representing almost white light). From eqs. (8.7) and (5.3) we obtain instead of eq. (5.5)

$$G_x(\omega) = (\hbar\tau/\pi m)\big(|\omega|^3 + f(\omega)\big)/[(\omega^2 - \omega_0^2)^2 + \tau^2\omega^6]\,. \tag{8.8}$$

This shows that the interaction between the oscillator and the radiation is in agreement with eq. (8.5) for any function $f(\omega)$ which varies little in a frequency interval of the order $\gamma\omega_0$.

As a conclusion, there is a definite discrepancy between SED and QED in the prediction of the line width of the oscillator except for the first excited state. It is remarkable that, in this case, the quantum prediction cannot be correct because for high enough excitation energy the line width would increase indefinitely according to eq. (8.5). However, it is well known that for high n the quantum result must approach the classical one, which is zero width if the damping is neglected or just eq. (8.6) if the damping is taken into account. We have here an interesting case of inadequacy of the Weisskopf-Wigner theory (if not of QED).

9. – Discussion.

We have studied in some detail the harmonic oscillator according to stochastic electrodynamics. The predictions of this theory agree rather closely with those of quantum electrodynamics. In view of this result, it seems very improbable that the agreement is accidental. On the other hand, there are definite differences in some predictions besides sharp differences in the interpretation of some similar results. This means that both theories are experimentally differentiable, at least in principle. Nevertheless, the most easily testable results agree fairly well and, therefore, a crucial experiment seems difficult to perform. On the other hand, the simple system studied in this paper—the harmonic oscillator—is too singular to make clear whether stochastic electrodynamics has a real possibility to become a substitute of quantum theory. In any case, the convenience of further research on this line is strongly supported.

Appendix A

Calculation of integrals.

$$A(T/\tau) \equiv \int_0^\infty \frac{[1 - \cos(\omega T)]\,\mathrm{d}\omega}{\omega(1 + \tau^2\omega^2)} \equiv \int_0^\infty \frac{[1 - \cos(Tx/\tau)]\,\mathrm{d}x}{x(1 + x^2)}\,.$$

We are interested only in the behaviour of this integral for $T/\tau \to \infty$ and $T/\tau \to 0$. We write

$$A = A_1 + A_2\,, \qquad A_1 \equiv \int_0^{\sqrt{\tau/T}} \frac{[1-\cos(Tx/\tau)]\,\mathrm{d}x}{x(1+x^2)}\,, \qquad A_2 \equiv \int_{\sqrt{\tau/T}}^{\infty} \frac{[1-\cos(Tx/\tau)]\,\mathrm{d}x}{x(1+x^2)}\,.$$

For $T/\tau \ll 1$, it is possible to approximate $\cos(Tx/\tau)$ by $1 - T^2x^2/2\tau^2$ in A_1 and to neglect unity compared with x^2 in A_2. Then, it follows that

$$A \simeq \int_0^{\sqrt{\tau/T}} \frac{(T^2x^2/2\tau^2)\,\mathrm{d}x}{x(1+x^2)} + \int_{\sqrt{\tau/T}}^{\infty} \frac{[1-\cos(Tx/\tau)]\,\mathrm{d}x}{x^3} \simeq \frac{T^2}{2\tau^2}\log\frac{\tau}{T}\,, \qquad T/\tau \to 0\,.$$

For $T/\tau \gg 1$, it is possible to neglect x^2 compared with unity in A_1 and $\cos(Tx/\tau)$ in A_2, so that we have

$$A \simeq \int_0^{\sqrt{\tau/T}} \frac{[1-\cos(Tx/\tau)]\,\mathrm{d}x}{x} + \int_{\sqrt{\tau/T}}^{\infty} \frac{\mathrm{d}x}{x(1+x^2)} \simeq \tau\log\frac{T}{\tau}\,, \qquad T/\tau \to \infty\,,$$

$$B(\omega_0 T, \gamma) \equiv \int_0^{\infty} \frac{\tau\omega_0^3\omega\cos(\omega T)\,\mathrm{d}\omega}{(\omega^2-\omega_0^2)^2 + \tau^2\omega^6} \equiv \int_0^{\infty} \frac{\gamma x\cos(\omega_0 Tx)\,\mathrm{d}x}{(x^2-1)^2 + \gamma^2 x^6}\,.$$

This integral is convergent for all $\gamma \gg 0$ and all $\omega_0 T$. In practice $\gamma \ll 1$, so that we calculate the integral in powers of γ up to first order. The zeroth order gives

$$B_0(\omega_0 T) \equiv \lim_{\gamma\to 0} B(\omega_0 T, \gamma) = (\pi/2)\cos(\omega_0 T)\,.$$

The first-order contribution can be calculated as

$$B_1(\omega_0 T) \equiv \lim_{\gamma\to 0} \gamma^{-1} B(\omega_0 T, \gamma) - B_0(\omega_0 T) =$$

$$= \int_0^{\infty} \frac{4x\cos(\omega_0 T) - (x+1)^2\cos(\omega_0 T)}{4(x^2-1)^2}\,\mathrm{d}x - \frac{1}{4}\cos(\omega_0 T)\,,$$

where the following identity was used:

$$B_0(\omega_0 T) = \int_{-\infty}^{\infty} \frac{\gamma\cos(\omega_0 T)\,\mathrm{d}x}{4(x-1)^2+\gamma^2} = \int_0^{\infty} \frac{\gamma\cos(\omega_0 T)\,\mathrm{d}x}{4(x-1)^2+\gamma^2} + \frac{\gamma}{4}\cos(\omega_0 T) + o(\gamma)\,.$$

The remaining integration is straightforward and we obtain

$$B(\omega_0 T, \gamma) = (\pi/2)\cos(\omega_0 T) - \\ - (\gamma/2)[1 + \omega_0 T \cos(\omega_0 T)\sinh(\omega_0 T) - \omega_0 T \sin(\omega_0 T)\cosh(\omega_0 T)] + o(\gamma) .$$

The integral $B(0, \gamma)$ can be calculated as the limit of $B(\omega_0 T, \gamma)$ for $T \to 0$, which gives

$$B(0, \gamma) = (\pi - \gamma)/2 + o(\gamma) ,$$

$$C(\omega_0 T, \gamma) = \int_0^\infty \frac{\tau\omega_0\omega^3 \cos(\omega T)\,\mathrm{d}x}{(\omega^2 - \omega_0^2)^2 + \tau^2\omega^6} = \int_0^\infty \frac{\gamma x^3 \cos(\omega_0 T x)\,\mathrm{d}x}{(x^2-1)^2 + \gamma^2 x^6} .$$

This integral can be easily evaluated by taking the second derivative of $B(\omega_0 T, \gamma)$ with respect to T. Hence, it follows that

$$C(\omega_0 T, \gamma) = (\pi/2)\cos(\omega_0 T) - (\gamma/2)[1 + 2\cos(\omega_0 T)\cosh(\omega_0 T) + 2\sin(\omega_0 t)\cdot \\ \cdot \sinh(\omega_0 T) - \omega_0 T \sin(\omega_0 T)\cosh(\omega_0 T) + \omega_0 T \cos(\omega_0 T)\sinh(\omega_0 T)] + o(\gamma) .$$

This expression does not have a finite limit for $T \to 0$, so that the integral $C(0, \gamma)$ must be evaluated directly. We use the identity

$$C(0, \gamma) = \int_0^\infty \frac{\gamma x^3 \mathrm{d}x}{(x^2-1)^2 + \gamma^2 x^6} - \int_0^\infty \frac{\gamma\,\mathrm{d}x}{4(x-1)^2 + \gamma^2} + \frac{\pi}{2} - \frac{1}{2}\operatorname{arc\,tg}(\gamma/2) .$$

Now, subtracting the integrals before performing the integration we have

$$C(0, \gamma) = \int_0^{\sqrt{1/\gamma}} \frac{\gamma(4x^3 - x^2 - 2x - 1)\,\mathrm{d}x}{4(x^2-1)^2} + \int_{\sqrt{1/\gamma}}^\infty \frac{\gamma\,\mathrm{d}x}{x + \gamma^2 x^3} + \frac{\pi}{2} - \frac{\gamma}{4} + o(\gamma) ,$$

where γ was neglected compared with $1/x$ in the first integral and unity was neglected compared with x in the second. Also, an integral giving a term of order $\gamma^{\frac{3}{2}}$ was not included. The remaining integrations are straightforward and we obtain

$$C(0, \gamma) = (\pi - \gamma)/2 + \gamma \log \gamma + o(\gamma) .$$

Appendix B

Ground state of the oscillator in quantum electrodynamics.

The Hamiltonian of an oscillator in nonrelativistic QED is

$$H = \sum_{\boldsymbol{k},\lambda} \omega_0 b^\dagger_{\boldsymbol{k},\lambda} b_{\boldsymbol{k},\lambda} + (\boldsymbol{p} - e\boldsymbol{A})^2/2m_0 + \tfrac{1}{2} m\omega_0 \boldsymbol{r}^2 , \tag{A.1}$$

where m_0 is the bare mass, e the charge and $m\omega_0^2$ the force constant of the oscillator, and we take $\hbar = c = 1$. The vector potential $\boldsymbol{A}$ is written, in the electric dipole approximation,

$$\boldsymbol{A} = \sum_{\boldsymbol{k},\lambda} \sqrt{2\pi/V\omega}\,(b_{\boldsymbol{k},\lambda} + b^{\dagger}_{\boldsymbol{k},\lambda})\,\boldsymbol{\varepsilon}_{\lambda}\,, \tag{A.2}$$

where $b^{\dagger}_{\boldsymbol{k},\lambda}$ ($b_{\boldsymbol{k},\lambda}$) is the creation (destruction) operator of photons with momentum $\boldsymbol{k}$ and polarization λ, $\boldsymbol{\varepsilon}_{\lambda}$ being the polarization vector of these photons; ω is the angular frequency of the photons (equal to $|\boldsymbol{k}|$ in our units) and V is the normalization volume. The bare mass can be related to the observable mass by means of [7]

$$m/m_0 = 1 + \mu(\alpha)\,, \qquad \mu(\alpha) = 4\alpha\Omega/3\pi m + o(\alpha)\,, \tag{A.3}$$

where Ω is a cut-off in high frequencies. The unperturbed Hamiltonian is

$$H_0 = H_{\mathrm{rad}} + \boldsymbol{p}^2/2m + \tfrac{1}{2}m\omega_0^2\boldsymbol{r}^2\,. \tag{A.4}$$

The perturbing Hamiltonian is best written by introducing the ladder operators $\boldsymbol{a}$, $\boldsymbol{a}^{\dagger}$ of the oscillator in the form

$$\boldsymbol{r} = (\boldsymbol{a}^{\dagger} + \boldsymbol{a})/\sqrt{2m\omega_0}\,, \qquad \boldsymbol{p} = i\sqrt{m\omega_0/2}(\boldsymbol{a}^{\dagger} - \boldsymbol{a})\,. \tag{A.5}$$

Hence, the perturbing Hamiltonian is

$$H_1 = -(\mu\omega_0/4)(\boldsymbol{a}^{\dagger} - \boldsymbol{a})^2 - (1+\mu)\sqrt{\alpha}\,\boldsymbol{p}\cdot\boldsymbol{A}/m + (1+\mu)\alpha\boldsymbol{A}^2/2m\,. \tag{A.6}$$

Let us calculate the correction to the ground-state energy of the oscillator up to first order in α. This correction has a first-order contribution in H_1 plus a second-order contrbution from the term $\boldsymbol{p}\cdot\boldsymbol{A}$, which is of order one-half in α. The energy correction results

$$\Delta E_1 = \mu\langle 0|\boldsymbol{p}^2/2m|0\rangle + (\alpha/m^2)\sum_{n} |\langle 0|\boldsymbol{p}\cdot\boldsymbol{A}|n\rangle|^2/(E_0 - E_n)\,. \tag{A.7}$$

In order to give contributions, the states $|n\rangle$ must be one-photon states with the oscillator in the first excited state. The term in $\boldsymbol{A}^2$ was neglected because it contributes the same constant energy to a free particle and to any state of the oscillator. A straightforward calculation gives

$$\Delta E_1 = (\alpha\omega_0^2/\pi m)\log(\Omega/\omega_0)\,. \tag{A.8}$$

The first-order correction in α to the ground-state vector of the oscillator is

$$|\psi_1\rangle = \sum_{n\neq 0} |n\rangle\langle n|H_1|0\rangle/(E_0 - E_n) + \\ + (\alpha/m^2)\sum_{l,n\neq 0} |n\rangle\langle n|\boldsymbol{p}\cdot\boldsymbol{A}|l\rangle\langle l|\boldsymbol{p}\cdot\boldsymbol{A}|0\rangle/(E_0 - E_l)(E_0 - E_n)\,. \tag{A.9}$$

With this correction, the ground-state vector is not normalized but it has the norm

$$(\langle 0| + \langle \psi_1|)(|0\rangle + |\psi_1\rangle) = 1 - (\alpha\omega_0^2/\pi m)[1 - \log(\Omega/\omega_0)] . \tag{A.10}$$

Now, we calculate the first-order correction to the expectation value of $H_{\text{osc}} \equiv H_0 - H_{\text{rad}}$, which is

$$\Delta\langle H_{\text{osc}}\rangle = \langle\psi_1|H_{\text{osc}}|\psi_1\rangle = (\alpha/m^2)\sum_{n\neq 0}\omega_0|\langle n|\boldsymbol{p}\cdot\boldsymbol{A}|0\rangle|^2/(E_0 - E_n)^2 = \tag{A.11}$$

$$= (\alpha\omega_0^2/\pi m)\log(1 + \Omega/\omega_0) - (\alpha\omega_0^2/\pi m)\,\Omega/(\omega_0 + \Omega) .$$

After this, we calculate the first-order correction to the expectation value of the operator M defined by

$$M = m\omega_0\boldsymbol{r}^2/2 - \boldsymbol{p}^2/2m = \omega_0(\boldsymbol{a}^{\dagger 2} + \boldsymbol{a}^2)/2 , \tag{A.12}$$

which is

$$\Delta\langle M\rangle = \langle 0|M|\psi_1\rangle = \omega_0\langle 0|a^2|\sum_{l,n\neq 0}|n\rangle\,[\mu\langle n|\boldsymbol{p}^2/2m|0\rangle]/(E_0 - E_n) + \tag{A.13}$$

$$+ (\alpha/m^2)\langle n|\boldsymbol{p}\cdot\boldsymbol{A}|l\rangle\langle l|\boldsymbol{p}\cdot\boldsymbol{A}|0\rangle/[(E_0 - E_l)(E_0 - E_n)] = (\alpha\omega_0^2/\pi m)\log(1 + \Omega/\omega_0) .$$

By addition and subtraction of eqs. (A.11) and (A.13) it is easy to obtain the first-order corrections to $\langle\boldsymbol{p}^2\rangle$ and $\langle\boldsymbol{r}^2\rangle$, which are given in eqs. (6.5) and (6.7).

APPENDIX C

Motion of coherent states of the quantum oscillator.

The excited states of the oscillator which are eigenstates of the unperturbed Hamiltonian can be represented by $|n_1 n_2 n_3\rangle$, where the three (integer) quantum numbers are associated to the three Cartesian co-ordinates. The probability per unit time for spontaneous emission of radiation from these states can be calculated by means of the *golden rule*. This gives

$$\Gamma = (4\omega_0^2\alpha/3)\sum_f |\langle f|\boldsymbol{r}|n_1 n_2 n_3\rangle|^2 = (2\alpha\omega_0^2/3m)(n_1 + n_2 + n_3) \tag{A.14}$$

(as in the preceding Appendix, we use units such that $\hbar = c = 1$).

It is a little more difficult to calculate the spontaneous emission from states which are not eigenstates of the unperturbed Hamiltonian. For a general state of the oscillator, the emission of radiation must be calculated by means

of time-dependent perturbation theory. This gives, in first order,

$$|\Psi_I(t)\rangle = \left(1 - i\int_0^t \mathrm{d}t' H_I(t')\right)|\Psi(0)\rangle\,, \tag{A.15}$$

where $|\Psi_I(t)\rangle$ represents the state of both the oscillator and the radiation field in the interaction representation. The vector $|\Psi(0)\rangle$ represents a general state of the oscillator and the vacuum of the field. After a time t, much larger than ω_0^{-1} but short enough for perturbation theory being valid, the amount of radiation emitted is

$$\langle\Psi_I(t)|H_{\mathrm{rad}}|\Psi_I(t)\rangle = (2\omega_0^2\alpha/3m)\,t\,\langle\Psi(0)|\boldsymbol{a}^\dagger\cdot\boldsymbol{a}|\Psi(0)\rangle = \gamma t\omega_0 E\,, \tag{A.16}$$

where E is the mean energy of the oscillator above the ground state. This formula, which can be derived from eq. (A.15) by straightforward calculation, is a generalization of the previous one.

Besides calculating the amount of radiation per unit time, it is interesting to follow more closely the time evolution of the quantum states in QED. In order to do this, we derive in the following an expression for the time dependence of the expectation value of any operator acting only on the degrees of freedom of the oscillator (*i.e.* commuting with the creation and destruction operators of photons). Let us consider that the oscillator is in a state $|\Psi(0)\rangle$ at time $t = 0$, the state at time t will be

$$|\Psi_I(t)\rangle = \left[1 - i\int_0^t \mathrm{d}t' H_I(t') - \int_0^t \mathrm{d}t' H_I(t')\int_0^{t'} \mathrm{d}t'' H_I(t'')\right]|\Psi(0)\rangle\,. \tag{A.17}$$

It is necessary to use second-order perturbation theory to obtain the state vector $|\Psi_I(t)\rangle$, correct up to first order in α. This is because the term $\boldsymbol{p}\cdot\boldsymbol{A}$ is of order one-half in α. In eq. (A.17), however, it was enough to have $|\Psi_I(t)\rangle$ calculated up to first order in perturbation theory because second order does not contribute to one-photon states and the two-photon states give a contribution of second order in α. If M is any time-independent operator (not necessarily an observable) which acts only on the degrees of freedom of the oscillator we have

$$\begin{aligned}\langle\Psi_I(t)|M|\Psi_I(t)\rangle = {} & \langle\Psi(0)|M|\Psi(0)\rangle + \\ & + (\omega_0^2\alpha t/3m)\langle\Psi(0)|(2\boldsymbol{a}^\dagger\cdot\boldsymbol{M}\boldsymbol{a} - \boldsymbol{a}^\dagger\cdot\boldsymbol{a}\boldsymbol{M} - \boldsymbol{M}\boldsymbol{a}^\dagger\cdot\boldsymbol{a})|\Psi(0)\rangle\,.\end{aligned} \tag{A.18}$$

This equation can be obtained from eq. (A.17) by a straightforward although lenghty calculation. It is to be noted that the equation is not valid for too short time intervals because the following replacement has being made in deriving it:

$$|1 - \exp[i(\omega - \omega_0)t]|/(\omega - \omega_0)^2 \simeq \pi\delta(\omega - \omega_0)\,. \tag{A.19}$$

This prevents us from deriving an equation like eq. (A.18) directly from the Heisenberg equation of motion.

We are now ready to study the evolution of the coherent states of the oscillator according to QED. The coherent states ([11]) are eigenstates of the ladder operator $\boldsymbol{a}$ defined in eq. (A.5). As the spectrum is nondegenerate, each coherent state can be labelled by its eigenvalue. In one dimension, the eigenvalue can be any complex number, and the following relations hold:

$$a|z\rangle = z|z\rangle\,, \qquad |z\rangle = \exp\left[-|z|^2/2\right] \sum_{n=0}^{\infty} \left(z^n/\sqrt{n!}\right)|n\rangle\,. \tag{A.20}$$

To study the change in energy with time, we put the oscillator Hamiltonian (see just before eq. (A.11)) instead of M in eq. (A.18) and we obtain

$$\langle H_{\mathrm{osc}}\rangle_t \equiv \langle \Psi_I(t)|H_{\mathrm{osc}}|\Psi_I(t)\rangle = (1 - 2\omega_0^2\alpha t/3m)\,E \simeq E\exp\left[-2\omega_0^2\alpha E t/3m\right], \tag{A.21}$$

where E is the energy above the ground state. This means that the energy decreases exponentially. Now, we choose M to be a power of a component a_x of the ladder operator, and we have

$$\langle \Psi_I(t)|a_x^n|\Psi_I(t)\rangle = z^n(1 - n\gamma t) \simeq z^n \exp\left[-n\gamma t\right]. \tag{A.22}$$

In particular, this shows that the coherent states remain eigenstates of the ladder operator, *i.e.* they remain coherent states. In fact, from eqs. (A.21) and (A.22) it follows that

$$\langle \Psi_I(t)|\boldsymbol{a}^\dagger \cdot \boldsymbol{a}|\Psi_I(t)\rangle = |\langle \Psi_I(t)|\boldsymbol{a}|\Psi_I(t)\rangle|^2, \tag{A.23}$$

which proves that $|\Psi_I(t)\rangle$ is an eigenstate of $\boldsymbol{a}$.

The evolution of the coherent states can be seen more intuitively going from the state vector in the interaction representation to the Schrödinger state vector by the usual relation

$$|\Psi_I(t)\rangle = \exp\left[iH_0 t\right]|\Psi_S(t)\rangle\,.$$

This is given by

$$\langle \Psi_S(t)|a|\Psi_S(t)\rangle \equiv \langle a\rangle_t = z\exp\left[-i\omega_0 t - \gamma t\right]. \tag{A.24}$$

We remember that the wave function of a coherent state can be represented by a minimum uncertainty wave packet whose centre is given by

$$x_0 = \sqrt{2/m\omega_0}\,\mathrm{Re}\,z\,, \qquad p_0 = \sqrt{2m\omega_0}\,\mathrm{Im}\,z\,. \tag{A.25}$$

This shows the analogy between the motion of the coherent states and eq. (7.7).

● RIASSUNTO (*)

L'elettrodinamica classica, insieme con l'ipotesi di esistenza di una radiazione universale di fondo, invariante rispetto a trasformazioni di Lorentz (elettrodinamica stocastica), è stata proposta come possibile alternativa all'elettrodinamica quantistica. Si ottengono secondo questa teoria le equazioni stocastiche del moto di una particella carica e si confrontano con quelle del moto browniano. Si considera uno sviluppo delle equazioni in funzione delle potenze della costante di struttura fine α. Si studia l'oscillatore armonico e si trova che esso esegue un moto armonico semplice notevolmente stabile in fase. L'ampiezza varia lentamente ed a caso. Si calcolano i valori medi dell'energia cinetica e dell'energia potenziale e si trova che essi concordano molto bene con i risultati dell'elettrodinamica quantistica fino al primo ordine in α. Si dimostra l'esistenza di stati eccitati che risultano molto simili agli stati coerenti dell'oscillatore quantistico. Il risultato del calcolo del rapporto di emissione spontanea coincide con quello fornito dall'elettrodinamica quantistica, ma le ampiezze delle linee non coincidono. Si forniscono argomenti che dimostrano che le ampiezze quantistiche delle linee, calcolate secondo la teoria di Weisskopf-Wigner, non possono essere corrette nel caso dell'oscillatore. Si ottiene inoltre, secondo l'elettrodinamica quantistica, un'espressione generale dell'evoluzione del valore atteso di qualsiasi osservabile dell'oscillatore.

(*) *Traduzione a cura della Redazione.*

Гармонический осциллятор в стохастической электродинамике.

Резюме (*). — Как возможная альтернатива квантовой электродинамнке, предлагается классическая электродинамика с гипотезой универсального, Лоренц-инвариантного, фонового излучения (стохастическая электродинамика). В соответствии с этой теорией выводятся стохастические уравнения движения заряженной частицы. Эти уравнения сравниваются с уравнениями броуновского движения. Рассматривается разложение этих уравнений по степеням постоянной тонкой структуры α. Исследуется гармонический осциллятор с тем результатом, что осциллятор совершает простое гармоническое движение, очень устойчивое по фазе. Амплитуда изменяется медленно и хаотически. Вычисляются средние величины кинетической и потенциальной энергии. Получается довольно хорошее согласие с результатами квантовой электродинамики вплоть до первого порядка по α. Показывается существование возбужденных состояний, которые оказываются когерентными состояниями квантового осциллятора. Вычисленная интенсивность спонтанного излучения согласуется с результатом квантовой электродинамики, но ширина линии не согласуется. Приводятдя аргументы, которые показывают, что квантовая, ширина линии, вычисленная согласно теории Вайсскопфа-Вигнера, не может быть правильной в случае осциллятора. Также выводится общее выражение для эволюции ожидаемой величины для любой наблюдаемой осциллятора в квантовой электродинамике.

(*) *Переведено редакцией.*