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A Novel Approach to the Theory of Shot Noise.

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Summary. — The theory of shot effect is examined on the basis of an inhomogeneous Poisson process. The «Poisson» parameter $\lambda(t)$ characterising the stochastic process is generalized in such a manner that λ itself becomes a random variable depending on the number and the position of events on the time axis. It is found that the number density of arrivals of electrons in the theory of shot effect has exactly the same behaviour as the process under consideration. Such a process is strongly non-Markovian and the calculation of moments and correlation functions of the output turns out to be difficult. However it is shown that a knowledge of the moments of and the correlation of events on the t-axis is sufficient to determine these functions. The conjecture of Rowland regarding the behaviour of the mean square of the cumulative response for shot effect is proved and in addition an explicit expression for the power spectrum of the response is derived. Other physical phenomena which can be explained on the basis of the stochastic model are cited.

1. – Introduction.

The probability theory of shot effect (1) originally proposed by CAMPBELL (2) and subsequently developed by RICE(3) gives an adequate description of the noise current when the system has attained stationarity. The generalization to the nonstationary case has been made by MOYAL (4) who has discussed the

(1) W. SCHOTTKY *Ann. Phys. (Leipzig)*, **57**, 541 (1918).
(2) N. CAMPBELL: *Proc. Camb. Phil. Soc.*, **15**, 117 (1909).
(3) S. O. RICE: *Bell. Sys. Tech. Journ.*, **23**, 282 (1944).
(4) J. E. MOYAL: *Journ. Roy. Statist. Soc.*, B **11**, 150 (1949).

voltage fluctuations arising from shot noise. The basic idea in all these treatments of shot noise has been the recognition of the experimentally observed phenomenon that the shot effect consists of a series of pulses of current (1 electron = 1 pulse) and that one can only observe the response of the circuit to the current pulses and not the pulses themselves. If $\Phi(t)$ (as a function of time) is the response to a single pulse at a time t after the occurrence of the pulse (at $t=0$) we are interested in the cumulative response given by

$$r(t) = \sum_i \Phi(t-t_i)H(t-t_i)\,, \tag{1.1}$$

where t_i is the time of occurrence of the i-th pulse.

For historical reasons, let us consider the voltage fluctuations in an anode circuit. In particular if the anode circuit has an inductance L as well as a resistance R and capacity C, the voltage $V(t)$ due to the arrival of an electron at time $t=0$ is given by (see for example ROWLAND [5])

$$\begin{cases} V(t) = \frac{1}{2}(-\varepsilon/C)\big[(1+R/2i\omega L)\exp[-(R/L-i\omega)t] + \\ \qquad\qquad + (1-R/2i\omega L)\exp[-(R/L+i\omega)t]\big] & t>0\,, \\ \quad = 0 & t<0\,, \end{cases} \tag{1.2}$$

where ω is given by

$$\omega = (1/CL - R^2/4L^2)^{\frac{1}{2}}\,. \tag{1.3}$$

Defining the Fourier transform of $V(t)$ as

$$W(\nu) = (2\pi)^{-\frac{1}{2}}\int_0^\infty V(t)\exp[i\nu t]\,\mathrm{d}t\,, \tag{1.4}$$

we notice that the Fourier transform of the response function $\varphi(t)$ is nothing but $W(\nu)G(\nu)$ where $G(\nu)$ is the amplification factor set for the frequency $\nu/2\pi$.

Thus a study of $r(t)$ or rather the probability distribution of $r(t)$ leads us towards a good understanding of the physical phenomenon. Hence the earliest attempts were directed towards the calculation of the first two moments of $r(t)$. CAMPBELL [2] proved that in the case of a stationary system, the mean

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[5] E. N. ROWLAND: *Proc. Camb. Phil. Soc.*, **32**, 580 (1936).

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and fluctuation about the mean of $r(t)$ are given by (*)

$$\varepsilon\{r(t)\} = \lambda\int_0^\infty \Phi(t')\,\mathrm{d}t', \tag{1.5}$$

$$\{\varepsilon[r(t) - \mathscr{E}\{r(t)\}]^2\} = \lambda\int_0^\infty [\Phi(t')]^2\,\mathrm{d}t', \tag{1.6}$$

where λ is the mean density of arrivals of electrons at the anode. CAMPBELL assumed, as is reasonably to be expected, that the number of electron arrivals at the anode is governed by a Poisson law which states that the probability that there are n arrivals in a time interval $(0, t)$ is $\pi(n, t)$ where

$$\pi(n, t) = \exp[-\lambda t](\lambda t)^n/n!\,. \tag{1.7}$$

We observe that while (1.5) and (1.6) are true only for the stationary system, results can be generalized with equal facility to the nonstationary case (when t is finite) (see for example refs. (4) and (6)). However (1.5) and (1.6) do not explain the shot effect completely when we take into account the limitations due to space-charge effect. In fact HULL and WILLIAMS (7) made intensive measurements of the shot voltage soon after CAMPBELL proposed his formulae and found that the measured shot voltage fell below even 40 percent of the theoretical value. Meanwhile JOHNSON (8) who made some measurements of shot voltage in space-charge limited tubes, pointed out that the expected value of the shot voltage should be calculated as for temperature limited currents, by assuming an internal resistance of the valve. Based on these experimental findings as well as the investigations of MOULLIN (9), ROWLAND (10) formulated the problem in a precise form. He assumed that electrons might with specific probabilities, have lives of any length on the anode system during which time each of them add $(-\epsilon/C)\exp[-(t-t_i)/RC]$ to the anode potential of the valve whose anode to earth capacity is C and feed resistance, R. In addition, there is an effective internal resistance ϱ of the valve, defined by assuming that a variation of the anode potential causes a variation $1/\epsilon\varrho$ times

(*) Throughout this paper we shall use the symbol ε to denote the expectation value of the quantity within the brackets.

(6) S. K. SRINIVASAN and P. M. MATHEWS: *Proc. Nat. Inst. Sci. (India)*, **22** A, 369 (1956).

(7) A. W. HULL and N. H. WILLIAMS: *Phys. Rev.*, **25**, 147 (1925).

(8) J. B. JOHNSON: *Phys. Rev.*, **26**, 71 (1925).

(9) E. B. MOULLINS: *Proc. Roy. Soc.*, **147** A, 100 (1934).

(10) E. N. ROWLAND: *Proc. Camb. Phil. Soc.*, **33**, 344 (1937).

as great, in the probable density of arrival of electrons. Thus the arrival of each electron must decrease the probable density of further arrivals by an amount $(1/C\varrho)\exp[-(t-t_i)/CR]$. If N_0 is the density of arrivals at time $t=0$, then $N(t)$ the arrival density at any time $t>0$, is given by

$$\text{(1.8)} \qquad \begin{cases} N(t) = N_0 - \sum\limits_i \alpha(t-t_i)\,, & N_0 - \sum\limits_i \alpha(t-t_i) > 0\,, \\ \quad = 0 \text{ otherwise}, & \end{cases}$$

where

$$\text{(1.9)} \qquad \alpha(t-t_i) = (1/C\varrho)\exp[-(t-t_i)/CR\,.$$

With the modification of the density of arrivals as given by (1.8), it is clear that the probability that there are n arrivals in the interval $(0, t)$ is no longer given by a simple Poisson law as (1.7). The calculations of Rowland relating to the mean and mean square of the response were based on the behaviour of $N(t)$ as given by (1.8). The expressions for the mean and mean square values of stationary $r(t)$ in terms of an infinite series of integrals under the exponential were, though not fomidable, sufficiently complicated. These difficulties were in fact overcome by ROWLAND who obtained simple expressions for the first two moments of stationary $r(t)$. However, there were some limitations (*) in the final results obtained by ROWLAND. During the process of integration Rowland ignored the possibility of $N(t)$ dropping down to negative values and there was no mechanism of achieving it in any refined form of calculation of the integrals. In spite of all this, Rowland's results can be regarded as a decisive achievement in the theory of shot-effect for inclusion of all the realistic features of shot noise. In fact it is precisely for this reason, that a considerable part of the book of MOULLIN [13] is devoted to the presentation and discussion of Rowland's theory.

Twenty five years have passed since the first exciting theoretical attempts to explain the shot noise and the publication of Moullin's classical account [13] of voltage fluctuations. In spite of many new developments of probability theory, this aspect of the problem has not been given any attention. Even in the work of MOYAL [14] which explained many of the outstanding physical phenomena from a fairly rigorous probability point of view, we find only some

(*) These do not include the criticism of J. M. WHITTAKER [11] which has been squarely met with by Rowland himself [see ref. [12]].

[11] J. M. WHITTAKER: *Proc. Camb. Phil. Soc.*, **34**, 329 (1938).

[12] E. N. ROWLAND: *Proc. Camb. Phil. Soc.*, **34**, 329 (1938).

[13] E. B. MOULLINS: *Spontaneous Fluctuations of Voltage* (Oxford, 1938).

[14] M. S. BARTLETT: *Stochastic Processes* (Cambridge, 1955), p. 54.

generalizations of Campbell's theorem for the nonstationary case in terms of a Poisson law given by (1.7). It is the object of the present contribution to explain shot noise from the point of view of stochastic processes of continuous parametric systems.

Section **2** of the paper deals with the formulation of the stochastic process governing the number of arrivals of electrons at the anode and its connection with the response function, which is the main object of interest. In Sect. **3**, we shall discuss the probability frequency function and moments of the number density of electron arrivals. We then use these results to obtain the correlation of events on the time axis. The correlation and moments of the response function are dealt with in the final Section of the paper.

2. – Formulation of the problem.

At the outset, we wish to observe that the stochastic process governing the number of electron arrivals is of a more general nature and is capable of explaining other types of phenomenon as well. Hence we shall formulate the problem in general terms. Let us first consider a stochastic process of the inhomogeneous Poisson type with the parameter $\lambda(t)$ characterizing the process being nonnegative, continuous and bounded function of the parameter t with respect to which the process progresses. The probability that n events occur between 0 and t is given by

$$P(n, t) = \exp[-\Lambda(t)][\Lambda(t)]^n/n!\,, \qquad \Lambda(t) = \int_0^t \lambda(t')\,\mathrm{d}t'. \tag{2.1}$$

Such a type of Poisson process can be used to describe a number of physical processes like electron emission in a counter and age-dependent birth and death processes. In these processes, the parameter $\lambda(t)$ depends on t only and is independent of the number of events that have occurred prior to t and the position of the earlier events on the t-axis. Processes in which $\lambda(t)$ depends on the number of events that have occurred prior to t have received some attention (see BARTLETT [14]). However shot effect falls under the last category, since an electron arriving at the anode between t_1 and $t_1 + \mathrm{d}t_1$ diminishes the probability of any further arrival at a later time t_2 by $b\exp[-a(t_2 - t_1)]$ where a and b are some positive physical constants depending on the system. We shall assume that the process is switched on at $t = 0$ when the probability of the occurrence of an event in the infinitesimal interval $(0, \Delta)$ is $\lambda_0\Delta$ (λ_0 being a constant). Thus the first event happens between t_1 and $t_1 + \mathrm{d}t_1$ with probability $\exp[-\lambda_0 t_1]\lambda_0\,\mathrm{d}t$, while the probability of occurrence of the second event between t_2 and $t_2 + \mathrm{d}t_2$ is given by

$$(2.2)\qquad P_2(t_1, t_2)\,\mathrm{d}t_2 =$$
$$= \exp\left[-\int_{t_1}^{t_2}[\lambda_0 - b\exp[-a(t'-t_1)]\,\mathrm{d}t'\right]\cdot[\lambda_0 - b\exp[-a(t_2-t_1)]\,\mathrm{d}t_2](^*)\,.$$

Let us denote the parameter characterising the process by $\lambda(t)$ in analogy with the inhomogeneous Poisson process. The parameter $\lambda(t)$ *is no longer a* deterministic function of t but depends on the various random values of t at which the events have occurred. A typical realized value of $\lambda(t)$ corresponding to the events that have occurred at $t_1, t_2, \ldots, t_n$ is given by

$$(2.3)\qquad \lambda^R(t) = \lambda_0 - b\sum_{i=1}^{n}\exp[-a(t-t_i)]\,.$$

The probability measure corresponding to the above realized value can be calculated using (2.2).

For such a process, a number of questions can be raised. However, in view of the complexity of the problem arising from the non-Markovian behaviour of the process, we shall be content with the following:

i) the probability frequency function of λ and

ii) the correlation of events occurring on the t-axis.

In Sect. 4, we shall indicate how we can obtain some significant information regarding any process with the help of the moments of λ and the correlation of events.

Let $\pi(\lambda, t)$ (**) be the probability frequency function of $\lambda(t)$ so that $\pi(\lambda, t)\,\mathrm{d}\lambda$ denotes the probability that $\lambda(t)$ has a value between λ and $\lambda + \mathrm{d}\lambda$ at t. We proceed to obtain the Kolmogorov equation (see ref. ([15])) satisfied by $\pi(\lambda, t)$. Let us increase t by Δ between t and $t+\Delta$ either an event occurs or it does not occur. In the latter case, $\lambda(t)$ increases deterministically during the interval $(t, t+\Delta)$ as can be seen from (2.3), the rate of increase being given by

$$(2.4)\qquad \mathrm{d}\lambda/\mathrm{d}t = a(\lambda_0 - \lambda)\,.$$

(*) $P_2(t_1, t_2)$ is a conditional probability frequency function.

(**) Throughout this paper we shall use the symbol π to denote any probability frequency function, the distinction between two different probability frequency functions being apparent from the context.

([15]) W. FELLER: *An Introduction to Probability Theory and its Applications* (New York, 1957).

If on the other hand an event occurs between t and $t+\Delta$, λ suddenly diminishes by b. Using these results, we obtain

$$\pi(\lambda, t+\Delta)\,\mathrm{d}\lambda = (1-\lambda\Delta)\pi[\lambda - a(\lambda_0-\lambda)\Delta, t]d[\lambda-(\lambda_0-\lambda)a\Delta] + \\ + \pi(\lambda+b, t)(\lambda+b)\Delta\,\mathrm{d}\lambda + 0(\Delta)\,. \tag{2.5}$$

Proceeding to the limit as $\Delta \to 0$ we obtain

$$\frac{\partial\pi(\lambda,t)}{\partial t} = (a-\lambda)\pi(\lambda,t) - a(\lambda_0-\lambda)\frac{\partial\pi(\lambda,t)}{\partial\lambda} + (\lambda+b)\pi(\lambda+b,t)\,, \tag{2.6}$$

(2.6) is true only if $\lambda>0$. When $\lambda<0$ it is easy to see that $\pi(\lambda, t)$ satisfies the equation

$$\frac{\partial\pi(\lambda,t)}{\partial t} = a\pi(\lambda,t) - a(\lambda_0-\lambda)\frac{\partial\pi(\lambda,t)}{\partial\lambda} + (\lambda+b)\pi(\lambda+b,t)\,. \tag{2.7}$$

However, in such a case λ cannot be a probability magnitude (*). This difficulty can be overcome by defining λ' by

$$\begin{cases} \lambda' = \lambda & \text{for} \quad \lambda>0\,, \\ = 0 & \text{otherwise}\,. \end{cases} \tag{2.8}$$

We observe that it is λ' that has probability significance and in any problem we have to deal with only the moment of λ'. It is indeed difficult to solve for $\pi(\lambda, t)$ explicitly from (2.5). However, it is possible to obtain the moments of λ'. Defining

$$p(n,t) = \int_{-b}^{\infty}\pi(\lambda,t)\,\lambda'\,\mathrm{d}\lambda\,, \tag{2.9}$$

we obtain

$$\frac{\partial p(n,t)}{\partial t} = -nap(n,t) + na\lambda_0 p(n-1,t) + \sum_{i=1}^{n}\binom{n}{i}p(n-i+1,t)(-b)^i\,, \tag{2.10}$$

with the conditions

$$p(0,t)=1\,, \qquad p(n,0)=\lambda_0^n\,. \tag{2.11}$$

(*) From (2.3) it is clear that there is a nonzero probability of λ taking a value in the range $(-b, 0)$. The author is very much thankful to Dr. K. J. SRIVASTAVA, for a discussion regarding this point.

The first few moments can be explicitly calculated:

$$(2.12)\qquad p(1,t) = a\lambda_0/(a+b) + b\lambda_0 \exp[-(a+b)t]/(a+b)\,.$$

$$(2.13)\qquad p(2,t) = b^2\lambda_0(2\lambda_0 - a - 2b)\exp[-2(a+b)t]/2(a+b)^2 + \\ + b\lambda_0(2a\lambda_0 + b^2)\exp[-(a+b)t]/(a+b)^2 + a\lambda_0(2a\lambda_0+b^2)/2(a+b)^2\,,$$

$$2.14)\qquad p(3,t) = \{\lambda_0^3 - (a\lambda_0+b^2)(2a\lambda_0+b^2)(3b+a)/2(a+b)^3 - \\ - b^2\lambda_0(2\lambda_0 - a - 2b)a\lambda_0 + b^2)/2(a+b)^2 + b^4\lambda_0/2(a+b)^2 + \\ + b^3a\lambda_0/3(a+b)^2\}\exp[-3(a+b)t] + 3b^2\lambda_0(2\lambda_0 - a - 2b)\cdot \\ \cdot(a\lambda_0 + b^2)\exp[-2(a+b)t]/2(a+b)^3 + \{3b\lambda_0(a\lambda_0+b^2)(2a\lambda_0+b^2)/2(a+b)^3 - \\ - b^4\lambda_0/2(a+b)^2\}\exp[-(a+b)t] - \frac{b^3\lambda_0 a}{3(a+b)^2} + \frac{a\lambda_0(a\lambda_0+b^2)(2a\lambda_0+b^2)}{2(a+b)^3}\,.$$

The moments have a simple form if we proceed to the limit as t tends to infinity. Defining $p(1)$, $p(2)$, $p(3)$ as the limit of $p(1,t)$, $p(2,t)$ and $p(3,t)$ respectively, we find

$$(2.15)\qquad p(1) = a\lambda_0/(a+b)\,,$$

$$(2.16)\qquad p(2) = [p(1)]^2 + b^2p(1)/2(a+b)\,,$$

$$(2.17)\qquad p(3) = [p(1)]^3 + b^2[p(1)]^2/2(a+b) - ab^2p(1)/3(a+b) + b^2p(2)/(a+b)\,.$$

In the special case $b=0$, we find

$$(2.18)\qquad p(n,t) = p(n) = \lambda_0^n\,,$$

(2.18) is consistent with the fact that the parameter λ is no longer random and that the value of λ at any time should be equal to its initial value λ_0.

3. – Correlation of events on the t-axis.

So far we have dealt with the probability distribution and the moments of the «Poisson» parameter $\lambda(t)$. However in order to give an adequate description of the process we need the correlation of events on the t-axis. The correlation of events is fully described by the product density functions of RAMAKRISHNAN [16] (see also ref. [17]). Let $f_m(t_1, t_2, \ldots, t_m)$ be the product

[16] A. RAMAKRISHNAN: *Proc. Camb. Phil. Soc.*, **46**, 595 (1950).
[17] S. K. SRINIVASAN and K. S. S. IYER: *Nuovo Cimento*, **33**, 273 (1964).

density of events of degree m in t-space. Then $f_m(t_1, t_2, \ldots, t_m)\,\mathrm{d}t_1\,\mathrm{d}t_2 \ldots \mathrm{d}t_m$ represents the joint probability that an event occurs between t_1 and $t_1 + \mathrm{d}t_1$ an event between t_2 and $t_2 + \mathrm{d}t_2, \ldots$ and an event between t_m and $t_m + \mathrm{d}t_m$ irrespective of the number of events occuring elsewhere. If we denote by $n(t)$ the number of events that have occurred between 0 and t the m-th moment of $n(t)$ is given by (see ref. (13))

$$(3.1)\qquad \varepsilon\{[n(t)]^m\} = \sum_{i=1}^{m} C_i^m \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \ldots \int_0^t \mathrm{d}t_i\, f_i(t_1, t_2, \ldots, t_i)\,,$$

where C_i^m are known co-efficients that have nothing to do with any particular process. Thus the moments of $n(t)$ can be obtained from the above correlations of events. In this Section we shall be concerned with the explicit evaluation of the correlations of the first few orders.

We note that in order to obtain $f_1(t)$ a knowledge of $\pi(\lambda, t)$ is necessary. Using elementary probability arguments, we find

$$(3.2)\qquad f_1(t)\,\mathrm{d}t = \int_0^\infty \pi(\lambda, t)\,\lambda\,\mathrm{d}\lambda\,\mathrm{d}t\,.$$

Thus we have

$$(3.3)\qquad f_1(t) = p(1, t)\,.$$

The mean number of events that have occurred between 0 and t is given by

$$(3.4)\qquad \varepsilon\{n(t)\} = \int_0^\infty f_1(t)\,\mathrm{d}t = a\lambda_0 t/(a+b) - b\lambda_0\big(1 - \exp[-(a+b)t]\big)/(a+b)^2\,.$$

To obtain the mean square of events, we must obtain $f_2(t_1, t_2)$. In view of the non-Markovian nature of the process, it is convenient to introduce the function $\pi(\lambda_2, t_2 | \lambda_1, t_1)$ where $\pi(\lambda_2, t_2 | \lambda_1, t_1)\,\mathrm{d}\lambda_2$ represents the probability that $\lambda(t)$ has a value between λ_2 and $\lambda_2 + \mathrm{d}\lambda_2$ at t_2 given that $\lambda(t)$ had a value λ_1 at t_1 and a value λ_0 initially. Then it is easy to find

$$(3.5)\qquad f_2(t_1, t_2) = \int_{\lambda_1}\int_{\lambda_2} \pi(\lambda_1, t_1)\,\lambda_1\,\mathrm{d}\lambda_1 \cdot \pi\big(\lambda_2, t_2 | \lambda_1 - b, t_1\big)\,\lambda_2\,\mathrm{d}\lambda_2 =$$

$$= \int_{\lambda_1} \varepsilon\{\lambda(t_2) | \lambda_1 - b, t_1\}\,\lambda_1 \pi(\lambda_1, t_1)\,\mathrm{d}\lambda_1\,,$$

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where $\varepsilon\{\lambda(t_2)\,|\,\lambda_1 - b,\ t_2\}$ represents the conditional moments of λ at t_2 given that λ had a value $\lambda_1 - b$ at t_1. Once we obtain this conditional moment, we can calculate $f_2(t_1, t_2)$ explicitly.

Using an argument similar to these in Sect. 2 we find that $\pi(\lambda_2, t_2\,|\,\lambda_1, t_1)$ satisfies the equation

$$\frac{\partial \pi(\lambda_2, t_2\,|\,\lambda_1, t_1)}{\partial t_2} = -a(\lambda_0 - \lambda_2)\frac{\partial \pi(\lambda_2, t_2\,|\,\lambda_1, t_1)}{\partial \lambda_2} + (a - \lambda_2)\pi(\lambda_2, t_2\,|\,\lambda_1, t_1) + (b + \lambda_2)\,\pi(\lambda_2 + b, t_2\,|\,\lambda_1, t_1)\,, \tag{3.6}$$

with the initial condition

$$\pi(\lambda_2, t_1\,|\,\lambda_1, t_1) = \delta(\lambda_2 - \lambda_1)\,. \tag{3.7}$$

(3.6) is very similar to (2.6) and hence cannot be solved explicitly. However we can obtain the conditional moments of λ and those are precisely the quantities that will be needed for the calculation of the correlation functions. Defining

$$p(n, t_2\,|\,\lambda_1, t_1) = \varepsilon\{[\lambda(t_2)]^n\,|\,\lambda_1, t_1\}\,, \tag{3.8}$$

we obtain

$$\frac{\partial p(n, t_2\,|\,\lambda_1, t_1)}{\partial t_2} = -nap(n_1, t_2\,|\,\lambda_1, t_1) + na\lambda_0 p(n-1, t_2\,|\,\lambda_1, t_1) + \sum_{i=1}^{n}\binom{n}{i} p(n-i+1, t_2\,|\,\lambda_1, t_1)\,(-b)^i\,. \tag{3.9}$$

The first few moments can be explicitly calculated. They are given by

$$p(1, t_2\,|\,\lambda_1, t_1) = a\lambda_0[1 - \exp[-(a+b)(t_2 - t_1)]]/(a+b) + \lambda_1 \exp[-(a+b)(t_2 - t_1)]\,, \tag{3.10}$$

$$\begin{aligned} p(2, t_2\,|\,\lambda_1, t_1) &= a\lambda_0(2a\lambda_0 + b^2)/2(a+b)^2 + \\ &+ [\lambda_1^2 - (2a\lambda_0 + b^2)(2\lambda_1 - a\lambda_0/(a+b))/2(a+b)]\exp[-2(a+b)t_2 - t_1) + \\ &+ (2a\lambda_0 + b^2)(\lambda_1 - a\lambda_0/(a+b))\exp[-(a+b)(t_2 - t_1)]/(a+b)\,. \end{aligned} \tag{3.11}$$

Thus from (3.10) and (3.5), we obtain

$$\begin{aligned} f_2(t_1, t_2) &= p(2, t_1)\exp[-(a+b)(t_2 - t_1)] + p(1, t_1)\cdot \\ &\cdot [a\lambda_0(1 - \exp[-(a+b)(t_2 - t_1)]/((a+b) - b\exp[-(a+b)(t_2 - t_1)]]\,. \end{aligned} \tag{3.12}$$

In an exactly similar way, we can obtain $f_3(t_1, t_2, t_3)$ using eqs. (3.10) and (3.11). $f_3(t_1, t_2, t_3)$ is given by

$$
\begin{aligned}
(3.13)\qquad & f_3(t_1, t_2, t_3) = p(3, t_1)\exp[-(a+b)(t_3-t_1+t_2-t_1)] + p(2, t_1)\cdot \\
& \cdot[p(1, t_3-t_2)\exp[-(a+b)(t_2-t_1)] - 2b\exp[-(a+b)(t_3-t_1+t_2-t_1)] + \\
& + (2a\lambda_0+b^2)\exp[-(a+b)(t_3-t_1)]\{1-\exp[-(a+b)(t_2-t_1)]\}/(a+b) - \\
& - (b+\lambda_0)\exp[-(a+b)(t_3-t_1)]] + p(1, t_1)[\{-(b+\lambda_0)p(1, t_3-t_2) + \\
& + (b^2-\lambda_0^2)\exp[-(a+b)(t_3-t_1)]\}\exp[-(a+b)(t_2-t_1)] + \\
& + p(1, t_3-t_2)p(1, t_2-t_1) - (b+\lambda_0)(2a\lambda_0+b^2)\exp[-(a+b)(t_3-t_1)]\cdot \\
& \cdot\{1-\exp[-(a+b)(t_2-t_1)]\}/(a+b) + p(2, t_2-t_1)\exp[-(a+b)(t_3-t_2)] + \\
& + (b+\lambda_0)^2\exp[-(a+b)(t_3-t_1)] - (b+\lambda_0)p(1, t_2-t_1)\exp[-(a+b)(t_3-t_2)]]\ .
\end{aligned}
$$

The mean square number of events is given by

$$
(3.14)\qquad \varepsilon\{[n(t)]^2\} = \varepsilon\{n(t)\} + 2\int_0^t \mathrm{d}t_1\int_{t_1}^{t_2} f_2(t_1, t_2)\,\mathrm{d}t_2\ .
$$

The factor 2 appears in the right-hand side of (3.14) since t_1 and t_2 run over the entire domain 0 to t and $f_2(t_1, t_2)$ as given by (3.12) is defined only for $t_2 > t_1$.

An interesting feature that emerges from (3.12) is the existence of the limit of $f_2(t_1, t_2)$ where both t_1 and t_2 tend to infinity in such a manner that $t_2 - t_1$ remains a constant τ. Thus we have

$$
(3.15)\qquad \lim f_2(t_1, t_2) = [a\lambda_0/(a+b)]^2 - a\lambda_0 b(2a+b)\exp[-(a+b)\tau]/2(a+b)^2\ .
$$

A similar expression for $f_3(t_1, t_2, t_3)$ can be obtained under the limit when each of the variables t_1, t_2, t_3 tend to infinity in such a way that $t_2 - t_1$ and $t_3 - t_2$ remain constants.

4. – Moments and correlations of the cumulative response.

As has been explained in Sect. **1**, if $\varphi(t)$ is the response to a single pulse, then $\varphi(t)$ is given by

$$
(4.1)\qquad \varphi(t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} W(x)G(x)\exp[itx]\,\mathrm{d}x\ .
$$

On the basis of (4.1) and (1.8), Rowland made the pioneer calculations for the mean square response. However the possibility of $N(t)$, the density of arrivals, taking negative values could not be circumvented. ROWLAND ([10]) first proved that

$$\lim_{\substack{t_1\to\infty \\ t_2\to\infty}} \ \lim_{t_2-t_1=\tau} f_2(t_1, t_2) = \bar{N}^2 - (\bar{N}/C\varrho)\exp\left[-\left[\tau\left(\frac{1}{CR}+\frac{1}{C\varrho}\right)\right]\right] \tag{4.2}$$

and in a subsequent note has pointed out that (4.2) is incorrect and should be replaced by

$$\lim f_2(t_1, t_2) = \bar{N}^2 - (2\bar{N}/2\varrho + R)\exp\left[-\left[\tau\left(\frac{1}{CR}+\frac{1}{C\varrho}\right)\right]\right]. \tag{4.3}$$

This was a conjecture based on the experiments of MOULLIN. (4.3) is identical with (3.15) if we notice that a and b are to be replaced by $1/C\varrho$ and $1/CR$, respectively. Thus the results of the previous Section fill up the gap anticipated by Rowland and provide a rigorous proof of his results. Moreover the method of approach explained in Sect. **2** and **3** wherein we have averaged over ensembles making use of the fact that probability magnitudes can never be negative removes once for all doubts regarding the correctness of the formula (4.3).

The results of the previous Section enable us to obtain the correlation of the response function at two different times after the system has attained stationarity. Towards this end, we notice that if $N(t)$ represents the number of pulses up to time t, then the cumulative response is given by

$$r(t) = \int_0^t \mathrm{d}N(\tau)\,\varphi(t-\tau)\,, \tag{4.4}$$

where $\varphi(t)$ is given by (4.1). The correlation of $r(t)$ is given by (see for example ref. ([17]))

$$\varepsilon\{r(t_1)r(t_2)\} = \int_0^{t_1}\int_0^{t_2} \varepsilon\{\mathrm{d}N(\tau_1)\,\mathrm{d}N(\tau_2)\}\,\varphi(t_1-\tau_1)\,\varphi(t_2-\tau_2)\,. \tag{4.5}$$

The integrand in (4.5) can be expressed in terms of the product densities of the pulses if we take into account the degeneracy arising from the overlapping of the intervals $\mathrm{d}\tau_1$ and $\mathrm{d}\tau_2$. Thus we can use the general formula for the correlation obtained by SRINIVASAN and VASUDEVAN ([18]) for the treatment

([18]) S. K. SRINIVASAN and R. VASUDEVAN: (to be published).

of Barkhausen noise

$$(4.6)\qquad \varepsilon\{r(t_1)r(t_2)\} = \int\limits_0^{\min(t_1t_2)} f_1(\tau_1)\,\varphi(t_1-\tau_1)\,\varphi(t_2-\tau_1)\,\mathrm{d}\tau_1 + \\ + \int\limits_0^{t_1}\int\limits_0^{t_2}\varphi(t_1-\tau_1)\,\varphi(t_2-\tau_2)\,f_2(\tau_1,\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,.$$

We are interested in the limiting form of the right hand side of (4.6) when t_1 and t_2 tend to infinity while t_2-t_1 remains fixed and is equal to C. It is shown in Appendix B of ref. ([18]) that the Fourier transform of the left-hand side of (4.6) which is only a function of the single argument $|t_2-t_1|$ can be expressed in terms of the Fourier transform of $\Phi(t)$ and of the limiting form of $f_2(t_1, t_2)$ (see (4.3)) which is again a function of the single argument t_2-t_1. Thus $r(\omega)$ the power spectrum of the response (*) (which is nothing but the Fourier transform of the correlation) is given by

$$(4.7)\qquad r(\omega) = (2\pi)^{-\frac{1}{2}}a\lambda_0/(a+b)\,|\varphi(\omega)|^2 + 2\pi(a\lambda_0/a+b)^2 RlR(\omega)\,|\varphi(\omega)|^2$$

where $R(\omega)$ is the Fourier transform of $f_2(t_1, t_2)$.

5. – Concluding remarks.

Finally we wish to make a few general remarks on the applicability of the stochastic process described in Sect. **2** to other physical situations. An important phenomenon which can be described by such a stochastic process is the Barkhausen noise. Recently interest has been evinced in the correlation and power spectrum of Barkhausen noise. The work of MAZZETTI ([19]) deserves special mention in this connection. In the following paper by SRINIVASAN and VASUDEVAN ([18]), the model of Mazzetti is improved on the basis of the present non-Markovian process. Another example of a stochastic process of this type is provided by the photon correlations in the recent experiments of HANBURY-BROWN and TWISS ([20]). In the treatment of photon correlations it is assumed that electron emission is essentially a Poisson process, the Poisson parameter being governed by some probability distribution function (see for

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(*) We use the same symbol φ to denote the function as well as its Fourier transform.

([19]) P. MAZZETTI: *Nuovo Cimento*, **25**, 1322 (1962); **31**, 88 (1964).

([20]) R. HANBURY-BROWN and R. Q. TWISS: *Phil. Mag.*, **45**, 663 (1954); *Proc. Roy. Soc.* (*London*), **242** A, 300 (1957); **243** A, 291 (1957).

example MANDEL [21], MANDEL, WOLF and SUDARSHAN [22]). There is ample scope for improving the usual formula for intensity correlations on the basis of the results obtained in this paper. Lastly some models of one-dimensional fluid binary mixtures (see for example ref. [23]) can be dealt with on the basis of the presents results. Since the interparticle potential is of the exponential type, the process will very well fit in with our mode of description. However in this case, the problem is more difficult since we have to obtain higher-order conditional correlation functions. Nevertheless the present formulation can be used to obtain the second-order correlation function and other results based on it.

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(21) L. MANDEL: *Proc. Phys. Soc.*, **72**, 1037 (1958).
(22) L. MANDEL, E. C. G. SUDARSHAN and E. WOLF: *Proc. Phys. Soc.*, **84**, 435 (1964).
(23) R. KIKUCHI: *Journ. Chem. Phys.*, **23**, 2327 (1955).

RIASSUNTO (*)

Si esamina la teoria dell'effetto shot sulla base di un processo di Poisson inomogeneo. Si generalizza il parametro di Poisson $\lambda(t)$, che caratterizza il processo stocastico, in modo che λ diventi una variabile casuale dipendente dal numero e dalla posizione degli eventi sull'asse dei tempi. Si trova che nella teoria dell'effetto shot la densità del numero degli arrivi di elettroni ha esattamente lo stesso comportamento del processo in esame. Tale processo è fortemente non markoviano e il calcolo dei momenti e delle funzioni di correlazione risulta difficile. Si mostra, tuttavia, che la conoscenza dei momenti e della correlazione degli eventi sull'asse dei tempi è sufficiente a determinare le funzioni cercate. Si dimostra esatta la congettura di Rowland riguardante il comportamento del valor quadratico medio della risposta cumulativa per l'effetto shot e si deriva inoltre un'espressione esplicita per lo spettro di potenza della risposta. Si citano altri fenomeni fisici che si possono spiegare sulla base del modello stocastico.

(*) *Traduzione a cura della Redazione.*