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# On the Solution of Certain Singular Integral Equations of Quantum Field Theory.

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**Summary.** — In the study of complex phenomena involving π-mesons (e.g. simple and double photoproduction, radiative scattering, double production, etc.) one encounters singular integral equations linking the matrix elements and phase-shifts. We give here a general method of resolution of these equations which furnishes the following results: *a*) the simplest type is integrable by quadratures; *b*) the general type is reducible to a Fredholm equation. We give also the solution of a system of coupled integral equations and of the particular one which occurs in the double production problem. This last case may be integrated by quadratures.

## 1. – Introduction.

We want to study singular integral equations of the following type

$$\varphi(x) = f(x) + \frac{1}{\pi}\int_1^\infty \frac{h^*(x')\varphi(x')}{x'-x-i\varepsilon}\,\mathrm{d}x' + \int K(x'x)\varphi(x')\,\mathrm{d}x' \,, \tag{1.1}$$

where $h(x)$ has the form $e^{i\delta}\sin\delta$ (real $\delta$) and $K(xx')$ is a regular kernel.

This type of equation frequently occurs in the static theory of Chew-Low-Wick ([1,2]) for instance when one studies processes initiated or ending by a π-nucleon system. Such equations are thus fundamental in the problems of simple ([1]) and double photoproduction, radiative scattering ([3]), production of

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([1]) G. F. Chew and F. E. Low: *Phys. Rev.*, **101**, 1579 (1956).

([2]) C. G. Wick: *Rev. Mod. Phys.*, **27**, 339 (1955).

([3]) B. Bosco: *Nuovo Cimento*, **5**, 1361 (1957).

mesons by meson [4,6]. However, we suppose that the interest of such a problem is not completely restricted to the static approximation.

Equation (1.1) is not of Fredholm type although analogous in form and, for instance, classical results such as Fredholm alternative are not applicable. Moreover, due to the kernel singularity, these equations do not lend themselves simply to approximation methods and numerical integration. This point does not seem to have been clearly stressed in literature and errors in the choice of approximation methods have been made concurrently (cf. ref. [6]). As the mathematical problem seemingly will keep its importance, we want to give to it in this article a form upon which the classical methods of approximation and the general theorems will be applicable. By the way, we shall be able to show in particular cases the existence of a solution and its non-uniqueness, and shall determine what conditions of a physical nature allow complete uniqueness. In fact the method can be generalized to a quite wide variety of problems which we shall indicate.

In Sect. **2** we give the general solution of the simplest equation of type (1.1). We study the uniqueness problem when certain conditions of physical nature are imposed. Sect. **3** deals with the reduction of a quite general case to Fredholm's one; the problem of the equivalence between these two forms is not treated. Sect. **4** generalizes this reduction to systems of coupled integral equations. Finally, we treat in Sect. **5** a particular system of coupled equations which is encountered in the problem of $\pi$-meson production by mesons [4,6]. This last case is seen to reduce to simple quadratures, just as the problem of Sect. **2**.

## 2. – Fundamental equation.

The simplest equation of type (1.1) is

$$\varphi(x) = f(x) + \frac{1}{\pi}\int_1^\infty \frac{h^*(x')\varphi(x')}{x'-x-i\varepsilon}\,\mathrm{d}x', \tag{2.1}$$

here $f(x)$ is a given function which we suppose bounded between 1 and $\infty$. $h(x)$ is a given function of the form $\exp[i\delta]\sin\delta$, where $\delta(x)$ is bounded. Whe choose arbitrarily the determination of $\delta$ which tends to 0 at infinity (supposing this choice possible) and $\delta(1)=k\pi$. $\delta$ tends to zero at least as quickly as $x^{-1}$ when $x$ tends to infinity. $\varphi(x)$ is the unknown function. In order to solve (2.1) we shall use a method which is a generalization of the

(4) J. Fukuda and J. S. Kovacs: *Phys. Rev.*, **104**, 1784 (1956).
(5) L. S. Rodberg: *Phys. Rev.*, **106**, 1090 (1957).
(6) R. Omnès: *Nuovo Cimento*, **6**, 780 (1957).

method introduced by N. I. MUSKHELISHVILI (7) for the solution of Hilbert problem in elasticity, the essential difference being that here the kernel contains $h(x)$ and is not a simple Cauchy kernel. We shall have to introduce functions of a variable $z$ defined in a complex plane cut along the interval $(1, \infty)$ and shall call $G(x+)$ (resp. $G(x-)$) the limit of a function $G(z)$ when $z$ tends to $x$ upon (resp. under) the cut. In the following, the cut will be called $L$.

Let us define the function

$$F(z) = \frac{1}{2\pi i}\int_L \frac{h^*(x')\varphi(x')}{x'-z}\,\mathrm{d}x', \tag{2.2}$$

which implies

$$\frac{1}{\pi}\int_L \frac{h^*(x')\varphi(x')}{x'-x-i\varepsilon}\,\mathrm{d}x' = 2iF(x+), \tag{2.3a}$$

$$\varphi(x) = h^{*-1}_{(x)}[F(x+)-F(x-)]. \tag{2.3b}$$

Equation (2.1) takes now the form

$$\exp[-2i\delta]F(x+) - F(x-) = f(x)h^*(x), \tag{2.4}$$

where we have used $1-2ih^* = \exp[-2i\delta]$. Let us now put

$$F(z) = \Phi(z)\Omega(z), \tag{2.5}$$

where the function $\Omega(z)$ is defined by the condition

$$\exp[-2i\delta]\Omega(x+) - \Omega(x-) = 0. \tag{2.6}$$

This last equation admits a solution (which is found by taking the logarithm of (2.6))

$$\Omega(z) = \exp[u(z)]$$

with

$$u(z) = \frac{1}{\pi}\int_L \frac{\delta(\zeta)}{\zeta - z}\,\mathrm{d}\zeta,$$

if we define

$$\varrho(x) = \frac{1}{\pi}P\int_1^\infty \frac{\delta(\zeta)}{\zeta - x}\,\mathrm{d}\zeta, \tag{2.7}$$

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(7) MUSKHELISHVILI: *Trud. Tbil. Mat. Inst.*, **10**, 1 (1941), cited in S. G. MIKHLIN: *Integral Equations* (London, 1957), p. 126 ff.

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we have

$$\Omega(x+) = \exp[\varrho + i\delta]; \quad \Omega(x-) = \exp[\varrho - i\delta]. \tag{2.8}$$

Equation (2.4) may now be transformed in a relation for $\Phi(z)$ which is of the Hilbert type

$$\Phi(x+) - \Phi(x-) = f(x)h^*(x)\Omega^{-1}(x-) = f(x)\sin\delta(x)\exp[-\varrho(x)], \tag{2.9}$$

which has a solution

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)\sin\delta(\zeta)\exp[-\varrho(\zeta)]}{\zeta - z}\,d\zeta: \tag{2.10}$$

one may now derive $\varphi(x)$ by (2.1) or (2.3*b*) which give evidently the same result

$$\varphi(x) = \left[f(x)\cos\delta(x) + \frac{1}{\pi}\exp[\varrho(x)]\cdot P\int_1^\infty \frac{f(\zeta)\sin\delta(\zeta)\exp[-\varrho(\zeta)]}{\zeta - x}\,d\zeta\right]\exp[i\delta(x)]. \tag{2.11}$$

It is an easy task to verify that every step of this method is correct, provided the written integrals converge and that (2.11) is truly a solution of (2.1).

It is important to point out that (2.11) is not the only solution of (2.1) for one may add to it any solution of the homogeneous equation

$$\varphi_0(x) = \frac{1}{\pi}\int_L \frac{h^*(x')\varphi(x')}{x' - x - i\varepsilon}\,dx', \tag{2.12}$$

such a solution of (2.12) may yet be defined by (2.3) and a relation analogous to (2.5)

$$F_0(z) = \Phi_0(z)\Omega(z), \tag{2.13}$$

where $\Phi_0(z)$ must now verify

$$\Phi_0(x+) - \Phi_0(x-) = 0. \tag{2.14}$$

This last relation shows that $\Phi_0(z)$ is an analytic function in the whole complex plane except eventually at the points 1 and $\infty$ where it may have singularities. If we exclude essential singularities, the general solution of (2.1) appears as

$$(2.15)\qquad \begin{cases} \varphi'(x) = \varphi(x) + L(x) \exp\left[\varrho(x) + i\,\delta(x)\right], \\ L(x) = \dfrac{P(x)}{(x-1)^n}, \end{cases}$$

where $P(x)$ and $n$ are arbitrary polynomial and integer. Generally a solution will be completely determined by its asymptotic behaviour in the neighbourhood of 1 and $\infty$ for this behaviour by (2.11) and (2.15) determines $n$ and $P(x)$. Particularly, for

*a*) $\qquad k = 0 \qquad$ or $\qquad \delta(1) = 0\,,$

*b*) $\qquad k = -1 \qquad$ or $\qquad \delta(1) = -\pi; \quad |\delta(\varepsilon) + \pi| \sim \varepsilon^\beta \qquad \beta > 1\,,$

(2.11) is the only solution of (2.1) regular in the neighbourhood of 1 and which tends to 0 at infinity (see appendix). These cases are precisely the more interesting in practical use since, in the context of Chew and Low model, case *a*) may be adopted for the small phase-shifts and case *b*) for $\delta_3$ with $\beta = \frac{3}{2}$. Let us recall here that we have taken the determination of $2\delta$ which is 0 at infinity. The choice $k = -1$ is equivalent to suppose only one resonance for the $(\frac{3}{2}\,\frac{3}{2})$ pion-nucleon state. Let us stress that hypotheses *a*) and *b*) simplify happily the calculations but are absolutely not essential to the success of this approach. If, effectively, *a*) and *b*) are not verified by $\delta$, it is an easy task to determine the singularities of the integrals appearing in (2.7) and (2.11) using the first terms of the Taylor expansion of $\delta$ near 1, and to choose in a unique fashion an integer $n$ and a polynomial $P(x)$ in (2.15) in order to have a regular solution in the neighbourhood of 1 and $\infty$. We shall consider this solution as the interesting one although we do not want to enter here in the difficult problem of the choice of physical criteria for a solution (see ref. ([8-10])).

Let us now remark that, for $f(x)$ real, the phase of $\varphi(x)$ is $\delta$ which is intimately connected with a theorem by FUBINI, NAMBU and WATAGHIN ([11]).

Finally, in the more general case where $h$ has not the form $e^{i\delta}\sin\delta$ our method is yet applicable in principle, the essential condition being now that $|1 - 2ih^*|$ must not be zero on $L$. One may yet write $1 - 2ih^* = \exp[-2i\gamma]$, where $\gamma$ is now complex. The method used by MUSKHELISHVILI ([7]) in the resolution of Hilbert's problem is apparently a particular case (*).

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([8]) L. CASTILLEJO, R. H. DALITZ and F. J. DYSON: *Phys. Rev.*, **101**, 453 (1956).

([9]) F. J. DYSON: *Phys. Rev.*, **106**, 157 (1957).

([10]) R. HAAG: *Nuovo Cimento*, **5**, 203 (1957).

([11]) S. FUBINI, Y. NAMBU and V. WATAGHIN: to be published.

(*) *Note added in proof:* This general case was known to MUSKHELISHVILI and treated in his book "*Singular Integral Equations*", Groningen, 1953; this was kindly pointed out to us by J. LASCOUX.

## 3. – Equation reducible to Fredholm's type.

Let us consider the equation

$$\varphi(x) = f(x) + \frac{1}{\pi}\int_1^\infty \left[\frac{h^*(x')}{x' - x - i\varepsilon} + K(x'x)\right]\varphi(x')\,\mathrm{d}x' , \tag{3.1}$$

where the kernel $K(xx')$ is regular. Including $\int K\varphi$ in the inhomogeneous part, one finds a solution analogous to (2.11) where $f(x)$ is now replaced by

$$f(x) + \frac{1}{\pi}\int K(x'x)\varphi(x')\,\mathrm{d}x' ,$$

if we suppose $\varphi(x)$ bounded and continuous and that $K(xx')$ verifies $K(xx') \to 0$ when $x \to \infty$, and is submitted to a Lipshitz condition, one may invert the order of integrations in

$$\left\{\begin{aligned} & P\int\frac{\lambda(\zeta)}{\zeta - x}\int K(x'x)\varphi(x')\,\mathrm{d}x' , \\ & \lambda(\zeta) = \sin\delta(\zeta)\exp[-\varrho(\zeta)] , \end{aligned}\right. \tag{3.2}$$

which leads to

$$\varphi(x) = \mu(x) + \frac{1}{\pi}\int_1^\infty N(x'x)\varphi(x')\,\mathrm{d}x' , \tag{3.3}$$

where

$$\mu(x) = \left[f(x)\cos\delta(x) + \frac{1}{\pi}\exp[\varrho(x)]P\int_1^\infty\frac{f(\zeta)\lambda(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\right]\exp[i\delta(x)] , \tag{3.4}$$

$$N(x'x) = \left[K(x'x)\cos\delta(x) + \exp[\varrho(x)]P\int\frac{K(x'\zeta)\lambda(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\right]\exp[i\delta(x)] . \tag{3.5}$$

Here we shall not try to determine in detail what are necessary conditions for $K$ and $h$ in order that equation (3.3) be of Fredholm type: we think it to be a matter of interest only in each particular case. The solution we have obtained must evidently be regular at 1 and $\infty$ and one may repeat here the arguments given in the preceding section.

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## 4. – The case of a system of coupled integral equations.

Let us consider the system

$$\varphi_i(x) = f_i(x) + \frac{1}{\pi}\int_1^\infty \left[\frac{h_i^*(x')\varphi_i(x')}{x'-x-i\varepsilon} + \sum_j K_{ij}(x'x)\varphi_j(x')\right] \mathrm{d}x' , \tag{4.1}$$

by the method of the Sect. **3**, this may be put eventually in Fredholm form by

$$\varphi_i(x) = \mu_i(x) + \frac{1}{\pi}\sum_j \int N_{ij}(x'x)\varphi_j(x')\,\mathrm{d}x' , \tag{4.2}$$

where

$$\left\{\begin{aligned} \mu_i(x) &= \left[f_i \cos\delta_i + \frac{1}{\pi}\exp[\varrho_i]P\int\frac{f_i(x')\lambda_i(x')}{x'-x}\,\mathrm{d}x'\right]\exp[i\,\delta_i(x)] , \\ N_{ij}(x'x) &= [K_{ij}(x'x)\cos\delta_i(x') + U_{ij}(x'x)]\exp[i\,\delta_i(x)] , \end{aligned}\right. \tag{4.3}$$

$$U_{ij}(x'x) = \exp[\varrho_i(x)]P\int\frac{K_{ij}(x'\zeta)\lambda_i(\zeta)}{\zeta - x}\,\mathrm{d}\zeta . \tag{4.4}$$

## 5. – Equations for the production of mesons by mesons.

When one studies the reaction nucleon$+\pi\to$nucleon$+2\pi$ by the methods of Chew-Low-Wick, one obtains equations of the following type for reduced matrix elements (*)

$$\varphi_i(xx_1x_2) = f_i(xx_1x_2) + \frac{1}{\pi}\int_1^\infty \frac{h_i^*(x')\varphi_i(x'x_1x_2)}{x'-x-i\varepsilon}\,\mathrm{d}x' + \sum_{j=1}^{8} A_{ij}C_{ij}(x_1x_2) , \qquad (i = 1, ..., 8); \tag{5.1}$$

where $A_{ij}$ are real numbers and

$$C_j(x_1x_2) = \frac{1}{\pi}\int\varphi_j^*(x'x_1x_2)h_j(x')\left[\frac{1}{x'-x_1-i\varepsilon} + \frac{1}{x'-x_2-i\varepsilon}\right] , \tag{5.2}$$

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(*) These equations reproduce equations (5.1) of reference [6] where we have put, in order to save writing $\varphi_i = T_{JL}$ and we have taken into account relation $\overline{T}_{JL} = (-)^{2J}T_{JL}$ which follows by time-reversal invariance.

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equation (5.1) is in fact a particular case of (2.1) and its solution is given by

$$\varphi_i(xx_1x_2) = \varphi_i^{(1)}(xx_1x_2) + \varphi_i^{(2)}(xx_1x_2)\,, \tag{5.3}$$

where

$$\varphi_i^{(1)}(xx_1x_2) = \left[f_i(xx_1x_2)\cos\delta_i(x) + \frac{1}{\pi}\exp\left[\varrho_i(x)\right]\cdot \right. \\ \left. \cdot P\int_1^\infty \frac{f_i(\zeta x_1x_2)\lambda_i(\zeta)}{\zeta - x}\,d\zeta\right]\exp\left[i\,\delta_i(x)\right]\,, \tag{5.4}$$

$$\varphi_i^{(2)}(xx_1x_2) = \sum_j A_{ij}C_j(x_1x_2)u_i(x)\,, \tag{5.5}$$

$$u_i(x) = \left[\cos\delta_i(x) + \frac{1}{\pi}\exp\left[\varrho_i(x)\right]P\int\frac{\lambda_i(\zeta)}{\zeta - x}\,d\zeta\right]\exp\left[i\,\delta_i(x)\right]\,, \tag{5.6}$$

if we define the operations $D_i[\psi(xx_1x_2)]$ which transforms a function $\psi$ of $(x, x_1, x_2)$ in a function of $x_1$ and $x_2$ only by

$$D_i[\psi] = \int\psi^*(x'x_1x_2)h_i(x')\left[\frac{1}{x' - x_1 - i\varepsilon} + \frac{1}{x' - x_2 - i\varepsilon}\right]dx'\,, \tag{5.7}$$

$$D_i[\varphi_i] = C_i(x_1x_2) \tag{5.8}$$

and apply it to (5.4), we obtain

$$C_i(x_1x_2) = D_i[\varphi_i^{(1)}] + \sum_j A_{ij}C_j^*D_i[u_i]\,. \tag{5.9}$$

In (5.9) $D_i[\varphi_i^{(1)}]$ and $D_i[u_i]$ may be explicitely calculated and one may easily solve (5.9) for the $C_j$ and bring them in (5.3–6). It is seen that the solution so obtained involves only quadratures, which is indeed an unexpected simple result.

## 6. – Conclusions.

We have given a method for the resolution of integral equations which present themselves in quantum field theory and, in particular, in the model of Chew-Low and Wick. Following the difficulties involved in the considered

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problem, one is led to an explicit solution by quadratures or to non-singular integral equations. The method may be generalized to a quite wide lot of other cases. An explicit and suggesting example is given by the equations for production of mesons.

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## APPENDIX

In this appendix, we want to study the convergence of integrals appearing in equations (2.8) and (2.11). In this respect, let us consider

$$I(x) = P\int_0^\infty \frac{F(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\,. \tag{A.1}$$

Here we have replaced the limit 1 by 0 in order to simplify developments and $F(\zeta)$ is a bounded and derivable function such that

$$F(\zeta) = \frac{A}{\zeta^\alpha} + O\left(\frac{1}{\zeta^\alpha}\right), \qquad \zeta > Z,\ \alpha > 0, \tag{A.2}$$

$$F(\zeta) = k + B\zeta^\beta + O(\zeta^\beta)\,, \qquad \zeta \sim 0,\ \beta > 0. \tag{A.3}$$

Let us study $I(x)$ for $x$ near to 0 and infinitely great.

a) *x great.*

Let us put

$$I = I_1 + I_2\, I_1 = \int_0^Z \frac{F(\zeta)}{\zeta - x}\,\mathrm{d}\zeta\,, \tag{A.4}$$

$$I_2 = P\int_Z^\infty \frac{F(\zeta)}{\zeta - x}\,\mathrm{d}\zeta,\,,$$

for $x \gg Z I_1$ is of order $\mathrm{const}/x + O(1/x)$. If we define $\alpha = n + \alpha'$ $(0 \leqslant \alpha' < 1)$

it comes

$$(A.5)\qquad I_2 = A\sum_{p=2}^{n}\frac{(-)^{n-p+1}}{x^{n-p+1}Z^{p+\alpha'-1}(p+\alpha'-1)} + \\ + (-)^{n+1}\frac{A}{x^n}P\int_Z^\infty\frac{1}{\zeta^{\alpha'}(\zeta-x)}\,d\zeta + P\int_Z^\infty\frac{O(1/\zeta^{\alpha})}{\zeta-x}\,d\zeta\,.$$

By the boundedness of the derivative the last integral is $O(x^{-1})$ and the preceding one is bounded by $Ax^{-n}\log(x-Z)Z^{-\alpha'}\log Z$. The results of Table I follow.

TABLE I.

| $\zeta, x$ | Cases | Order of $I(x)$ | Order of $\exp[I(x)]$ |
|---|---|---|---|
| $\infty$ | $\alpha > 1$ | $x^{-1}$ | 1 |
| | $\alpha = 1$ | $x^{-1}\log x$ | 1 |
| | $\alpha < 1$ | ? | ? |
| 0 | $k \neq 0$ | $-k\log x$ | $x^{-k}$ |
| | $k = 0$ | 1 | 1 |

b) *x near to* 0.

The method is analogous, one uses (A.3) and separates the parts of $I$ due to $k$, to $\zeta^\beta$ and $O(\zeta^\beta)$, which gives, term-by-term

$$I(x) = -k\log x + \text{const} + O(x)\,,$$

from which results of Table I follow.

In Table I, we give the asymptotic values of $I(x)$ and $\exp[I(x)]$. Table II

TABLE II.

| $\zeta, x$ | Cases | | Order of $\delta$ | Order of $e^{\varrho}$ | Order of $\nu$ | Order of $J$ |
|---|---|---|---|---|---|---|
| $\infty$ | $\alpha > 1$ | | $\zeta^{-\alpha}$ | 1 | $\zeta^{-\alpha}$ | 1 |
| | $\alpha = 1$ | | | 1 | | 1 |
| | $\alpha < 1$ | | | ? | | ? |
| 0 | $k \neq 0$ | $\beta < -k$ | 1 | $x^{-k}$ | $\zeta^{\beta+k}$ | ? |
| | | $\beta = -k$ | 1 | $x^{-k}$ | | $\log x$ |
| | | $\beta > -k$ | 1 | $x^{-k}$ | | 1 |
| | $k = 0$ | | $\zeta^{\beta}$ | 1 | | 1 |

is a direct application of the results of Table I to $\delta(\zeta)$ and $\nu(\zeta) = f(\zeta) \sin \delta(\zeta) \cdot \exp[-\varrho(\zeta)]$ where $f(\zeta)$ is bounded, and finally of $J_{(x)} = P\int \nu(\zeta)\, d\zeta/(\zeta - x)$.

One sees, as indicated in the text, that if $\beta > -k$, the integral of (2.11) converges and if $k \leqslant 0$ $\exp[\varrho(x)]$ is everywhere bounded.

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## RIASSUNTO (*)

Nello studio dei fenomeni complessi interessanti i mesoni $\pi$ (ad es. fotoproduzione semplice e doppia, scattering radiativo, produzione doppia, ecc) si incontrano equazioni integrali che collegano gli elementi di matrice coi spostamenti di fase. Diamo qui un metodo generale per la soluzione di queste equazioni che conduce ai seguenti risultati: *a*) il tipo più semplice è integrabile per quadrature; *b*) il tipo generale è riducibile a una equazione di Fredholm. Diamo anche la soluzione di un sistema di equazioni integrali accoppiate e di quella particolare equazione che interviene nel problema della produzione doppia. Quest'ultimo caso può essere integrato per quadrature.

(*) *Traduzione a cura della Redazione.*

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