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A Monte-Carlo Method to Calculate Multiple Phase Space Integrals - I.

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1. – Introduction.

In the region of primary (lab) energies of the order of 1 to 100 GeV, the number of particles (mesons, heavy mesons, anti-nucleons, hyperons) created in a collision is of the order of 1 to 10. Consequently, pure thermodynamical calculations fail and if one wishes to apply a theory like Fermi's [1] one has to calculate phase space integrals. We do not consider here the physical side of the problem, *i.e.* the validity of this type of theory. We only try to solve the mathematical problem of calculating the phase space integral, which has not been achieved in a simple and reliable manner so far. The integral, for n emerging particles with total energy E and total momentum P, is given by [2]

$$(1) \qquad \varrho_n(E, P) = (2\pi\hbar)^{-(3n-3)} \int \mathrm{d}\boldsymbol{p}_1 \dots \mathrm{d}\boldsymbol{p}_n \, \delta(\boldsymbol{P} - \sum_{i=1}^{n} \boldsymbol{p}_i) \, \delta(E - \sum_{i=1}^{n} \sqrt{p_i^2 + m_i^2}) \, .$$

This integral does not depend on the direction of P. In the center-of-mass system (C.M.S.) we have $P = 0$. In what follows, we shall consider frequently

(1) E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950).

(2) The phase space integral is of course a function also of $m_1, m_2, \dots m_n$. As a shorthand notation we write $\varrho_n(E, P)$ instead of $\varrho_{m_1 \dots m_n}(E, P)$.

instead of (1) the unnormalized phase space integral $\varrho_n^*(E, P)$

$$\varrho_n^*(E, P) = (2\pi\hbar)^{3n-3} \cdot \varrho_n(E, P) . \tag{2}$$

The main difficulty in all attempts to perform the integrations have been the square roots $\sqrt{p_i^2+m_i^2}$. The exact integration is possible only up to $n = 3$, for higher numbers of particles one has to integrate numerically. In principle this can be done by repeated convolution since

$$\varrho_n^*(E, P) = \int \mathrm{d}\boldsymbol{p}_n \varrho_{n-1}^* \left(E - \sqrt{p_n^2 + m_n^2}, \boldsymbol{P} - \boldsymbol{p}_n\right) \tag{3}$$

and this is in fact the way BLOCK [3] has done the first few integrations. But for larger numbers n the repeated numerical convolution becomes either very inaccurate or prohibitively long. Several authors have tried approximations [4], but in many cases it is difficult to estimate the error. In unfavourable circumstances, the result can be wrong by a factor five, even if for other cases the error is only some 10%.

We may hope that by keeping the square root (instead of approximating it by either p_i or $p_i^2/2m_i$ as in most of the other calculations) but using a Monte-Carlo method (M.C.) for the integration, we may achieve an accuracy of a few percent. The method has the advantage that it gives in the course of calculating $\varrho_n^*(E, 0)$ automatically the energy spectra of each kind of particles.

Exact spectra and phase space densities should be useful in extracting from future experiments a suitable representation of the matrix elements of interaction by simple functions of energy, numbers of particles, etc. These functions (the « interaction volume » in the simplest case) may be considered as parameters of the theory whose order of magnitude and general behaviour can be concluded from physical arguments but which still remain somewhat arbitrary. We feel that such a theory is not very satisfactory, but it may, as a half-empirical procedure, lead to good qualitative predictions about the reactions of elementary particles between $1 \div 100$ GeV. Questions of this kind have turned out to be interesting since accelerators are under construction, which will yield protons of 25 and more GeV.

In all that follows, we treat the particles as if they were distinguishable and consequently each one has a « label »: 1, 2, ..., n. In any application one has then to correct ϱ_n by suitable factors [5].

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[3] M. M. BLOCK: *Phys. Rev.*, **101**, 796 (1956).

[4] *a*) R. MILBURN: *Rev. Mod. Phys.*, **27**, 1 (1955); *b*) S. BELEN'KIJ *et al.*: *Usp. Fiz. Nauk*, **62**, 1 (1957); *c*) G. E. A. FIALHO: *Phys. Rev.*, **105**, 328 (1957) and various other papers quoted in *a*) and *b*).

[5] Y. YEIVIN and A. DE-SHALIT: *Nuovo Cimento*, **1**, 1146 (1956).

2. – Preparation of $\varrho_n^{\cdot\cdot}(E, P)$ for the application of MC.

First of all, we do as much analytical evaluation of the integral as possible and apply MC only at the end.

We put

$$\boldsymbol{p}_i = p_i \boldsymbol{e}_i \; ; \qquad |\boldsymbol{e}_i| = 1$$

$$\mathrm{d}\boldsymbol{p}_i \equiv p_i^2 \, \mathrm{d}p_i \, \mathrm{d}\boldsymbol{e}_i = p_i^2 \, \sin\theta_i \, \mathrm{d}\theta_i \, \mathrm{d}\varphi_i \, \mathrm{d}p_i$$

and have

$$(4) \quad \varrho(E_n^*, P) =$$

$$= \int_0^\infty \dots \int_0^\infty \mathrm{d}p_1 \dots \mathrm{d}p_n \, p_1^2 \dots p_n^2 \, \delta\Big(E - \sum_{i=1}^n \sqrt{p_i^2 + m_i^2}\Big) \int_{\text{all directions}} \dots \int \delta\Big(\boldsymbol{P} - \sum_{i=1}^n p_i \boldsymbol{e}_i\Big) \, \mathrm{d}\boldsymbol{e}_1 \dots \mathrm{d}\boldsymbol{e}_n \, .$$

In this way the whole integral is split into two parts.

The function

$$(5) \qquad w_n(P, p_1 \dots p_n) = \Big(\frac{1}{4\pi}\Big)^n \int \delta\Big(\boldsymbol{P} - \sum p_i \boldsymbol{e}_i\Big) \, \mathrm{d}\boldsymbol{e}_1 \dots \mathrm{d}\boldsymbol{e}_n$$

has a very simple meaning: $w_n(P, p_1 \dots p_n) \, \mathrm{d}\boldsymbol{P}$ is the probability that for given $p_1 \dots p_n$, but random directions $\boldsymbol{e}_1 \dots \boldsymbol{e}_n$ the resultant vector $\sum p_i \boldsymbol{e}_i$ lies in the neighbourhood $\mathrm{d}P$ of P. It is normalized such that

$$(6) \qquad \int w_n(P, p_1 \dots p_n) \, \mathrm{d}\boldsymbol{P} = 4\pi \int_0^\infty P^2 w_n(P, p_1 \dots p_n) \, \mathrm{d}P = 1 \, ,$$

and it depends in fact only on $|P|$. We shall consequently refer to this function as the « random walk function ». An explicit derivation of this result and some further properties of the function $w_n(P, p_1 \dots p_n)$ are found in the Appendix.

Introducing now the energies by

$$(7) \qquad \varepsilon_i = \sqrt{p_i^2 + m_i^2} \, , \qquad p_i(\varepsilon_i) = \sqrt{\varepsilon_i^2 - m_i^2} \, ,$$

and defining

$$(8) \qquad u_i(\varepsilon) = \begin{cases} \varepsilon\sqrt{\varepsilon^2 - m_i^2} & \varepsilon > m_i \\ 0 & \varepsilon \leqslant m_i \end{cases} ; \quad v_n(P, \varepsilon_1 \dots \varepsilon_n) = w_n(P, p_1 \dots p_n) \, ,$$

we may write (4) in the center of mass system (CMS) as

$$(9) \qquad \varrho_n^*(E, 0) = (4\pi)^n \int_{m_1}^\infty \mathrm{d}\varepsilon_1 \dots \int_{m_n}^\infty \mathrm{d}\varepsilon_n \, u_1(\varepsilon_1) \dots u_n(\varepsilon_n) \, v_n(0, \varepsilon_1 \dots \varepsilon_n) \, \delta\Big(E - \sum \varepsilon_i\Big) \, .$$

A further transformation from the total energies to the kinetic energies

$$t_i = \varepsilon_i - m_i \; ; \qquad \sum m_i = M \; ; \qquad E - M = T \tag{10}$$

gives

$$\begin{cases} \varrho_n^*(E, 0) = (4\pi)^n \int_0^\infty \mathrm{d}t_1 \dots \int_0^\infty \mathrm{d}t_n \, \Phi(t_1 \dots t_n) \, \delta(T - \sum t_i) \; , \\ \Phi(t_1 \dots t_n) = u_1(t_1 + m_1) \dots u_n(t_n + m_n) \cdot v_n(0, t_1 + m_1, \dots t_n + m_n) \; . \end{cases} \tag{11}$$

A last transformation gives the final form to which MC will be applied:

$$\begin{cases} t_1 = T_1 \\ t_2 = T_2 - T_1 \\ t_n = T_n - T_{n-1} \end{cases} \tag{12}$$

leads to

$$\varrho_n^*(E, 0) = (4\pi)^n \int_0^\infty \mathrm{d}T_1 \int_{T_1}^\infty \mathrm{d}T_2 \dots \int_{T_{n-2}}^\infty \mathrm{d}T_{n-1} \int_{T_{n-1}}^\infty \mathrm{d}T_n \, \Phi(T_1, \, T_2 - T_1, \dots) \, \delta(T - T_n) \; ,$$

where now the δ-function drops out by integrating over T_n:

$$\boxed{\varrho_n^*(E, 0) = (4\pi)^n \int_0^T \mathrm{d}T_1 \int_{T_1}^T \mathrm{d}T_2 \dots \int_{T_{n-2}}^T \mathrm{d}T_{n-1} \Phi(T_1, \, T_2 - T_1, \dots \, T_n - T_{n-1}) \; .} \tag{13}$$

3. - The MC approach.

We consider the following process: We choose independently and at random $n - 1$ numbers between 0 and T in such a way that the probability density for the single numbers is constant over the whole interval; we then order them according to magnitude.

With N such ordered samples $(T_1^{(i)}, \, T_2^{(i)} \dots T_{n-1}^{(i)})$; $(i = 1 \dots N)$ we have

$$\int_0^T \mathrm{d}T_1 \int_{T_1}^T \mathrm{d}T_2 \dots \int_{T_{n-2}}^T \mathrm{d}T_{n-1} \, \Phi(T_1, \, T_2 - T_1, \dots) = \tag{14}$$

$$= \frac{T^{n-1}}{(n-1)!} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \Phi(T_1^{(i)}, \, T_2^{(i)} - T_1^{(i)}, \dots) \; .$$

To prove the truth of (14), we note that the probability that the i-th ordered (*) sample fills just the cells ΔT_1, $\Delta T_2, \ldots, \Delta T_{n-1}$ is (see Fig. 1)

$$(n-1)!\frac{\Delta T_1}{T}\cdot\frac{\Delta T_1}{T}\ldots\frac{\Delta T_{n-1}}{T}=\frac{(n-1)!}{T^{n-1}}\,\Delta T_1\Delta T_2\ldots\Delta T_{n-1}$$

and N times this probability is the total number of ordered samples filling these cells — they all lead to the same value of φ. Hence (14) holds.

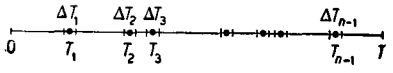

Fig. 1. – For the proof of (14).

This is the principle — it remains to write the formulae in a more convenient form.

It should be mentioned that one may hope for a fairly rapid convergence, since the sampling process has a tendency to draw preferably samples with $T_1 \approx T_2 - T_1 \approx \ldots \approx T - T_{n-1}$ because for large n the points $T_i^{(j)}$ cluster around the value $(i/n)T$. Fortunately, it happens that the integrand has a maximum for those values of the variables that nearly correspond to equipartition of kinetic energies. Thus the samples tend automatically to pick out the most important terms of the sum and to neglect the small ones. This is the main advantage of MC applied to this problem as compared to a simple multidimensional numerical integration. The problem of convergence will be treated in the following paper.

4. – The computing programme.

From (14) we have with (13)

$$\varrho_n^*(E,0)=\frac{(4\pi)^n}{(n-1)!}\,T^{n-1}\cdot\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\Phi(T_1^{(i)},\,T_2^{(i)}-T_1^{(i)}\ldots)\,, \tag{16}$$

where $T_1^{(i)}\ldots T_{n-1}^{(i)}$ is the i-th ordered sample. Going back to the variables ε_j, each ordered sample $T_1^{(i)}\ldots T_{n-1}^{(i)}$ yields a sample $\varepsilon_1^{(i)}\ldots\varepsilon_n^{(i)}$:

$$(\varepsilon_1^{(i)},\,\varepsilon_2^{(i)},\ldots,\,\varepsilon_n^{(i)})\equiv(T_1^{(i)}+m_1,\,T_2^{(i)}-T_1^{(i)}+m_2,\ldots,\,T-T_{n-1}^{(i)}+m_n) \tag{17}$$

and

$$\Phi(T_1,\,T_2-T_1,\ldots T-T_{n-1})\equiv\varepsilon_1\sqrt{\varepsilon_1^2-m_1^2}\ldots\varepsilon_n\sqrt{\varepsilon_n^2-m_n^2}\cdot v_n(0,\,\varepsilon_1\varepsilon_2\ldots\varepsilon_n)\,.$$

(*) The ordering is responsible for the factor $(n-1)!$

For v_n we take the form (cf. A.9))

$$v_n(0, \varepsilon_1 \dots \varepsilon_n) = -\frac{1}{(n-3)!\,\pi \cdot 2^{n+1}} \cdot$$

$$\cdot \frac{1}{\sqrt{\varepsilon_1^2 - m_1^2} \dots \sqrt{\varepsilon_n^2 - m_n^2}} \sum_{\sigma_2 \dots \sigma_n} \left[\sum \sigma_1 \dots \sigma_n \left\{ \operatorname{sg}\left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2}\right) \right\} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2}\right)^{n-3} \right] .$$

We have therefore

$$\Phi(T_1, T_2 - T_1, \dots T - T_{n-1}) = \frac{1}{2^{n+1} \cdot \pi \cdot (n-3)!}\, \psi_n(\varepsilon_1 \dots \varepsilon_n)$$

with

$$\psi_n(\varepsilon_1 \dots \varepsilon_n) = -\varepsilon_1 \cdot \varepsilon_2 \dots \varepsilon_n \sum_{\sigma_2 \dots \sigma_n} \left[\sum \sigma_1 \dots \sigma_n \left\{ \operatorname{sg}\left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2}\right) \right\} \left(\sum_i \sigma_i \sqrt{\varepsilon_i^2 - m_i^2}\right)^{n-3} \right] \tag{18}$$

With this definition

$$\varrho_n^*(E, 0) = \frac{(E-M)^{n-1} \cdot (2\pi)^{n-1}}{(n-1)!\,(n-3)!} \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \psi_n(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \dots, \varepsilon_n^{(i)}) . \tag{19}$$

The samples $\varepsilon_1^{(i)}$, $\varepsilon_2^{(i)} \dots \varepsilon_n^{(i)}$ are constructed according to (17) and at the same time one calculates the corresponding set of square roots $\sqrt{\varepsilon_j^{(i)2} - m_j^2}$.

5. - The spectrum.

Another advantage of this MC method for solving the phase space problem is that by the very nature of the computation process one gets easily without further calculations the energy spectrum for each kind of particle considered, more accurately: that part of the spectrum which is determined by phase space considerations.

We consider here the (unnormalized) spectrum of the r-th particle, which because of the assumed individuality means a quite definite one. From (3) we see that in the C.M.S. ($P = 0$) the quantity

$$\mathrm{d}p_r \cdot 4\pi p_r^2 \cdot \varrho_{n-1}^*(E - \sqrt{p_r^2 + m_r^2},\, p_r) \tag{20}$$

is proportional to the number of particles « r » with momentum between p_r and $p_r + \mathrm{d}p_r$.

The function $\varrho_n^{(r)}(E, 0, \varepsilon_r)$, defined as follows:

$$\varrho_n^{(r)}(E, 0, \varepsilon_r) = 4\pi\varepsilon_r \sqrt{\varepsilon_r^2 - m_r^2} \cdot \varrho_{n-1}^*(E - \varepsilon_r, \sqrt{\varepsilon_r^2 - m_r^2}) \tag{21}$$

allows then a similar interpretation.

Now from (9)

$$(22)\quad \varrho^*_{n-1}(E-\varepsilon_r,\, p_r) = (4\pi)^{n-1}\int_{m_1}^{\infty} d\varepsilon_1 \dots \int_{m_{r-1}}^{\infty} d\varepsilon_{r-1} \int_{m_{r+1}}^{\infty} d\varepsilon_{r+1} \dots \int_{m_n}^{\infty} d\varepsilon_n \cdot$$
$$\cdot u_1(\varepsilon_1) \dots u_{r-1}(\varepsilon_{r-1})\, u_{r+1}(\varepsilon_{r+1}) \dots u_n(\varepsilon_n) \cdot v_{n-1}(p_r, \varepsilon_1, \dots, \varepsilon_{r-1}, \varepsilon_{r+1}, \dots, \varepsilon_n) \cdot$$
$$\cdot \delta\Big(E - \varepsilon_r - \sum_{i \neq r} \varepsilon_i\Big) .$$

Furthermore with $\varepsilon_r = \sqrt{p_r^2 + m_r^2}$ from (A.7)

$$(23)\qquad v_{n-1}(p_r, \varepsilon_1 \dots \varepsilon_{r-1}, \varepsilon_{r+1} \dots \varepsilon_n) = v_n(0, \varepsilon_1 \dots \varepsilon_n) .$$

Since also $\delta(E - \varepsilon_r - \sum_{i \neq r} \varepsilon_i) = \delta(E - \sum_i \varepsilon_i)$, we see that the differential spectrum of particle « r », namely $\varrho_n^{(r)}(E, 0, \varepsilon_r)$ is obtained from (9) if $d\varepsilon_r$ is taken outside the integrals, *i.e.*, if the integration over ε_r is omitted. From this follows then the almost obvious way to calculate $\varrho_n^{(r)}(E, 0, \varepsilon_r)$.

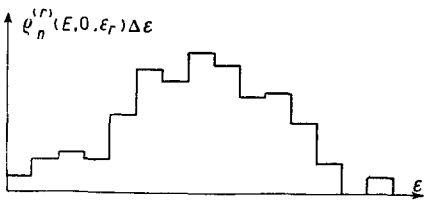

Fig. 2. – Typical form of a MC spectrum.

For each kind « r » of particle a set of Z storages is provided and the total energy E is divided into Z intervals $\Delta\varepsilon = E/Z$, each cell corresponding to one storage.

At the same time as the samples $\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)}$ are prepared, one looks in each sample for the value of $\varepsilon_r^{(i)}$ and determines in which of the cells $\Delta\varepsilon$ it lies. Then one adds into the corresponding storage the number $\psi_n(\varepsilon_1^{(i)} \dots \varepsilon_n^{(i)})$ (see (19)).

After completion of this distribution of ψ's (which takes place of course during the calculation of $\varrho^*(E, 0)$) into the cells, the whole distribution is multiplied by

$$\frac{1}{N} \cdot \frac{(E-M)^{n-1}(2\pi)^{n-1}}{(n-1)!\,(n-3)!}$$

and the result is $\varrho_n^{(r)}(E, 0, \varepsilon_r)$, the spectrum for the particle « r ». Indeed: this distribution means nothing more than the omission of the integration over $d\varepsilon_r$ and represents the integrand as a function of ε_r.

The accuracy will of course be smaller than for $\varrho_n^*(E, 0)$ itself, because of the subdivision of the total statistical information, but one obtains without additional work at least a rough idea of the spectra.

In a following paper we shall give details of the actual performance of such a calculation including estimates of errors and a flow diagram for an electronic computor. A brief report concerning our experiences in practical calculations will be added.

APPENDIX

The random walk function.

Given a set of n vectors with fixed lengths $p_1 \dots p_n$ but random directions $\boldsymbol{e}_1 \dots \boldsymbol{e}_n$. What is the probability $w_n(\boldsymbol{P}, p_1 \dots p_n)\,\mathrm{d}\boldsymbol{P}$, that the resultant $\sum_{i=1}^{n} p_i \boldsymbol{e}_i$ lies in the neighbourhood $\mathrm{d}P$ of P?

First we make two simplifying remarks:

1) Since each $\boldsymbol{e}_i$ has a random direction, the same holds for the resultant. $w(\boldsymbol{P}, p_1 \dots p_n)$ can only depend on $|P|$. We write therefore $w_n(P, p_1 . p_n)$.

2) The order in which the vectors are added is irrelevant, w_n must be symmetric in $p_1 \dots p_n$. Even more: the probability density for n vectors $\boldsymbol{p}_1 \dots \boldsymbol{p}_n$ to have a resultant $\boldsymbol{P}$ must be the same as for the $n+1$ vectors $\boldsymbol{P}, \boldsymbol{p}_1 \dots \boldsymbol{p}_n$ to have the resultant zero. Hence

$$w_{n+1}(0, p_1 \dots p_n, P) = w_n(P, p_1 \dots p_n)$$

and since the left hand side is symmetric in p, this must be true also. P for the r.h.s.

We now derive Eq. (5).

The required probability is obviously proportional to the number of possible realizations, *viz.*

$$w_n(P, p_1 \dots p_n) = K_n \int \delta(\boldsymbol{P} - \sum p_i \boldsymbol{e}_i)\, \mathrm{d}\boldsymbol{e}_1 \dots \mathrm{d}\boldsymbol{e}_n \,, \tag{A.1}$$

where we have imposed the condition by means of a δ-function and integrate thus over all possible ways of choosing directions, which give the wanted length of the resultant. K_n is a normalizing factor to be determined later.

Introducing a Fourier representation for the δ-function

$$\delta(\boldsymbol{P} - \sum_i p_i \boldsymbol{e}_i) = \left(\frac{1}{2\pi}\right)^3 \int \exp\left[i\boldsymbol{\lambda}(\boldsymbol{P} - \sum p_i \boldsymbol{e}_i)\right] \mathrm{d}\boldsymbol{\lambda} \,, \tag{A.2}$$

we find

$$w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} \int \mathrm{d}\boldsymbol{\lambda}\, \mathrm{d}\boldsymbol{e}_1 .. \mathrm{d}\boldsymbol{e}_n \exp\left[i\boldsymbol{\lambda}(\boldsymbol{P} - \sum p_i \boldsymbol{e}_i)\right]$$

or

$$w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} \int \mathrm{d}\boldsymbol{\lambda} \exp\left[i\boldsymbol{\lambda}\boldsymbol{P}\right] \cdot \prod_{j=1}^{n} \left[\int \mathrm{d}\boldsymbol{e} \exp\left[-ip_j \boldsymbol{\lambda}\boldsymbol{e}\right] \right] . \tag{A.3}$$

With polar co-ordinates $\mathrm{d}\boldsymbol{e} = 2\pi \sin\theta\, \mathrm{d}\theta$ one has

$$\int \mathrm{d}\boldsymbol{e} \exp\left[-ip_j \boldsymbol{\lambda}\boldsymbol{e}\right] = \frac{4\pi \sin p_j \lambda}{p_j \lambda} \,.$$

Since the product does no longer depend on the direction of $\boldsymbol{\lambda}$ we may integrate $\exp[i\boldsymbol{\lambda}\boldsymbol{P}]$ also over all directions of $\boldsymbol{\lambda}$ with the same result and find

$$(A.4) \qquad w_n(P, p_1 \dots p_n) = \frac{K_n}{(2\pi)^3} (4\pi)^{n+1} \int_0^\infty \lambda^2 \, \mathrm{d}\lambda \, \frac{\sin P\lambda}{P\lambda} \cdot \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda} .$$

The normalization requires

$$(A.5) \qquad \lim_{P_0 \to \infty} \int_0^{P_0} w_n(P, p_1 \dots p_n) \cdot 4\pi P^2 \, \mathrm{d}P = 1 .$$

(It is necessary to consider this limit, because we have reversed the order of integration and thereby obtained a singular function.) We obtain

$$4\pi \int_0^{P_0} w_n(P, p_1 \dots p_n) P^2 \, \mathrm{d}P = \frac{K_n (4\pi)^{n+2}}{(2\pi)^3} \int_0^\infty \lambda \, \mathrm{d}\lambda \, \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda} \left[\frac{\sin P_0\lambda}{\lambda^2} - \frac{P_0}{\lambda} \cos P_0\lambda \right] .$$

Now the second term in the bracket makes the integral vanish for $P_0 \to \infty$ whereas the first one gives

$$4\pi \int_0^{P_0} w_n(P, p_1 \dots p_n) P^2 \, \mathrm{d}P = \frac{K_n (4\pi)^{n+2}}{(2\pi)^3} \cdot \frac{\pi}{2} \cdot \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\lambda \, \frac{\sin \lambda P_0}{\lambda} \, \frac{\sin \lambda p_1}{\lambda p_1} \dots \frac{\sin \lambda p_n}{\lambda p_n} .$$

Since

$$\lim_{P_0 \to \infty} \frac{1}{\pi} \frac{\sin \lambda P_0}{\lambda} = \delta(\lambda) ,$$

we find

$$4\pi \int_0^\infty w_n(P, p_1 \dots p_n) P^2 \, \mathrm{d}P = (4\pi)^n \cdot K_n = 1 .$$

Hence the normalized random walk function is

$$(A.6) \quad w_n(P, p_1 \dots p_n) = \left(\frac{1}{4\pi} \right)^n \int \delta(\boldsymbol{P} - \sum p_i \boldsymbol{e}_i) \, \mathrm{d}\boldsymbol{e}_1 \dots \mathrm{d}\boldsymbol{e}_n =$$

$$= \frac{1}{2\pi^2} \int_0^\infty \lambda^2 \, \mathrm{d}\lambda \, \frac{\sin P\lambda}{P\lambda} \, \frac{\sin p_1\lambda}{p_1\lambda} \dots \frac{\sin p_n\lambda}{p_n\lambda} .$$

We observe at once the complete symmetry not only with respect to $p_1 \dots p_n$ but also with respect to $P, p_1 \dots p_n$, which all may be arbitrarily permuted. A simple consequence is the important equation

$$(A.7) \qquad w_n(0, p_1 \dots p_n) = w_{n-1}(p_j, p_1 \dots p_{j-1}, p_{j+1} \dots p_n) ; \qquad j = 1 \dots n ,$$

which we found already at the beginning from general arguments.

We now evaluate (A.6) for $P = 0$ (which, because of (A.7), is no restriction). Written with exponentials and the λ-integration going from $-\infty$ to ∞, (A.6) gives

$$w_n(0, p_1 \dots p_n) = \frac{1}{4\pi^2} \cdot \frac{1}{p_1 \dots p_n} \cdot \left(\frac{1}{2i}\right)^n \int_{-\infty - i\tau}^{\infty - i\tau} \frac{\mathrm{d}\lambda}{\lambda^{n-2}} \left(\exp[ip_1\lambda] - \exp[-ip_1\lambda]\right) \dots \left(\exp[ip_n\lambda] - \exp[-ip_n\lambda]\right) .$$

Since the integrand does in fact contain no singularity, we could shift the path by $-i\tau$ into the lower half plane. Evaluating the product and closing then the path in the upper or lower half plane according to where the integrand vanishes, we find for the integral

$$\int_{-\infty - i\tau}^{\infty - i\tau} \frac{\mathrm{d}\lambda}{\lambda^{n-2}} \sum_\sigma \sigma_1 \dots \sigma_n \exp[i\lambda \sum_i \sigma_i p_i] = \frac{2\pi i}{(n-3)!} \frac{\mathrm{d}^{n-3}}{\mathrm{d}\lambda^{n-3}} \left\{ \sum_\sigma \sigma_1 \dots \sigma_n \exp[i\lambda \sum_i \sigma_i p_i] \right\}_{\substack{\sum \sigma_i p_i \geqslant 0 \\ \lambda = 0}} .$$

Here every σ takes the two values $+1$ and -1 and the sum is over all 2^n sign combinations, except those for which $\sum \sigma_i p_i < 0$. (Because for these the path is closed in the lower half plane with residual zero). The differentiation gives finally

$$(A.8)\quad w_n(0, p_1 \dots p_n) = -\frac{1}{(n-3)!\, 2^{n+1}\pi} \cdot \frac{1}{p_1 \dots p_n} \left[\sum_\sigma \sigma_1 \dots \sigma_n \left(\sum_i \sigma_i p_i\right)^{n-3} \right]_{\sum \sigma_i p_i \geqslant 0} .$$

For practical calculation of the sum over σ we remark that to every sign combination there exists the inverse one and one of them gives $\sum_i \sigma_i p_i \geqslant 0$. We need therefore sum only over one half of all sign combinations provided that we reverse all signs if $\sum_i \sigma_i p_i < 0$. We may therefore fix $\sigma_1 \equiv +1$ and vary all the rest. This gives just one half of the possible sign combinations $\sigma_1 \dots \sigma_n$ and no two of them are the inverse of each other. Thus the whole class $[\sigma_1 = +1]$ is inverse to the whole class $[\sigma_1 = -1]$ and we shall now sum only over the former one. If for a certain combination $\sigma_1 \dots \sigma_n$ $(\sigma_1 = +1)$ one finds $\sum \sigma_i p_i < 0$, one has to change all signs $\sigma_i \to \sigma_i' = -\sigma_i$ and to add

$$\sigma_1' \sigma_2' \dots \sigma_n' \left(\sum \sigma_i' p_i\right)^{n+3} = (-)^n \sigma_1 \dots \sigma_n \cdot (-)^{n-3} \left(\sum \sigma_i p_i\right)^{n-3} = -\sigma_1 \dots \sigma_n \left(\sum \sigma_i p_i\right)^{n-3} .$$

That means: If $\sum \sigma_i p_i < 0$, we have to subtract $\sigma_1 \dots \sigma_n \left(\sum \sigma_i p_i\right)^{n-3}$ instead of adding it. This is expressed simply by writing

$$(A.9)\quad w_n(0, p_1 \dots p_n) = -\frac{1}{(n-3)!\, 2^{n+1}\pi} \cdot \frac{1}{p_1 \dots p_n} \left[\sum_{\substack{\sigma_2 \dots \sigma_n \\ \sigma_1 = +1}} \sigma_1 \dots \sigma_n \cdot \mathrm{sg}\left(\sum \sigma_i p_i\right) \left(\sum \sigma_i p_i\right)^{n-3} \right] ,$$

where

$$\operatorname{sg} x = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x \leqslant 0 \end{cases}$$

is the sign function (*).

An evident consequence of the meaning of the random walk function is that it must vanish identically as soon as any p_j is larger than the sum of all the others:

$$w_n(0, p_1 \dots p_n) \equiv 0 \quad \text{if} \quad 2p_j - \sum_{i=1}^{n} p_i \quad (\text{for any } j = 1 \dots n). \tag{A.10}$$

As this is not obvious in the present form (A.9) of the function, we prove it explicitely:

Because of the symmetry one may take $j = 1$. The square bracket of (A.9) must vanish.

i) We observe that if $2p_1 \geqslant \sum p_i$, one has

$$\operatorname{sg} \left(\sum \sigma_i p_i\right) \equiv +1 .$$

ii) We expand the power in writing ($\sum'$ means $i \neq 1$)

$$\sum_{\sigma} \sigma_2 \dots \sigma_n (p_1 + \sum{}' \sigma_i p_i)^{n-3} = \sum_{\sigma} \sigma_2 \dots \sigma_n \sum_{\Sigma i_k = n-3} [p_1^{i_1} (\sigma_2 p_2)^{i_2} \dots (\sigma_n p_n)^{i_n}] =$$

$$= \sum_{\Sigma i_k = n-3} p_1^{i_1} p_2^{i_2} \dots p_n^{i_n} \sum_{\sigma} \sigma_2^{i_2+1} \sigma_3^{i_3+1} \dots \sigma_n^{i_n+1} .$$

iii) From $\sum_{k=1}^{n} i_k = n - 3$ follows that not all $i_k \neq 0$. Let $i_j = 0$. Then one has ($\sum''$ means: $\sigma_j^{i_j+1}$ omitted)

$$\sum_{\sigma} = \sum_{\sigma_j = \pm 1} \sigma_j \left(\sum{}'' \sigma_2^{i_2+1} \dots \sigma_n^{i_n+1}\right) = \sum{}' - \sum{}'' = 0 . \qquad \textit{q.e.d.}$$

The evaluation of (A.9) seems very easy on an electronic computer, if one

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(*) Ambiguities arise if accidentally a certain $\sum \sigma_i p_i = 0$. In fact this gives a difficulty only for $n = 3$, since in that case $w(0, p_1 p_2 p_3)$ is a discontinuous function because of the triangle inequality. As a pure convention we put $\operatorname{sg}(0) = -1$. For $n > 3$ one has in such a case $\left(\sum \sigma_i p_i\right)^{n-3} = 0$, which settles the question. In the programme of an electronic computor, however, this does not help the machine to know what to do with the sg(0), so we shall adopt the same definition for all n.

proceeds in the following way:

a) One calculates $\sum_{j=1}^{n} p_i$, which corresponds to

$$\sigma_1 = \sigma_2 = \ldots = \sigma_n = 1 .$$

b) One varies the signs in such a way that in the order $\sigma_2\sigma_3 \ldots \sigma_n$ each σ keeps its sign as long as possible (see example):

$\sigma_1 \cdot \sigma_2 \ldots \sigma_n$	σ_2	σ_3	σ_4	σ_5	Corresponds to	Change against foregoing set $\sum \sigma_i p_i$
+	+	+	+	+	$p_1 + p_2 + p_3 + p_4 + p_5$	
−	+	+	+	−	$-2p_5$	
+	+	+	−	−	$-2p_4$	
−	+	+	−	+	$+2p_5$	
+	+	−	−	+	$-2p_3$	
−	+	−	−	−	$-2p_5$	
+	+	−	+	−	$+2p_4$	
−	+	−	+	+	$+2p_5$	
+	−	−	+	+	$-2p_2$	
−	−	−	+	−	$-2p_5$	
+	−	−	−	−	$-2p_4$	
−	−	−	−	+	$+2p_5$	
+	−	+	−	+	$+2p_3$	
−	−	+	−	−	$-2p_5$	
+	−	+	+	−	$+2p_4$	
−	−	+	+	+	$+2p_5$	

The example shows that from combination to combination there is always only one addition to do and the sign of $\sigma_1 \cdot \sigma_2 \ldots \sigma_n$ alternates systematically Besides the additions, the machine has to do the $(n-3)$rd power and to ob-

serve the sign of $\sum \sigma_i p_i$. The total number of additions goes therefore roughly with 2^{n-1} and after each addition (*i.e.* for each $\sum \sigma_i p_i$) the power is to be calculated, that is 2^{n-1} times a $(n-3)$rd power. We are unlikely to have to calculate cases where n is bigger than 12, the normal situation will have n running from 3 to 10 (only mesons produced) and from 3 to 5 or 6 (heavier particles produced). For $n=12$ one has thus ~ 2050 additions and 2050 times a 9-th power. Cases with $n=7$ may be most interesting. This gives ~ 150 additions and 150 times a 4-th power.

Since this number of operations has to be carried out for each sample of $p_1 \ldots p_n$, the calculation of $w_n(0, p_1 \ldots p_n)$ contributes a great deal to the total computing time.