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On The Theory of Higher Spin Fields (*).

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CONTENTS. —

Introduction.

Recent experiments (1) on strange particles have given some support for the possibility that elementary particles of spin higher than 1 may exist. In particular the observed angular correlation between the planes of production and decay of the hyperon Λ and Σ, and the anisotropy of the angular distribution of the Σ decay products, seem to indicate that the spins of these particles are $\frac{3}{2}$ or higher (2).

(*) Part of dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy from the University of California at Los Angeles.

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(1) W. D. WALKER and W. D. SHEPHARD: *Phys. Rev.*, **101**, 1810 (1956); W. B. FOWLER, R. P. SCHUTT, A. M. THORNDYKE and W. L. WHITTEMORE: *Phys. Rev.*, **91**, 1287 (1953); **93**, 861 (1594); **98**, 121 (1955); L. W. ALVAREZ, H. BRADNER, P. FALK-VAIRANT, I. D. GOW, A. H. ROZENFELD, F. T. SOLMITZ and R. D. TRIPP: *UCRL* 3583, University of California Radiation Laboratory, Berkeley 1956.

(2) At the time that the present work is being prepared for publication, evidence for the higher spin of the hyperons is weakening, see *Proceedings of the Rochester Conference of High Energy Physics*, 1957 (to be published).

The theory of higher spin fields proposed by FIERZ (3), and FIERZ and PAULI (4) has been simplified by the work of RARITA and SCHWINGER (5), GUPTA (6) and MOLDAUER and CASE (7). In spite of these simplifications, however, calculations are very lengthy, as illustrated by the calculation of the Compton scattering cross-section by MATTHEWS (8).

It is the aim of this note to present a new simple formulation of the Fierz-Pauli theory, and to analyze in some detail the structure of this theory for arbitrary spin.

All relativistic theories of free fields are based on the principle of invariance under the group of co-ordinate transformations known as the Lorentz group (9). This principle requires that the wave function form a basis for a representation of the Lorentz group. The simplest kind of field is defined by an irreducible representation and is said to describe an elementary particle, which provides an exact and natural definition of the latter concept.

The Lorentz group possesses two invariants

$$p^2 = p^\mu p_\mu\,,$$
$$p^2 S^2 = \tfrac{1}{2} L^{\mu\nu} L_{\mu\nu} p^2 - L^{\mu\lambda} L_{\mu\lambda} p_\nu p^\lambda\,.$$

In an irreducible representation these must be multiples of the identity. Define, therefore, two numbers m and s, such that

$$(a) \qquad p^2 = -m^2\,, \qquad \text{(Definition of mass)},$$

$$(b) \qquad S^2 = s(s+1)\,, \qquad \text{(Definition of spin)}.$$

In all the cases of physical interest m is a positive number or zero, and $2s$ is a positive integer or zero. In these cases there is only one representation of the Lorentz group for any given set of values of m and s (10), apart from an ambiguity in the choice of reflection operators (11). In the following the value $m = 0$ is excluded from consideration, unless otherwise stated.

In the first three sections conditions (a) and (b) are formulated in terms of conditions on the wave function, in the case of no external forces. A « spin

(3) M. FIERZ and W. PAULI: *Proc. Roy. Soc.*, A **173**, 211 (1939).
(5) W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1941).
(6) S. N. GUPTA: *Phys. Rev.*, **95**, 1334 (1954).
(7) P. A. MOLDAUER and K. M. CASE: *Phys. Rev.*, **102**, 279 (1956).
(8) J. MATTHEWS: *Phys. Rev.*, **102**, 270 (1956).
(9) The short term « Lorentz group » is used for the « Extended inhomogeneous Lorentz group », which includes translations and reflections.
(10) E. P. WIGNER: *Ann. of Math.*, **40**, 149 (1939).
(11) L. L. FOLDY: *Phys. Rev.*, **102**, 568 (1956).

projection operator » is introduced, which greatly simplifies the treatment of the subsidiary conditions. In Sect. **4-9** the Fierz-Pauli theory for electromagnetic interaction of particles of arbitrary spin is analysed, with the aid of the spin projection operator.

In the last sections polarization operators are introduced and applied to the calculations of angular distribution of hyperon decay products.

1. – The free fields.

The simplest irreducible representation of the Lorentz group is that for which $s = 0$. The wave function is a single function of p, and satisfies the condition (a)

$$(p^2 + m^2)\varphi(p) = 0 . \tag{1.1}$$

This is recognized as the Klein-Gordon equation.

Four functions $\varphi_\mu(p)$, which transform like the components of a vector, provide the basis for a reducible representation. Designating this representation by D, we have in fact

$$D = \mathrm{D}(1) \oplus D(0) ,$$

where $D(s)$ is the irreducible representation corresponding to the spin value s.

Irreducible representations corresponding to any other integral spin value may be formed by taking direct products of D with itself and expanding the products in Clebsch-Gordon series. For example

$$D \otimes D = D(2) \oplus 3D(1) \oplus 2D(0) .$$

The quantity that transforms according to $D^s = D \otimes D \otimes \ldots \otimes D$ is the tensor of rank s. The irreducible representation $D(s)$ is $(2s+1)$ dimensional. The wave equation (1) applies to each component of the wave function, while condition (b) requires that the projections of φ on the spaces of representations of lower spin values vanish. When $s = 2$, this condition may be written

$$\begin{cases} p^{\mu_1}\varphi_{\mu_1\mu_2} = 0 , \\ \varphi_{\mu_1\mu_2} = \varphi_{\mu_2\mu_1} , \\ g^{\mu_1\mu_2}\varphi_{\mu_1\mu_2} = 0 , \end{cases} \tag{1.2}$$

where $\varphi_{\mu_1\mu_2}$ is a tensor of rank 2. Equations (2) are equivalent to Eq. (6), and are referred to as subsidiary conditions ([12]). Similarly the general integral spin field is defined by the following set of equations:

$$(1.3a)\qquad (p^2+m^2)\varphi_{\mu_1\cdots\mu_s}=0\,,$$

$$(1.3b)\qquad \varphi_{\dots\mu_i\dots\mu_j\dots}=\varphi_{\dots\mu_j\dots\mu_i\dots}$$

$$(1.3c)\qquad p^{\mu_1}\varphi_{\mu_1\cdots\mu_s}=0\,,$$

$$(1.3d)\qquad g^{\mu_1\mu_2}\varphi_{\mu_1\mu_2\cdots\mu_s}=0\,,$$

where s is the spin.

If the spin is $s=n+\frac{1}{2}$, n integer, the wave function is a tensor of rank n, each component of which is a Dirac four spinor.
Equation (3*a*) is replaced by the Dirac equation ([13])

$$(1.3a^*)\qquad (\boldsymbol{p}+im)\varphi_{\mu_1\cdots\mu_n}=0\,,$$

and Eq. (3*d*) by the subsidiary condition

$$(1.3d^*)\qquad \gamma^{\mu_1}\varphi_{\mu_1\cdots\mu_n}=0\,.$$

Equation (3*a*) is a consequence of (3*a**), and the only additional information contained in (3*a**) concerns the choice of reflection operators ([11]).

2. – The spin projection operator.

It is convenient, temporarily, to express the subsidiary conditions (1.3*b*, *c*, *d*) or (1.3*b*, *c*, *d**) by the symbolic notation

$$(2.1)\qquad \eta_i\varphi=0\,,\qquad i=1,\,2,\,3\,.$$

([12]) The equivalence of Eqs. (1.2) and Eq. (*b*) may be understood by noting that the former equates to zero all the tensors of lower rank that can be formed from $\varphi_{\mu_1\mu_2}$. Thus, $g^{\mu_1\mu_2}\varphi_{\mu_1\mu_2}$ is a scalar, $p^{\mu_1}\varphi_{\mu_1\mu_2}$ is a vector, and the antisymmetric part of $\varphi_{\mu_1\mu_2}$ is a « six-vector », which transforms by the sum of two irreducible representations of spin 1. In Eqs. (3) below the components of $\varphi_{\mu_1}\dots$ are accounted for as follows: the 4^s components are reduced to $\binom{s+3}{3}$ by the symmetry condition. Subtracting the numbers $\binom{s+2}{3}$ and $\binom{s}{2}$ of conditions imposed by the other two subsidiary conditions leaves $2s+1$ independent components, as is appropriate for the spin 2 field.

([13]) Notation: $\boldsymbol{p}\equiv\gamma_\mu p^\mu$, where γ_μ are the four-by-four Dirac matrices, defined by the commutation relations

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}\,.$$

Similarly, for (1.3a) or (1.3a*)

$$\eta\varphi = 0\,. \tag{2.2}$$

Introduce an orthogonal projection operator:

$$\Theta = \bar{\Theta} = \Theta^2\,, \qquad \bar{\Theta} \equiv \gamma_4 \Theta^\dagger \gamma_4\,, \tag{2.3}$$

with the following properties: (i) if $\varphi_{\mu\ldots}$ is a tensor of rank s (or a spin-tensor of rank $s-\frac{1}{2}$), then $(\Theta\varphi)_{\mu\ldots} = \psi_{\mu\ldots}$ transforms according to $D(s)$ (i.e., $\eta_i\psi = 0$), (ii) if a specific (spinor-) tensor $\varphi_{\mu\ldots}$ transforms according to $D(s)$, then $\varphi_{\mu\ldots} = (\Theta\varphi)_{\mu\ldots}$, and (iii) if $\varphi_{\mu\ldots}$ satisfies Eq. (2), so does $(\Theta\varphi)_{\mu\ldots}$: Symbolically, omitting indices:

$$\eta_i(\Theta\varphi) \equiv 0\,, \qquad \text{(i)},$$

$$(\eta_i\varphi = 0) \rightarrow (\varphi = \Theta\varphi)\,, \qquad \text{(Condition of uniqueness) (ii)},$$

$$(\eta\varphi = 0) \rightarrow (\eta\Theta\varphi = 0)\,, \qquad \text{(Commutativity with } \eta\text{) (iii)}.$$

It will now be shown that the only non-trivial solution of (3) and (i) satisfies (ii) and (iii).

Uniqueness of Θ. In view of the complete symmetry of $(\Theta\varphi)_{\mu\ldots}$ in all indices, Θ must be constructed from the metric tensor, the vector p_μ and (for half-odd-integer spin) γ_μ:

$$\Theta^{\nu_1\ldots\nu_n}_{\mu_1\ldots\mu_n} = \sum_P \{a g^{\mu_1}_{\mu_1}\ldots g^{\nu_n}_{\mu_n} + b g_{\mu_1\mu_2} g^{\nu_1\nu_2}\ldots + \ldots + d\gamma_{\mu_1}\gamma^{\nu_1}\ldots + \ldots\}\,, \tag{2.4}$$

where the sum is over all permutations of the lower indices. The defining equations (i) may be written

$$\Theta^{\nu_1\ldots\nu_n}_{\ldots\mu_i\ldots\mu_j\ldots} = \Theta^{\nu_1\ldots\nu_n}_{\ldots\mu_j\ldots\mu_i\ldots}\,, \tag{2.5a}$$

$$p^{\mu_1}\,\Theta^{\nu_1\ldots}_{\mu_1\ldots} = 0\,, \tag{2.5b}$$

$$g^{\mu_1\mu_2}\Theta^{\nu_1\ldots}_{\mu_1\mu_2\ldots} = 0\,, \qquad 2s \text{ even}, \tag{2.5c}$$

$$\gamma^{\mu_1}\Theta^{\lambda_1\ldots}_{\mu_1\ldots} = 0\,, \qquad 2s \text{ odd}. \tag{2.5d}$$

Combination of Eqs. (4) and (5) gives

$$\Theta^{\nu_1\ldots}_{\mu_1\ldots}\,\Theta^{\lambda_1\ldots}_{\nu_1\ldots} = a\sum_P \Theta^{\lambda_1\ldots}_{\mu_1\ldots} = an!\,\Theta^{\lambda_1\ldots}_{\mu_1\ldots}\,.$$

Provided Θ is not identically zero, Eq. (3) yields

$$an! = 1 .$$

If $\varphi_{\mu\dots}$ satisfies all the subsidiary conditions, Eq. (4) gives

$$\Theta^{\nu_1\dots}_{\mu_1\dots}\varphi_{\nu_1\dots} = a\sum_p \varphi_{\mu_1\dots} = \varphi_{\mu_1\dots} ,$$

which proves that (ii) is a consequence of (3) and (i) unless Θ is identically zero. The uniqueness of Θ follows.

Commutativity of Θ with η. The consistency of Eqs. (1) with Eq. (2) requires that

$$(2.6) \qquad [\eta_i, \eta]\varphi \equiv \eta'_i\varphi = 0 .$$

If this condition is a consequence of Eq. (1) alone ([14]), the set (1) may be said to be complete with respect to η. If it is not, simply include the new equations (6) in the set (1), and repeat the process untill a complete set of subsidiary conditions is obtained. No generality is lost therefore, by assuming that the original set was complete. This is actually true of the set (1.3*b*, *c*, *d*) or (1.3*b*, *c*, *d**).

The commutativity of Θ with η is an immediate consequence of the completeness of the set of subsidiary conditions. The latter may be expressed by the equation

$$[\eta_i, \eta]\,\Theta = 0 ,$$

or

$$\eta_i[\Theta, \eta] = -\eta_i\eta\,\Theta = 0 .$$

Because of (ii), this means that

$$\Theta\,[\Theta, \eta] = [\Theta, \eta] .$$

Since Θ and either η or $i\eta$ are self-adjoint

$$[\Theta, \eta]^\dagger = \mp\,[\Theta, \eta] .$$

Applying the last two equations

$$[\Theta, \eta] = \Theta[\Theta, \eta] = \mp\,\Theta[\Theta, \eta]^\dagger = \mp\,\Theta\{\Theta[\Theta, \eta]\}^\dagger = \mp\,\Theta[\Theta, \eta]^\dagger\,\Theta = \Theta[\Theta, \eta]\Theta = 0 .$$

([14]) If, for example $\eta_1\varphi = \gamma^\mu\varphi_\mu$, $\eta_2\varphi = p^\mu\varphi_\mu$, $\eta = \boldsymbol{p} + im$, then $[\eta_1, \eta]\varphi \equiv \eta'\varphi = 2p^\mu\varphi_\mu = 2\eta_2\varphi = 0$.

This completes the proof that (iii) as well as (ii) are consequences of (i) and of Eq. (3). For integral spin (iii) is trivial; for half-odd integral spin it means that Θ commutes with $\boldsymbol{p}$.

The actual determination of Θ for arbitrary spin has been carried out in Ref. ([6]). The results are, for integral spin s

$$(2.7)\quad \Theta^{\beta_1\ldots\beta_s}_{\alpha_1\ldots\alpha_s} = \left(\frac{1}{s!}\right)^2 \sum_{\substack{P(\alpha)\\P(\beta)}} \left[\prod_{i=1}^{s} \Theta^{\beta_i}_{\alpha_i} + a_1^{(s)}\Theta_{\alpha_1\alpha_2}\Theta^{\beta_1\beta_2}\prod_{i=3}^{s}\Theta^{\beta_i}_{\alpha_i} + \ldots + \right.$$

$$\left. + \begin{cases} a^{(s)}_{(s/2)}\Theta_{\alpha_1\alpha_2}\Theta^{\beta_1\beta_2}\ldots\Theta_{\alpha_{s-1}\alpha_s}\Theta^{\beta_{s-1}\beta_s} \\ a^{(s)}_{(s-1)/2}\Theta_{\alpha_1\alpha_2}\ldots\Theta^{\beta_{s-2}\beta_{s-1}}\Theta^{\beta_s}_{\alpha_s} \end{cases}\right], \quad \begin{matrix} s \text{ even} \\ s \text{ odd} \end{matrix},$$

where

$$a_r^{(s)} = (-\tfrac{1}{2})^r s!\,\{r!\,(s-2r)!\,(2s-1)(2s-3)\ldots(2s-2r+1)\}^{-1}$$

$$\Theta_{\alpha_1\alpha_2} = g_{\alpha_1\alpha_2} - p_{\alpha_1}p_{\alpha_2}/p^2\,.$$

For half-odd-integral spin $s = n + \frac{1}{2}$

$$(2.8)\qquad \Theta^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n}(s) = \frac{2s+1}{4(s+1)}\,\gamma^\alpha\gamma_\beta\Theta^{\beta\beta_1\ldots\beta_n}_{\alpha\alpha_1\ldots\alpha_n}(n+1)\,.$$

This last formula provides a very simple proof of the commutativity of Θ with $\boldsymbol{p}$

$$[\Theta^{\beta_1\ldots\beta_n}_{\alpha_1\ldots\alpha_n}(s), \boldsymbol{p}] = \frac{2s+1}{2(s+1)}\,(\gamma^\alpha p_\beta - p^\alpha\gamma_\beta)\,\Theta^{\beta\beta_1\ldots\beta_n}_{\alpha\alpha_1\ldots\alpha_n}(n+1) = 0\,.$$

One of the main uses of the spin projection is through the formulae

$$(2.9)\qquad \sum \varphi_{\alpha_1\ldots}\bar{\varphi}^{\beta_1\ldots} = \Lambda^+\Theta^{\beta_1\ldots}_{\alpha_1\ldots}\,,$$

for half-odd integer spin, and

$$(2.10)\qquad \sum \varphi_{\alpha_1\ldots}\bar{\varphi}^{\beta_1\ldots} = \Theta^{\beta_1\ldots}_{\alpha_1\ldots}\,,$$

for integer spin, where the sums are over the positive energy solutions of the wave equation and the subsidiary conditions. Equation (2.9) has been used in Sect. **10** and in reference ([6]).

3. – Electromagnetic interaction ([16]).

The first attempt to introduce an interaction between the higher spin field and the Maxwell field was made by DIRAC ([17]), who started from Eq. (1.3)

([15]) R. E. BEHRENDS and C. FRONSDAL: *Phys. Rev.*, **106**, 345 (1957).

([16]) Only half-odd integer spins are considered in Sect. **3**-**6**. Summary results for integer spin are given in Sect. **7**.

([17]) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **155**, 447 (1936).

and carried out the substitution

$$p_\mu \to p_\mu - ieA_\mu . \tag{3.1}$$

Although this substitution represents the only known method of introducing a gauge-invariant direct interaction between A_μ and the particle field, it does not uniquely determine the form of the interaction. Indeed, different theories are obtained by starting from different, though equivalent, formulations of the free field theory.

It was shown by FIERZ and PAULI ([4]) that Dirac's method leads to inconsistent equations for the interaction of φ with an arbitrary external electromagnetic field. An alternative method was proposed, that avoids the inconsistency of the Dirac theory, without completely removing the ambiguity.

Although it is not certain that the Fierz-Pauli theory is self-consistent (in particular difficulties seem to present themselves with regard to quantization ([18])), it has been rather widely accepted. It is, therefore, of some interest to exhibit, in a simple way, the intimate relationship between the various formulations of the Fierz-Pauli theory, as well as the importance of the spin projection operator in this connection.

The theory of Fierz and Pauli is based on the requirement that all the field equations be derivable from a single Lagrangian variational principle ([19]). While this requirement is satisfied by any formulation of the free field theory, the same may not hold after the substitution (1) has been carried out. The method therefore consists of determining a Lagrangian for the free fields, and then carrying out the substitution in the Lagrangian.

Application of Eq. (2.5) shows that the wave equation and subsidiary condition (1.3) may be deduced very simply from the following equation

$$(\Theta \boldsymbol{p} + im)\varphi = 0 . \tag{3.2}$$

In the momentum representation p_μ is a c-number, so that Eq. (4.2) may be obtained from the Lagrangian

$$\mathfrak{L} = \overline{\varphi}(\Theta \boldsymbol{p} + im)\varphi \tag{3.3}$$

by variation with respect to φ. However, since $\Theta \boldsymbol{p}$ involves inverse powers of p^2, it does not seem possible to treat with conventional methods the theory

([18]) S. KUSAKA and J. W. WEINBERG: *University of California Doctoral Dissertation* of J. W. WEINBERG (Berkeley, 1940).

([19]) That is, all equations which are postulated « a priori » (i.e., before the variation of the Lagrangian is carried out.) « A priori » conditions on the wave function are permissible provided there are properly taken into account under the variation: the variations $\delta\varphi_{\mu_1} \ldots$ are not all independent. This was overlooked in the theory of Moldauer and Case.

which would result from applying (1) to Eq. (2). Indeed, it is a practical necessity that the Lagrangian be linear in p_μ.

The remainder of this section deals with the problem of replacing (3) by a Lagrangian which is linear in p_μ, with the objective of relating to each other the various existing formulations of the Fierz-Pauli theory. The discussion is limited to the case of spin $\frac{3}{2}$, since much of the litterature treats this case exclusively. In the following section a more general approach is taken.

The case of spin $\frac{3}{2}$. Eq. (2) may be reduced to the form of a first-order differential equation. The corresponding first order Lagrangian is equivalent to (3) in the sense that variation leads to the same equations (wave equation as well as subsidiary conditions) for φ_μ. After the substitution (1) has been carried out, this will no longer be true.

Eq. (2) may be written, omitting indices

$$-im\varphi = (1 - \tfrac{1}{4}\gamma\gamma)\boldsymbol{p}\Theta\varphi' \,, \tag{3.4}$$

where φ' is the projection of φ which satisfies $\gamma\cdot\varphi = 0$:

$$\varphi' \equiv (1 - \tfrac{1}{4}\gamma\gamma)\varphi \,. \tag{3.5}$$

The explicit expression for Θ is

$$\Theta = 1 - \frac{1}{3}\gamma\gamma - \frac{1}{3p^2}(p\gamma\boldsymbol{p} + \boldsymbol{p}\gamma p) \,. \tag{3.6}$$

When Eq. (6) is inserted into Eq. (4), the result may be written

$$-im\varphi = (1 - \tfrac{1}{4}\gamma\gamma)(\boldsymbol{p}\varphi' - p\psi) \,, \tag{3.7}$$

where

$$\psi \equiv \frac{2}{3p^2}\boldsymbol{p}p\cdot\varphi \tag{3.8}$$

is a four-component spinor field.

Multiplying Eq. (8) by $-im$, and substituting for $-im\varphi$ from Eq. (7), there results

$$-im\psi = -\tfrac{1}{2}\boldsymbol{p}\psi + \tfrac{1}{3}p\cdot\varphi' \,. \tag{3.9}$$

Eqs. (7), (9) are equivalent to the original set (1.3), and are seen to be identical to the equations of Fierz and Pauli, in the form obtained in the Appendix.

Later formulations of the Fierz-Pauli theory do not require the use of the auxiliary field ψ. In the following it will be seen how ψ can be eliminated, and a general theory obtained that includes, as special cases, the theories of Rarita and Schwinger ([5]), Harish-Chandra ([20]) and Moldauer and Case ([7]).

Equation (7) is equivalent to

$$-im\varphi' = (1 - \tfrac{1}{4}\gamma\gamma)(\boldsymbol{p}\varphi' - p\psi)\,, \tag{3.10}$$

$$\gamma\cdot\varphi = 0\,. \tag{3.11}$$

The vanishing of $p\cdot\varphi'$ and hence of ψ does not depend on Eq. (11), but may be inferred from Eqs. (9), (10). Since the latter do not involve the projection

$$\varphi'' = \tfrac{1}{4}\gamma\gamma\cdot\varphi\,,$$

which is orthogonal to φ', it is possible to identify the field ψ (which vanishes by virtue of Eqs. (9), (10)), with the field $\gamma\cdot\varphi = \gamma\cdot\varphi''$ (which vanishes by virtue of Eq. (11)). Thus the auxiliary field ψ may be eliminated by writing

$$\psi = a\gamma\cdot\varphi\,, \qquad a \neq 0. \tag{3.12}$$

This makes Eq. (11) superfluous, while Eqs. (9), (10) become, respectively

$$-im\varphi'' = -\frac{1}{8}\gamma\boldsymbol{p}\gamma\cdot\varphi'' + \frac{1}{12a}\gamma p\cdot\varphi'\,, \tag{3.13}$$

$$-im\varphi' = (1 - \tfrac{1}{4}\gamma\gamma)(\boldsymbol{p}\varphi' - ap\gamma\cdot\varphi'')\,. \tag{3.14}$$

Eqs. (13), (14) may be replaced by any linear combination of them

$$k(3.13) + (3.14)\,, \qquad k \neq 0, \tag{3.15}$$

as is seen by multiplying first by $\frac{1}{4}\gamma\gamma$ and then by $(1 - \frac{1}{4}\gamma\gamma)$

$$\tfrac{1}{4}\gamma\gamma(3.15) = k(3.13)\,,$$

$$(1 - \tfrac{1}{4}\gamma\gamma)(3.15) = (3.14)\,.$$

([20]) Harsh-Chandra: *Phys. Rev.*, **71**, 793 (1947). The theories of References ([5]) and ([7]), as well as the part of Reference ([20]) which is of interest in the present connection, are all equivalent to the Fierz-Pauli theory. The latter, however, suffers from a cumbersome formulation (see Appendix).

When Eq. (15) is written out in full, and multiplied by $\bar{\varphi}$, the Lagrangian

$$(3.16)\qquad \mathcal{L} = \bar{\varphi}\left\{im\left(1-\frac{1}{4}(1-k)\gamma\gamma\right) + \boldsymbol{p} - \left(\frac{1}{2}+a\right)p\gamma + \right.$$
$$\left. + \left(-\frac{1}{2}+\frac{k}{12a}\right)\gamma p + \left(\frac{3}{8}+\frac{a}{4}-\frac{k}{8}-\frac{k}{48a}\right)\gamma\boldsymbol{p}\gamma\right\}\varphi$$

is obtained.

In order that the equation derived from (16) by variation of φ be the adjoint of Eq. (15), the Lagrangian must be Hermitian. This imposes the condition

$$(3.17)\qquad k = -12\,aa^* \neq 0\,,$$

and (16) reduces to, with $A = -a - \frac{1}{2} \neq -\frac{1}{2}$,

$$(3.18)\qquad \mathcal{L} = \bar{\varphi}\{(\boldsymbol{p}+im) + Ap\gamma + A^*\gamma p + \tfrac{1}{2}(3AA^* + A + A^* + 1)\gamma\boldsymbol{p}\gamma -$$
$$- im(3AA^* + \tfrac{3}{2}A + \tfrac{3}{2}A^* + 1)\gamma\gamma\}\varphi\,.$$

This is a two-parameter family of Lagrangians, and reduces to a one-parameter family which is identical to that obtained by MOLDAUER and CASE ([7]), when A is taken to be real. The Rarita-Schwinger ([5]) theory is obtained by the special choice $A = -\frac{1}{3}$. From the theory of Harish-Chandra ([20]), which describes particles of mixed spins, $\frac{1}{2}$ and $\frac{3}{2}$, PETRAS ([21]) has extracted the spin $\frac{3}{2}$ part. Petras' Lagrangian, which is not Hermitian, corresponds to the special choice $k=1$, $a = -\frac{1}{2} \pm \sqrt{\frac{1}{3}}$.

The complete generality of (18) is not guaranted by the above analysis, but is proved rigorously in Sect. **5**. The proof rests on some general theorems concerning first order wave equations for arbitrary spin; these are obtained in Sect. **4** for half-odd integer spin, and in Sect. **7** for integer spin.

4. – The minimal condition.

The most general Euler-Lagrange equation which is obtained from a Hermitian; Lorentz invariant Lagrangian, and which is of the first order in p_μ, and linear in φ, may be written

$$(4.1)\qquad (p^\mu\alpha_\mu + im\beta)\varphi = 0\,,\qquad \alpha_\mu^+ = \alpha_\mu\,,\qquad \beta^+ = \beta\,,$$

where the matrices α_μ and β are form-invariant, numerical matrices ([22]).

([21]) M. PETRAS: *Czech. Journ. Phys.*, **5**, 2 (1955).
([22]) Note that in the spin $\frac{1}{2}$ case, $\alpha_\mu = \gamma_\mu$, $\beta = 1$.

The wave function φ transforms according to some reducible representation D_φ of the Lorentz group. The transformations consist of an « orbital part » and a « spin part », and the two parts commute. Therefore the spin matrices are, by themselves, a representation of the Lorentz group; $D_\varphi^{(s)}(L)$ say. This may be written as a direct sum of irreducible representations $D(i, L)$, with $i = \frac{1}{2}, \frac{3}{2}, \ldots, s$, (we are dealing in this section with half-odd-integer spin only)

$$(4.2) \qquad \begin{cases} D_\varphi^{(s)}(L) = D(s, L) \oplus D(s-1, L) \oplus \ldots \oplus D(s-1, L) \oplus D(s-2, L) \oplus \ldots \\ \qquad\qquad = D(s, L) \oplus \sum_{i=\frac{1}{2}}^{s-1} n_i D(i, L) , \end{cases}$$

The spin value s appears once, while the number n_i is the multiplicity of the spin value i.

Commutation relations for the matrices α_μ and β will now be derived from the following requirements. First, that Eq. (1) be equivalent to the following two equations

$$(4.3) \qquad (1 - \Theta)\varphi = 0 ,$$

$$(4.4) \qquad (\boldsymbol{p} + im)\, \Theta\varphi = 0 .$$

Here $\Theta\varphi$ is the part of φ which transforms according to the first term of Eq. (2). Thus Eq. (3) is the subsidiary condition which requires that the spin be unique, and Eq. (4) is the wave-equation which we impose on $\Theta\varphi$ only. Notice that the representation is not necessarily in terms of Dirac spinors, so that Eq. (4) means that there is a wave equation which would take that form *if* such a representation were used.

Eq. (4) is the condition that the mass be unique, but reflects also the choice of reflection operators. Specifically, it is invariant under the substitution

$$(4.5) \qquad \Theta\varphi(p_4, \boldsymbol{p}) \to \gamma_4\, \Theta\varphi(p_4, -\boldsymbol{p}) .$$

The reason for demanding this invariance is simply the success of the Dirac equation for spin $\frac{1}{2}$. As long as one is discussing free fields only, there is no reason to extend invariance under (5) to hold for the entire wave-function. However, when the electromagnetic interaction is introduced, Eq. (3) will no longer hold. Analogy with the spin $-\frac{1}{2}$ case therefore suggests that Eq. (1) be invariant under a transformation of the form

$$(4.6) \qquad \varphi(p_4, \boldsymbol{p}) \to \gamma_4 \varphi(p_4, -\boldsymbol{p}) .$$

The choice of γ_4 in Eq. (6) is not imperative, but any other choice consistent with Eq. (5) is easily seen to be equivalent for our purpose. An alternative way of stating the invariance under (6) is to require that, if a representation in terms of 4-spinors is used, the α_μ and β be expressible in terms of γ_μ-matrices and the metric tensor (i.e. no σ-matrices or γ_5 will appear).

Thus, the requirements from which commutation relations for α_μ and β will be derived are the equivalence of Eq. (1) to the set (3), (4), plus the invariance of Eq. (1) under (6).

A component of φ may be labelled by the number i, referring to the representation $D(i, L)$ (see Eq. (2)), a number t_i, $(1 \leqslant t \leqslant n_i)$ which refers to one of the n_i representations $D(i, L)$, a number s_i taking on $2i+1$ values and a sign (the sign of the energy). The two latter refer to the $2(2i+1)$ components of $D(i, L)$. Hence

$$\varphi = |t_i, i, s_i, \varepsilon\rangle , \qquad \varepsilon = \pm . \tag{4.7}$$

It is clear that, by definition

$$\langle \varepsilon, s_i, i, t_i | \Theta | t'_i, i', s'_i, \varepsilon' \rangle = \delta_{i,s}\delta_{i',s}\delta_{t_i t'_i}\delta_{s_i s'_i}\delta_{\varepsilon\varepsilon'} . \tag{4.8}$$

Consider the sub-group L^0 of L under which a given momentum vector p^0 is invariant. This was called the little group by WIGNER [10], who proved that $D^{(s)}_\varphi(L^0)$ may be written as the following direct sum of irreducible representations

$$D^{(s)}_\varphi(L^0) = D^+(s, L^0) \oplus D^-(s, L^0) \oplus \sum_{i=\frac{1}{2}}^{s-1} n_i [D^+(i, L^0) \oplus D^-(i, L^0)] . \tag{4.9}$$

Form-invariance of α_μ and β under L, means that $\alpha_\mu p^{0\mu}$ and β commute with $D^{(s)}_\varphi(L^0)$. By Schur's lemma:

$$\langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i', s'_i, \varepsilon' \rangle = \delta_{i,i'}\delta_{s_i,s'_i}\langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i, s_i, \varepsilon' \rangle \tag{4.10}$$

and similarly for β. Invariance under (6) gives (as is easily seen in the rest-system of $p^{0\mu}$)

$$\langle \varepsilon, s_i, i, t_i | \alpha_\mu p^{0\mu} | t'_i, i, s_i, \varepsilon' \rangle = (\boldsymbol{p}^0)_{\varepsilon\varepsilon'} a(i)_{t_i t'_i} ,$$

$$\langle \varepsilon, s_i, i, t_i | \beta | t'_i, i, s_i, \varepsilon' \rangle = \delta_{\varepsilon\varepsilon} b(i)_{t_i t'_i} .$$

Formally

$$\alpha_\mu p^\mu = \boldsymbol{p} \otimes [a(s) \oplus \sum_{i=\frac{1}{2}}^{s-1} a(i)_{t_i t'_i}] , \tag{4.11}$$

$$\beta = [b(s) \oplus \sum_{i=\frac{1}{2}}^{s-1} b(i)_{t_i t'_i}] , \tag{4.12}$$

where $a(i)_{t_i t_i'}$ and $b(i)_{t_i t_i'}$ are numerical matrices diagonal in the dimension i. Here the superscript «0» on $p^{0\mu}$ has been dropped, since Eqs. (11) and (12) are explicitely covariant.

Comparison of Eqs. (8), (11) and (12) gives immediately (since $a(s)$ and $b(s)$ are one-dimensional)

$$\alpha_\mu p^\mu \Theta = \Theta \alpha_\mu p^\mu = \boldsymbol{p} \Theta a(s) , \tag{4.13}$$

$$\beta\Theta = \Theta\beta = \Theta b(s) . \tag{4.14}$$

Hence Eq. (11) may be written as two separate equations

$$[\boldsymbol{p} a(s) + im\, b(s)]\Theta\varphi = 0 , \tag{4.15}$$

$$[\alpha_\mu p^\mu + im\beta](1-\Theta)\varphi = 0 . \tag{4.16}$$

Up to this point the form-invariance of α_μ and β plus invariance of Eq. (1) under (6) have been exploited. There remains the requirement that Eq. (1) be equivalent to Eqs. (3) and (4). In view of the above this means, first, that

$$a(s) = b(s) = 1 , \tag{4.17}$$

and, second, that Eq. (16) reduce to Eq. (3), i.e. that the secular determinant

$$\mathrm{Det}\,[(\alpha_\mu p^\mu + im\beta)(1-\Theta)] \tag{4.18}$$

must be non-zero for all values of m. It is seen that this is impossible for $m=0$, so that this case must be excepted. Next, β must be non-singular, or more precisely, $\beta\varphi$ must have the same number of components as φ. Then there exists a matrix defined by

$$\beta^{-1}\beta\varphi = \varphi .$$

The determinant (4.18) may then be replaced by

$$\mathrm{Det}\,[(\Gamma_\mu p^\mu + im)(1-\Theta)] , \qquad \Gamma_\mu = \beta^{-1}\alpha_\mu . \tag{4.19}$$

According to Eqs. (11) and (12),

$$\Gamma_\mu p^\mu = \boldsymbol{p} \otimes [1 \oplus \sum_{i=\frac{1}{2}}^{s-1} C(i)_{t_i t_i'}] , \qquad C(i) = b^{-1}(i)a(i), \tag{4.20}$$

and (4.18) is the product of the determinants

$$\mathrm{Det}\,[\boldsymbol{p} \otimes C(i)_{t_i t_i'} + im\,\delta_{t_i t_i'}] , \qquad i = s,\ s-1, \ldots, \tfrac{1}{2}. \tag{4.21}$$

These determinants are non-zero only if they are all of the form

$$(im)^{n_i}\,.$$

But every matrix satisfies its characteristic equation, hence

$$[C(i)]^{n_i}=0\,. \tag{4.22}$$

This is the main result. It may be written in a number of ways, the most striking being Eq. (27) below. Conditions of this form are referred to as minimal conditions.

It has been proved that the necessary and sufficient conditions that Eq. (1), with form invariant α_μ and β, be equivalent to Eqs. (3) and (4), and be invariant under (6), are that the matrices satisfy the conditions

$$\Gamma_\mu p^\mu\,\Theta=\Theta\Gamma_\mu p^\mu=\boldsymbol{p}\Theta\,, \tag{4.23}$$

$$\beta\Theta=\Theta\beta=\Theta\,, \tag{4.24}$$

$$\Gamma_\mu p^\mu=\boldsymbol{p}\otimes[1\oplus\sum_{i=\frac{1}{2}}^{s-1}C(i)_{t_it_i'}]\,, \tag{4.25}$$

$$\beta=1\oplus\sum_{i=\frac{1}{2}}^{s-1}b(i)_{t_it_i'}\,, \tag{4.26}$$

$$(\Gamma_\mu p^\mu)^{\bar n}\,\Theta=(\boldsymbol{p})^{\bar n}\,\Theta\,, \tag{4.27}$$

where

$$\Gamma_\mu=\beta^{-1}\alpha_\mu\,, \tag{4.28}$$

and $\bar n$ is the largest of the numbers n_i. Eqs. (23) and (24) are simple restatements of Eqs. (13), (14) and (17). Eq. (27) is completely equivalent to Eq. (22).

Note. – If the wave-function is a tensor spinor of rank $s-\frac{1}{2}$, and symmetric in all the tensor indices, $\bar n=n_{\frac{1}{2}}=s+\frac{1}{2}$, so that

$$(\Gamma_\mu p^\mu)^{s+\frac{1}{2}}=(\boldsymbol{p})^{s+\frac{1}{2}}\,\Theta\,.$$

This can be realized only if $s=\frac{1}{2}$ or $\frac{3}{2}$, since only then is the right-hand side a polynomial in p_μ. This proves a statement by KUSAKA and WEINBERG ([18]), that a symmetric tensor-spinor of rank 2 is insufficient to describe the spin $\frac{5}{2}$ field by a first order wave equation. If, however, a general (i.e., not symmetric) spinor-tensor is used, $\bar n$ will always satisfy the minimum requirement of making the right-hand side of Eq. (27) a polynomial in p_μ.

5. – Wawe equations for spin $\frac{3}{2}$

In Sect. **3** it was seen that the spinor-tensor φ_μ is adequate for the construction of a first-order wave equation for spin $\frac{3}{2}$, and a Lagrangian (Eq. (3.18)) was constructed. With the aid of the results of the preceeding section, the complete generality of that Lagrangian will be demonstrated.

The reduction of φ_μ according to irreducible representations of the Lorentz group may be carried out as follows

$$\varphi_\mu = \begin{bmatrix} \Theta_\mu{}^\nu & \varphi_\nu \\ A^\nu & \varphi_\nu \\ B^\nu & \varphi_\nu \end{bmatrix},$$

with

$$A^\nu\varphi_\nu = \sqrt{(1/12)}\,(\gamma^\nu - 4p^\nu \boldsymbol{p}/p^2)\varphi_\nu\,,$$

$$B^\nu\varphi_\nu = \tfrac{1}{2}\,\gamma^\nu\varphi_\nu\,.$$

Here $\Theta\varphi$ is the spin $\frac{3}{2}$ part, having 8 independent components, and the other two parts have each the four components appropriate to spin $\frac{1}{2}$ fields. Furthermore

$$A\cdot B = B\cdot A = 0\,,$$

$$A\cdot A = B\cdot B = 1\,.$$

In this representation the explicit form of $\Gamma_\lambda p^\lambda$, as given by (4.25) is

$$(\Gamma_\lambda p^\lambda) = \Theta\boldsymbol{p} + AC_{11}\boldsymbol{p}A + AC_{12}\boldsymbol{p}B + BC_{21}\boldsymbol{p}A + BC_{22}\boldsymbol{p}B\,. \tag{5.1}$$

It is now a simple matter to impose the requirements 1), that $\Gamma_\lambda p^\lambda$ be of the first order in p_μ, and 2), that the square of the matrix C_{ij} be zero, as required by Eq. (4.27). Next the most general form of the matrix β is written down

$$\beta = 1 + \text{const}\,\gamma\gamma\,. \tag{5.2}$$

It is immediately found that the Lagrangian

$$\mathfrak{L} = \overline{\varphi}\{\beta\Gamma_\lambda p^\lambda + im\beta\}\varphi\,,$$

with $\Gamma_\lambda p_\gamma$ and β given by (5.1) and (5.2), is identical to the one found in

Sect. 3 (Eq. 3-16). This constitutes a rigorous proof that the latter is indeed the most general Lagrangian which is linear in p_μ, and from which all the field equations may be derived by a single variation, when the wave equation for the spin $\frac{3}{2}$ field is taken to be the tensor-spinor φ_μ. In particular (3.18) is the most general hermitian Lagrangian, under these conditions.

6. – Algebra of the Γ-matrices.

Commutation-relations may be derived for the matrices $\Gamma_\mu = \beta^{-1}\alpha_\mu$ from the minimal condition (4.27) and from (4.23), (4.24). In terms of Γ_μ

$$\left\{\begin{aligned} &(\Gamma_\mu p^\mu)^{\bar n} = \boldsymbol{p}^{\bar n}\,\Theta\,, \\ &(\Gamma_\mu p^\mu)\,\Theta = \Theta\,(\Gamma_\mu p^\mu) = \boldsymbol{p}\Theta\,, \end{aligned}\right. \tag{6.1}$$

or

$$(\Gamma_\mu p^\nu - \boldsymbol{p})(\Gamma_\nu p^\nu)^{\bar n} = 0\,. \tag{6.2}$$

In the rest system

$$(\Gamma_4 - \gamma_4)\Gamma_4^{\bar n} = 0\,. \tag{6.3}$$

In these equations $\bar n$ is the largest number of fields of given spin that appear in the wave-function φ. The esixtence of a relation of the form of Eq. (3) was derived by KUSAKA and WEINBERG [18], who did not give the present definition of the number $\bar n$. Neither was Eq. (1) given by these authors. The weaker condition

$$(\Gamma_4^2 - 1)\Gamma_4^{\bar n} = 0\,, \tag{6.4}$$

has been given by HARISH-CHANDRA [20] and by UMEZAWA and VISCONTI [22]. Although the methods of these authors differ somewhat from each other, the basis of the argument is in each case the requirement that every component of the wave-function satisfy the Klein-Gordon equation. In the present analysis Eq. (4), as well as the stronger condition (2) have been derived from the requirements that the solutions of Eq. (1) describe particles of unique spin, and that a unitary parity operator exist.

In the rest system $\Theta = \Gamma_4^{\bar n}$, by Eq. (1). Hence $\Gamma_4^{\bar n}$ is an idempotent. This can also be seen by iteration of Eq. (4).

Since the p_μ are arbitrary, conditions more general than Eq. (3) may be deduced from Eq. (2)

$$\sum_{P(\mu)} (\Gamma_\mu - \gamma_\mu)\Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar n}} = \sum_{P(\mu)} \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar n}}(\Gamma_\mu - \gamma_\mu) = 0\,, \tag{6.5}$$

and the weaker conditions

$$\sum_{P(\mu)} (\Gamma_\mu \Gamma_{\mu'} - \delta_{\mu\mu'})\Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}} = \sum_{P(\mu)} \Gamma_{\mu_1} \dots \Gamma_{\mu_{\bar{n}}}(\Gamma_\mu \Gamma_{\mu'} - \delta_{\mu\mu'}) = 0 , \tag{6.6}$$

where the sums are over all permutations of the indices. Special examples of Eqs. (5), (6) are needed for the calculation in Sect. **8** of magnetic moments

$$\Gamma_4^{\bar{n}} (\Gamma_\mu - \gamma_\mu) \Gamma_4^{\bar{n}} = 0 , \tag{6.7}$$

$$P_+[\Gamma_i(\sum_{i=0}^{\bar{n}} \Gamma_4{}^i - \tfrac{1}{2}\Gamma_4^{\bar{n}})\Gamma_j + \Gamma_j(\sum_{i=0}^{\bar{n}} \Gamma_4{}^i - \tfrac{1}{2}\Gamma_4^{\bar{n}})\Gamma_i]P_+ = \delta_{ij} , \tag{6.8}$$

$$\Gamma_4 P_\pm = \pm P_\pm , \tag{6.9}$$

where $i, j = 1, 2, 3$ and

$$P_\pm = \tfrac{1}{2}(1 \pm \Gamma_4)\Gamma_4^{\bar{n}} = \tfrac{1}{2}\Gamma_4^{\bar{n}}(1 \pm \Gamma_4) . \tag{6.10}$$

7. – Integer spin fields.

In the case of integer spin, not first-order but second order wave equations are considered

$$(\alpha_{\mu\nu} p^\mu p^\nu + m^2 \beta)\varphi = 0 . \tag{7.1}$$

In the simplest non-trivial case, that of spin 2, it proves sufficient to work with a symmetric, traceless second rank tensor $\varphi_{\mu\nu}$, and one auxiliary scalar field ψ. Starting from the wave equation (in terms of an arbitrary 2-nd rank tensor)

$$(\Theta p^2 + m^2)\varphi = 0 ,$$

a field φ' that satisfies the algebraic subsidiary conditions, is introduced

$$\varphi'_{\mu\nu} = [\tfrac{1}{2}(\delta^\lambda_\mu \delta^\varrho_\nu + \delta^\varrho_\mu \delta^\lambda_\nu) - \tfrac{1}{4}\delta_{\mu\nu}\delta^{\lambda\varrho}]\varphi_{\lambda\varrho} . \tag{7.2}$$

The terms in Θp^2 containing powers of $1/p^2$ are absorbed into a new scalar field ψ, which is subsequently identified with $\varphi_\mu{}^\mu$: There result two equations that may be added in analogy with Eqs. (3.13), (3.14), to give a 2-parameter family of Lagrangians.

Considerations very analogous to those of Sect. **4** may be applied to the integer spin case. The main results of Sect. **4** are contained in Eqs. (4.23-27), and have the following analogues, respectively

(7.3) $$(p\cdot\alpha\cdot p)\Theta = \Theta(p\cdot\alpha\cdot p) = p^2\Theta\,,$$

(7.4) $$\beta\Theta = \Theta\beta = \Theta\,,$$

(7.5) $$p\cdot\Gamma\cdot p = p^2[1\oplus \sum_{i=0}^{s-1} C(i)_{t_i t_i'}]$$

(7.6) $$\beta = [1\oplus \sum_{i=0}^{s-1} b(i)_{t_i t_i'}]\,,$$

(7.7) $$(p\cdot\Gamma\cdot p)^{\bar{n}} = p^{2\bar{n}}\,\Theta\,,$$

where

(7.8) $$\Gamma_{\mu\nu} = \beta^{-1}\alpha_{\mu\nu}\,,$$

and β^{-1} is defined by

$$\beta^{-1}\beta\varphi = \varphi\,.$$

These results may be applied to the derivation of the most general wave equation for spin 2, in terms of the wave function $\varphi_{\mu\nu}$. For a traceless, symmetric $\varphi_{\mu\nu}$, the dimensions of the $C(i)$ matrices are $n_1 = n_0 = 1$. Thus $\bar{n} = 1$, and Eq. (7) cannot be satisfied. Relaxing the trace condition on $\varphi_{\mu\nu}$, there results $n_1 = 1$, $n_0 = \bar{n} = 2$. Alternatively, the symmetry condition may be given up, so that $n_1 = \bar{n} = 3$, $n_0 = 1$.

In the general case of integer spin s, the wave function may be taken to be a tensor of rank s. A completely symmetric tensor cannot be used, since then $\bar{n} = n_0 = \frac{1}{2}(s+2)$ or $\frac{1}{2}(s+1)$, according to whether s is even or odd, respectively. But this does not make the right hand side of Eq. (5) a polynomial, except when $s = 2$. If, instead $\varphi_{\mu\nu\dots}$ is completely traceless, $n_{s-i} = \binom{s}{i}$, which in general is much larger than what is required by Eq. (5).

From Eqs. (3), (4), (5) commutation relations may be derived in the form

$$\sum_{P(\mu)} \Gamma_{\mu_1\mu_2}\dots\Gamma_{\mu_{\bar{n}-1}\mu_n}(\Gamma_{\mu\mu'} - g_{\mu\mu'}) = 0\,.$$

8. – Magnetic moment of fermions.

The magnetic moment of particles defined by an equation of the form

(8.1) $$[\Gamma_\mu\Pi^\mu + im]\varphi = 0\,,\qquad \Pi^\mu = p^\mu - ieA^\mu\,,$$

was considered by HARISH-CHANDRA [24]. HARISH-CHANDRA finds, in the non-

([23]) H. UMEZAWA and A. VISCONTI, see H. UMEZAWA: *Quantum Field Theory* (Amsterdam 1956).

([24]) HARISH-CHANDRA: *Proc. Roy. Soc.*, A **195**, 195 (1948).

relativistic approximation, that Eq. (1) reduces to

$$\left\{(\Pi^4 + im) - \frac{1}{2im}\Pi^i\Pi^j P_+ \Gamma_i (P_- + 2\sum_{i=0}^{\bar{n}-1}\Gamma_4^i P_0)\Gamma_j\right\} P_+\varphi = 0\,, \tag{8.2}$$

where $P_\pm$ were defined by Eq. (6.10) and

$$P_0 = 1 - \Gamma_4^{\bar{n}}\,.$$

This result relies on Eq. (6.4) only, and the number $\bar{n}$ remains unspecified in Harish-Chandra's theory.

Noting that the form invariance of Γ_μ requires that

$$\Gamma_i = \Gamma_4 S_{4i} - S_{4i}\Gamma_4\,, \tag{8.3}$$

where S_{4i} are the spin transformation matrices, there follows by a generalization of a technique due to Petras

$$\begin{aligned} P_+\Gamma_i(P_- + 2\sum_{i=0}^{\bar{n}-1}\Gamma_4^i P_0)\Gamma_j P_+ &= \tfrac{1}{2}P_+(\Gamma_4 S_{4i} - S_{4i}\Gamma_4)(P_- + 2\sum_{j=0}^{\bar{n}-1}\Gamma_4^i P_0)\Gamma_j P_+ + \\ &+ \tfrac{1}{2}P_+\Gamma_i(P_- + 2\sum_{i=0}^{\bar{n}-1}\Gamma_4^i P_0)(\Gamma_4 S_{4j} - S_{4j}\Gamma_4)P_+ = P_+(S_{4i}\Gamma_j - \Gamma_j S_{4j})P_+\,. \end{aligned} \tag{8.4}$$

When this result is introduced into Eq. (2), that equation becomes

$$\left\{(\Pi^4 + im) - \frac{1}{2im}\Pi^i\Pi_i - F^{ij}M_{ij}\right\} P_+\varphi\, 0\,,$$

where the magnetic moment M_{ij} is given by

$$M_{ij} = \frac{e}{2m}P_+(S_{4i}\Gamma_j - S_{4j}\Gamma_i)P_+\,. \tag{8.5}$$

Any relativistic wave function may be considered as being composed of two parts as follows. One part contains the spin s field, and is a tensor-spinor of rank $n = s - \frac{1}{2}$. This part of the complete wave function φ (written without indices) shall be written $\varphi_{\mu_1 \dots \mu_n}$ (with indices). The other part of φ does not contain a spin s field, and must therefore vanish as a consequence of the field equations. This part will be written ψ. The projection $(\Theta\varphi)$ is the set of wave functions which satisfy the following conditions

$$\text{(i)} \qquad p^{\mu_1}\varphi_{\mu_1\ldots\mu_n} = 0\,,$$

$$\text{(ii)} \qquad \gamma^{\mu_1}\varphi_{\mu_1\ldots\mu_n} = 0\,,$$

$$\text{(iii)} \qquad \varphi_{\ldots\mu_i\ldots\mu_j\ldots} = \varphi_{\ldots\mu_j\ldots\mu_i\ldots}\,,$$

$$\text{(iv)} \qquad \psi = 0\,.$$

Let an infinitesimal Lorentz transformation be carried out

$$\begin{aligned} p^\mu &\to p'^\mu\,, \\ \varphi(p) &\to \varphi(p') + \mathrm{d}\omega S_{4i}\,\varphi(p)\,. \end{aligned} \tag{8.6}$$

When these substitutions are made in Eqs. (i)-(iv), there follows that

$$(\eta_r\varphi = 0) \to (\eta_r S_{4i}\varphi = 0)\,.$$

Here $\eta_r\varphi = 0$, $r = 1, 2, 3$ stands for the three conditions (ii), (iii) and (iv). By the definition of Θ this may be written

$$\eta_r S_{4i}\Theta = 0\,,$$

or, in the rest system

$$\eta_r S_{4i} P_+ = 0\,. \tag{8.7}$$

With respect to (iv), this means that the part ψ of the wave function may be ignored in the calculation of the magnetic moment. Then the Γ_i-matrices must be expressed in terms of the invariant matrices $g_{\alpha\beta}$, γ_α, and $\varepsilon_{\alpha\beta\gamma\delta}$. When an expression of this form is introduced into (5), it is immediately recognized that, because of Eq. (7), the only surviving term is that which is diagonal in all tensor indices. By Eq. (6.7), this term is equal to γ_i. Hence

$$M_{ij} = (e/2m)P_+(S_{4i}\gamma_j - S_{4j}\gamma_i)P_+\,. \tag{8.8}$$

The explicit form of S_{4i} is (ignoring the part which acts on ψ)

$$(S_{4i})^{\beta_1\ldots}_{\alpha_1\ldots} = \tfrac{1}{2}\gamma_4\gamma_i\delta^{\beta_1}_{\alpha_1}\ldots + \sum_m \delta^{\beta_1}_{\alpha_1}\ldots(\delta_{4\alpha_m}\delta^{\beta_m}_i - \delta_{i\alpha_m}\delta_4^{\beta_m})\delta^{\beta_{m+1}}_{\alpha_{m+1}}\ldots$$

When this is inserted into (8), all but the first term is annihilated by the projection operator P_+, and there remains

$$M_{ij} = (e/2m)P_+\gamma_i\gamma_j P_+\,.$$

By means of some simple algebra

$$(P_+)^{\beta_1\dots}_{\alpha_1\dots}\gamma_i\gamma_j(P_+)^{\gamma_1\dots}_{\beta_1\dots} =$$
$$= (P_+)^{\beta_1\dots}_{\alpha_1\dots}(\delta^{\delta_1}_{\beta_1} - \tfrac{1}{2}(1 - 1/2s)\gamma_{\beta_1}\gamma^{\delta_1})\gamma_i\gamma_j(\delta^{\varepsilon_1}_{\delta_1} - \tfrac{1}{2}(1 - 1/2s)\gamma_{\delta_1}\gamma^{\varepsilon_1})(P_+)^{\gamma_1\dots}_{\varepsilon_1\delta_2\dots} =$$
$$= (P^+)^{\beta_1\dots}_{\alpha_1\dots}\frac{1}{s}[\tfrac{1}{2}\gamma_i\gamma_j\delta^{\delta_1}_{\beta_1}\dots - \sum_m \delta^{\delta_1}_{\beta_1}\dots(\delta_{i\beta_m}\delta^{\delta_m}_j - \delta_{j\beta_m}\delta^{\delta_m}_i)\dots](P_+)^{\gamma_1\dots}_{\delta_1\beta_1\dots} = \frac{1}{s}S_{ij}P_+ .$$

Hence the eigenvalues of the magnetic moment are given by

$$\langle M_{ij}\rangle = \frac{e}{2m}\,\frac{\langle S_k\rangle}{S}\,, \tag{8.10}$$

where $\langle S_k\rangle$ are the eigenvalues of the k-component of the spin, and $i, j, k = 1, 2, 3$ cyclically.

This formula has been given by MOLDAUER and CASE (7), but derived rigorously only for spin $\frac{3}{2}$ (see footnote (19)). PETRAS (21) gave the correct value for the magnetic moment of spin $\frac{3}{2}$ particles, although the Lagrangian underlying Petras theory is not Hermitian. Recently evidence concerning the spin of the muon has been obtained, which is partly based on Eq. (10). GARWIN, LEDERMAN and WEINRICH (29) have found that the gyromagnetic ratio of the muon is 2.00. The gyromagnetic ratio of particles obeying the Fierz-Pauli equation (1) is given by Eq. (10) to be $1/s$. Hence, if the muon is a Fierz-Pauli particle, its spin must be $\frac{1}{2}$.

9. – Polarization operators.

In addition to the spin projection operators Θ introduced in Sect. **2**, projection operators which select certain values of the z-component of the spin are needed. (For convenience polarization is always referred to the z-direction.) These will be referred to as polarization operators. The simplest example is encountered in the case of spin $\frac{1}{2}$. Polarization operators for this case have been used by MICHEL and WIGHTMAN (25), and BOUCHIAT and MICHEL (26) Defining

$$P(s) = \frac{1}{2s}(S_{\frac{1}{2}} + s)\,, \tag{9.1}$$

where $S_{\frac{1}{2}}$ is the spin operator with eigenvalues $\pm\frac{1}{2}$, it follows that

$$P(s)\psi(s') = \delta_{ss'}\psi(s')\,,$$

(25) L. MICHEL and A. S. WIGHTMAN: *Phys. Rev.*, **98**, 1190 (1955).

(26) C. BOUCHIAT and L. MICHEL: *Compt. Rend. Acad. Sci.*, **243**, 642 (1956). See also C. FRONSDAL and H. ÜBERALL, *Phys. Rev.*, to appear.

where ψ describes a pure spin state. The spin operator $S_{\frac{1}{2}}$ may be taken to be either

$$S'_{\frac{1}{2}} = \frac{1}{4i} S^{\mu\nu}\gamma_\mu\gamma_\nu \quad \text{or} \quad S''_{\frac{1}{2}} = \frac{i}{2}\gamma_5 S^\mu \gamma_\mu ,$$

where, if the momentum is directed along the z-axis

$$S^{\mu\nu} = \delta_1^\mu \delta_2^\nu - \delta_2^\mu \delta_1^\nu , \qquad S^\mu = \left\{0, 0, \frac{E}{m}, \frac{p}{im}\right\} .$$

It is easy to see that

$$S'_{\frac{1}{2}}(\boldsymbol{p} - im) = S''_{\frac{1}{2}}(\boldsymbol{p} - im)$$

so that the two operators are the same when applied to solutions of the Dirac equation. Previous authors have used the operator $S''_{\frac{1}{2}}$, here the alternative $S'_{\frac{1}{2}}$ will be found advantageous. (The prime will be dropped henceforth.)

The generalization of (1) to the case of a total spin of $n+\frac{1}{2}$ with z-component s is

$$P_n(s) = \mathrm{d}(n, s) \prod_{s' \neq s} (S_{n+\frac{1}{2}} - s') , \tag{9.2}$$

where s' runs over the eigenvalues of the z-component of the spin, and

$$(S_{n+\frac{1}{2}})^{\beta_1 \dots \beta_n}_{\alpha_1 \dots \alpha_n} = S_{\frac{1}{2}} \delta^{\beta_1}_{\alpha_1} \dots \delta^{\beta_n}_{\alpha_n} + \frac{1}{i} \sum_{m=1}^{n} \delta^{\beta_1}_{\alpha_1} \dots S^{\beta_m}_{\alpha_m} \dots \delta^{\beta_n}_{\alpha_n} . \tag{9.3}$$

The normalization factor may be found by noting that

$$S_{n+\frac{1}{2}} P_n(s) = s P_n(s) .$$

Hence

$$\mathrm{d}(n, s) = \Big\{\prod_{s' \neq s} (s - s')\Big\}^{-1} ,$$

and

$$P_n(s) = \Big\{\prod_{s' \neq s} (s - s')\Big\}^{-1} \prod_{s' \neq s} (S_{n+\frac{1}{2}} - s') . \tag{9.4}$$

The polarization operator commutes with the spin operator, as is seen by verifying that

$$p^{\alpha_1} (S_{n+\frac{1}{2}} \Theta)^{\beta_1 \dots}_{\alpha_1 \dots} = 0 ,$$

$$\gamma^{\alpha_1} (S_{n+\frac{1}{2}} \Theta)^{\beta_1 \dots}_{\alpha_1 \dots} = 0 .$$

Examples of (4) that will be used in the following section are, for spin $\frac{3}{2}$ and $\frac{5}{2}$, respectively

$$P_1(s) = \{2s(s^2 - 9/16s^2)\}^{-1}(S_{\frac{3}{2}} + s)(S_{\frac{3}{2}}^2 - 9/16s^2)\,, \tag{9.5}$$

$$P_2(s) = \{2s(2s^4 - 35s^2/4 + 225/64s^2)\}^{-1}(S_{\frac{5}{2}} + s)\cdot \cdot [S_{\frac{5}{2}}^4 + (s^2 - 35/4)S_{\frac{5}{2}}^2 + 225/64s^2]\,. \tag{9.6}$$

When only the magnitude, but not the sign of the spin component is relevant, the appropriate projection operators are

$$P_n(|s|) = P_n(s) + P_n(-s) = \{\prod_{|s'|\neq|s|}(s^2 - s'^2)\}^{-1}\prod_{|s'|\neq|s|}(S_{n+\frac{1}{2}}^2 - s'^2)\,.$$

10. – Angular distributions.

Calculations on angular distributions of the decay products in interactions involving particles of higher spin will be limited to one particular case: the decay of hyperons through the scheme

$$\mathcal{H} \to \mathcal{N} + \pi\,,$$

where $\mathcal{H}$ is a hyperon and $\mathcal{N}$ is a nucleon. Of all the elementary particles which are known or suspected to exist, the hyperons are most likely to have higher spin (1,2).

The angular distributions of the hyperon decay products have been calculated by ADAIR (27) and TREIMAN (28). The present calculations differ in two respects. First, ADAIR and TREIMAN assumed that parity is conserved in the decay. This assumption has been found to be inconsistent with observed facts, in recently performed experiments (30), and will not be made here. Second, earlier calculations have been carried out by means of Clebsch-Gordon coefficients, while the present method employs the spin projection operators and the polarization operators. Because the parity non-conserving interaction has a parity conserving part, a partial comparison with the results of ADAIR and TREIMAN can be made.

(27) R. K. ADAIR: *Phys. Rev.*, **100**, 1540 (1955).

(28) S. B. TREIMAN: *Phys. Rev.*, **101**, 1216 (1956).

(29) C. S. WU, E. AMBLER, R. W. HAYWARD, D. D. HOPPES and R. P. HUDSON: *Phys. Rev.*, **105**, 1413 (1957); R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **105**, 1415 (1957).

(30) *Venice Conference*, 1957.

The most general direct interaction is given by the following matrix element (no attention is paid to constant factors, since the interest is in the distributions)

$$K = \overline{\psi}_2(p_2)(1 + a\gamma_5)p_2 \dots p_2 P_n(s)\psi_1(p_1)\varphi(p_1 - p_2)\,,$$

where ψ_1, ψ_2 and φ are the wave functions of the hyperon (spin $n+\frac{1}{2}$, z-component s), the nucleon and the pion, respectively. Non conservation of parity is caused by interference of the two terms in the factor $(1+a\gamma_5)$. The transition probability, summed over the spins of the two fermions, is calculated by means of Eq. (2.9), and is found to be given by

$$|K|^2 = \mathrm{Tr}\{p_2 \dots p_2 P_n(s)\ \Theta(p_1)p_2 \dots p_2(\boldsymbol{p}_1 - im_1)(1 - a^*\gamma_5)(\boldsymbol{p}_2 - im_2)(1 + a\gamma_5)\}.$$

Using the subsidiary conditions, this can be reduced to

$$(10.1)\qquad |K|^2 \approx \left[\frac{1 + aa^*}{2}\, p_1\cdot p_2 - \frac{1 - aa^*}{2}\, m_1 m_2\right] T_1 + \frac{a + a^*}{2}\, T_2\,,$$

where

$$T_1 = \mathrm{Tr}\{p_2 \dots p_2 \prod_{|s'|\neq|s|} (S^2_{n+\frac{1}{2}} - s'^2)\ \Theta(p_1)p_2 \dots p_2\},$$

$$T_2 = \mathrm{Tr}\left\{p_2 \dots p_2 \frac{1}{s}\, S_{n+\frac{1}{2}} \prod_{|s'|\neq|s|} (S^2_{n+\frac{1}{2}} - s'^2)\,\Theta(p_1)p_2 \dots p_2\boldsymbol{p}_1\boldsymbol{p}_2\gamma_5\right\}.$$

The first trace, T_1, is the parity conserving part of the angular distribution. The second trace depends on the relative signs of s and the z-component of the nucleon momentum. This correlation between the directions of an axial vector and a polar vector is characteristic of a transition in which parity is not conserved.

The quantities T_1 and T_2 can be calculated by making use of the explicit form for the spin projection operators given by (2.7) and (2.8). The results are as follows: In the rest system of the hyperon Eq. (1) reduces to

$$(10.2)\qquad |K|^2 \sim \left[\frac{1 + aa^*}{2}\, p_1\cdot p_2 - \frac{1 - aa^*}{2}\, m_1 m_2\right] T_1 + \frac{a + a^*}{2}\,\frac{m_1}{2s}\,|\vec{p}_2|\cos\vartheta\, T_2'\,,$$

where ϑ is the angle between the hyperon spin and the nucleon momentum.

For spin $\frac{1}{2}$, T_1 and T_2' are equal to unity.

For spin $\frac{3}{2}$

$$(10.3)\qquad \begin{cases} T_1 = |\vec{p}_2|^2 \left\{\dfrac{7}{6} - \dfrac{3}{8s^2} - \cos^2\vartheta\right\}, \\[2ex] T_2' = |\vec{p}_2|^2 \left\{\dfrac{19}{6} - \dfrac{3}{8s^2} - 3\cos^2\vartheta\right\}. \end{cases}$$

For spin $\frac{5}{2}$ the results are

$$(10.4)\quad \begin{cases} T_1 = \dfrac{1}{5}|\vec{p}_2|^4\left\{\dfrac{17}{2}s^2 - \dfrac{97}{4} + \dfrac{225}{32s^2} - (8s^2+10)\cos^2\vartheta + 30\cos^4\vartheta\right\}, \\ T_2' = \dfrac{1}{5}|\vec{p}_2|^4\left\{\dfrac{49}{2}s^2 - \dfrac{17}{4} + \dfrac{225}{32s^2} - (24s^2+150)\cos^2\vartheta + 150\cos^4\vartheta\right\}. \end{cases}$$

If a hyperon is produced in a scalar interaction involving a nucleon and various particles of spin zero, the component of its spin in the direction of flight is $\pm\frac{1}{2}$. If the two components of the hyperon beam—that is the spin $+\frac{1}{2}$ component and the spin $-\frac{1}{2}$ component—decay incoherently and are of equal strength, or if the decay conserves parity (that is, if a is zero), the transition probability is given by T_1 alone. This case was considered by ADAIR [27] and TREIMAN [28]. Setting $s^2=\frac{1}{4}$ in Eqs. (3) and (4), the result is:

for spin $\frac{3}{2}$

$$T_1 \sim 1 + 3\cos^2\vartheta$$

and for spin $\frac{5}{2}$

$$T_1 \sim 1 - 2\cos^2\vartheta + 5\cos^4\vartheta\,.$$

These angular distributions are the same as those found by ADAIR and TREIMAN. The more complete results of this work are contained in Eqs. (2), (3) and (4).

* * *

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APPENDIX

The connection between the spinor representation used by FIERZ and PAULI, and the tensor representation employed in the presen paper, is illustrated by transforming the spin $\frac{3}{2}$ wave equations from one representation to the other. Even though the spinor representation is the more complicated one, it is useful for gaining new insight into the meaning of the subsidiary conditions, and the nature of the spin projection operator.

The van der Waerden spinors that form a basis for a spin $\frac{3}{2}$ representation of the proper elements of the Lorentz group are of rank 3. There are four kinds: a_{ABC}, $a^{\dot{A}}{}_{CB}$, $a^{\dot{A}\dot{B}}{}_{C}$ and $a^{\dot{A}\dot{B}\dot{C}}$. The improper elements of the Lorentz group transform a_{ABC} into $a^{\dot{A}\dot{B}\dot{C}}$, $a^{\dot{A}}{}_{BC}$ into $a^{\dot{A}\dot{B}}{}_{C}$, and vice versa. Thus, to represent the Lorentz group the set a_{ABC}, $a^{\dot{A}\dot{B}\dot{C}}$ or the set $a^{\dot{A}}{}_{BC}$, $a^{\dot{A}\dot{B}}{}_{C}$ is needed. These representations are reducible. The subsidiary conditions satisfied by the irreducible spin $\frac{3}{2}$ parts are

$$a_{ABC}\,, \quad a^{\dot{A}\dot{B}\dot{C}}\,, \quad \text{completely symmetric}\,, \tag{A-1}$$

$$a^{\dot{A}}{}_{BC} = a^{\dot{A}}{}_{CB}\,, \qquad a^{\dot{A}\dot{B}}{}_{C} = a^{\dot{B}\dot{A}}{}_{C}\,, \tag{A-2}$$

$$p^{B}_{\dot{A}} a^{\dot{A}}{}_{BC} = 0\,, \qquad p^{C}_{\dot{A}} a^{\dot{A}\dot{B}}{}_{C} = 0\,, \tag{A-3}$$

where

$$p_{A\dot{A}} = p_\mu \sigma^\mu{}_{A\dot{A}}\,,$$

and $\sigma_4 = iI$, I being the unit 2-by-2 matrix.

The first set possesses an advantage with regard to simplicity. The *spin projection operator is simply the complete symmetrizer*, and the redundant components may be completely eliminated. This advantage is off-set, however, by the non-existence of a first order wave equation connecting a_{ABC} and $a^{\dot{A}\dot{B}\dot{C}}$. For this reason DIRAC, and all subsequent authors, have considered the second set. The wave equation is

$$\begin{cases} m a^{\dot{A}}{}_{BC} = p_{B\dot{B}} a^{\dot{A}\dot{B}}{}_{C}\,, \\ m a^{\dot{A}\dot{B}}{}_{C} = p^{\dot{B}B} a^{\dot{A}}{}_{BC}\,. \end{cases} \tag{A-4}$$

DIRAC suggested introducing the electromagnetic interaction by means of the substitution

$$p_\mu \to p_\mu - ieA_\mu\,. \tag{A-5}$$

However, as pointed out by FIERZ and PAULI, this leads to serious difficulties. They therefore introduced two auxiliary fields, C_A and $C^{\dot{A}}$, with the help of which the free field equations (2), (3) and (4) can be derived from a single Lagrangian principle. There also follows from this that

$$C_A = C^{\dot{A}} = 0\,. \tag{A-6}$$

The substitution (5) may now be carried out in the Lagrangian, or in the Euler-Lagrange equations. Of Eqs. (2), (3), (4), (6) only Eq. (2) remains valid in the presence of an interaction.

The free field Euler-Lagrange equations are

$$\begin{cases} m a^{\dot{A}\dot{B}}{}_{C} = \frac{1}{2}(\delta^{\dot{A}}_{\dot{E}}\delta^{\dot{B}}_{\dot{F}} + \delta^{\dot{B}}_{\dot{E}}\delta^{\dot{A}}_{\dot{F}})(p^{\dot{E}D} a^{\dot{F}}{}_{CD} + p^{\dot{F}}_{C} C^{\dot{E}})\,, \\ m a^{\dot{C}}{}_{AB} = \frac{1}{2}(\delta^{E}_{A}\delta^{F}_{B} + \delta^{E}_{B}\delta^{F}_{A})(p_{E\dot{D}} a^{\dot{C}\dot{D}}{}_{F} + p^{\dot{C}}_{F} C_{E})\,, \\ m C_A = -\frac{1}{2} p_{A\dot{B}} C^{\dot{B}} + \frac{1}{6} p^{B}_{\dot{C}} a^{\dot{C}}{}_{AB}\,, \\ m C_{\dot{A}} = -\frac{1}{2} p^{\dot{A}B} C_B + \frac{1}{6} p^{C}_{\dot{B}} a^{\dot{A}\dot{B}}{}_{C}\,. \end{cases} \tag{A-7}$$

Introduce tensor indices by writing

$$a^{\dot{A}\dot{B}}{}_C = \sigma^{\mu\dot{A}}{}_C\, a^{\dot{B}}_\mu\,, \qquad a^{\dot{A}}_\mu = \tfrac{1}{2}\sigma_\mu{}^C{}_{\dot{B}}\, a^{\dot{A}\dot{B}}{}_C\,.$$

Eq. (7) then assumes the form

$$\begin{aligned} m a^{\dot{A}}_\mu &= (\eta^\nu_\mu)^{\dot{A}}_{\dot{E}}(p^{\dot{E}C} a_{\nu C} + p_\nu C^{\dot{E}})\,,\\ m a_{\mu A} &= (\eta^\nu_\mu)^E_A(p_{E\dot{C}} a^{\dot{C}}_\nu + p_\nu C_E)\,,\\ m C_A &= -\tfrac{1}{2} p_{A\dot{B}} C^{\dot{B}} + \tfrac{1}{3} p^\mu a_{\mu A}\,,\\ m C^{\dot{A}} &= -\tfrac{1}{2} p^{\dot{A}B} C_B + \tfrac{1}{3} p^\mu a^{\dot{A}}_\mu\,, \end{aligned}$$

where

$$\begin{aligned} (\eta^\nu_\mu)^{\dot{A}}_{\dot{E}} &= \tfrac{1}{2}\sigma_{\mu\dot{B}}^{\ C}\tfrac{1}{2}(\delta^{\dot{A}}_{\dot{E}}\delta^{\dot{B}}_{\dot{F}} + \delta^{\dot{B}}_{\dot{E}}\delta^{\dot{A}}_{\dot{F}})\,\sigma^{\nu\dot{F}}_{\ \ C} = \tfrac{1}{2}\,\delta^\nu_\mu\delta^{\dot{A}}_{\dot{E}} + \tfrac{1}{4}\sigma_{\mu\dot{E}}^{\ C}\sigma^{\nu\dot{A}}_{\ \ C}\,,\\ (\eta^\nu_\mu)^E_A &= \tfrac{1}{2}\delta^\nu_\mu\,\delta^E_A + \tfrac{1}{4}\sigma^E_{\mu\,\dot{C}}\sigma^{\nu\,\dot{C}}_{\ \ A}\,. \end{aligned}$$

Next introduce the Dirac 4-spinors. Suppressing spinor indices

$$\varphi_\mu = \begin{pmatrix} a_{\mu A} \\ a_\mu{}^{\dot{A}} \end{pmatrix}, \qquad \psi = \begin{pmatrix} iC_A \\ iC^{\dot{A}} \end{pmatrix},$$

and introducing the γ-matrices

$$\gamma_\mu = \frac{1}{i}\begin{pmatrix} & \sigma_{\mu\dot{A}B} \\ \sigma_\mu{}^{\dot{A}B} & \end{pmatrix},$$

the wave equations finally reduce to

$$\text{(A-8)}\qquad \begin{cases} -im\varphi_\mu = (\delta^\nu_\mu - \tfrac{1}{4}\gamma_\mu\gamma^\nu)(\boldsymbol{p}\varphi_\nu - p_\nu\psi)\,,\\ -im\psi \ = -\tfrac{1}{2}\boldsymbol{p}\psi + \tfrac{1}{3}p^\mu\varphi_\mu\,. \end{cases}$$

Equations (8) may also be written

$$\text{(A-9)}\qquad (p^\mu\Gamma_\mu + im)\chi = 0\,,$$

where

$$\chi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \psi \end{pmatrix}, \qquad p^\mu\Gamma_\mu = \left(\begin{array}{c:c} & -p_1 + \tfrac{1}{4}\gamma_1\boldsymbol{p} \\ (\delta^\nu_\mu - \tfrac{1}{4}\gamma_\mu\gamma^\nu)\boldsymbol{p} & -p_2 + \tfrac{1}{4}\gamma_2\boldsymbol{p} \\ & -p_3 + \tfrac{1}{4}\gamma_3\boldsymbol{p} \\ & -p_4 + \tfrac{1}{4}\gamma_4\boldsymbol{p} \\ \hdashline \tfrac{1}{3}p^1,\ \tfrac{1}{3}p^2,\ \tfrac{1}{3}p^3,\ \tfrac{1}{3}p^4 & -\tfrac{1}{2}\boldsymbol{p} \end{array}\right)$$

Equation (9) is of the same form as that obtained by GUPTA ([6]). Although χ has 20 components, as compared with the 16 components in Gupta's theory, a considerable simplification is achieved in writing the Γ_μ matrices in terms of the γ-matrices. In Gupta's theory the four matrices must be given expliticly. It should be emphasized that the three formulations of the Fierz-Pauli theory are completely equivalent. Equation (9) shall not be considered further, since a yet simpler formulation (Eq. (3.18)) exists.