The Fluctuations of Intensity of an Extended Light Source.

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(ricevuto l'11 Gennaio 1959)

Summary. — Extending calculations carried out before (3), the present paper is dealing with the exact statistics of the fluctuation of wave trains emitted by a source the extension of which cannot be neglected. As before the individual wave trains are supposed to decay exponentially, however, it is assumed here that the individual trains have different frequencies, intensities and polarization. Various experimentally observed effects, among them that obtained with the stellar interferometer by HANBURY BROWN and TWISS (1) can be quantitatively accounted for.

1. – A beam emitted by any light source shows fluctuations of intensity; the fluctuations are caused by the random superpositions of the wave trains emitted by the individual atoms of the source. This problem was dealt with phenomenologically by HANBURY BROWN and TWISS (1) and others; the quantum mechanical treatment was considered recently by MANDEL (2) and we have also dealt with this problem previously (3). However, so as to predict quantitatively effects caused by this fluctuation it is necessary to consider the problem in more detail.

In the first part of this paper we determine the simultaneous distribution functions of the electric vectors produced by the source at given times in given points of the receiver and their time derivatives. In the second part we shall determine certain experimentally observable quantities and discuss a number of effects.

(1) R. HANBURY BROWN and R. Q. TWISS: *Proc. Roy. Soc.*, A **242**, 300 (1957).
(2) L. MANDEL: *Proc. Roy. Soc.*, **72**, 1037 (1958).
(3) L. JÁNOSSY: *Nuovo Cimento*, **6**, 111 (1957).

2. – Consider a light source situated around a point P and a receiver situated around a point Q, we denote

$$\overrightarrow{PQ} = \boldsymbol{L} \qquad \text{and} \qquad L = |\boldsymbol{L}|\,.$$

The atoms of the source are situated in points $\boldsymbol{P}_i$, while points of the receiver may be denoted Q_k, we shall write

$$\overrightarrow{PP}_i = \boldsymbol{R}_i\,, \qquad \overrightarrow{QQ}_k = \boldsymbol{r}_k\,.$$

Thus the vector pointing from P_i to Q_k is given by

$$\boldsymbol{L}_{ik} = \boldsymbol{L} + \boldsymbol{r}_k - \boldsymbol{R}_i\,, \qquad L_{ik} = |\boldsymbol{L}_{ik}|\,. \tag{1}$$

We shall suppose that the atom in P_i is suddenly excited at an instant T_i and thus starts to emit an exponentially decaying wave band. The front of the wave band arrives at a time $T_i + L_{ik}/c$ in the point Q_k. The electric vector of the wave train in Q_k at a time t_k can be written

$$\boldsymbol{E}_i^{(k)} = \boldsymbol{E}_i e(\gamma t_{ik}) \cos(\omega_i t_{ik} + 2\pi\varphi_i) \tag{2}$$

with

$$t_{ik} = T_i - t_k + L_{ik}/c\,,$$

where γ is the damping constant of the emitting atom, ω_i the frequency emitted, φ_i the phase. $\boldsymbol{E}_i$ is a vector giving polarization and intensity of the emission, finally

$$e(x) = \begin{cases} e^{-x}\,, & x > 0\,, \\ 0\,, & x < 0\,. \end{cases} \tag{3}$$

Strictly speaking, the vector $\boldsymbol{E}_i$ depends also on the position of the point Q_k. We shall, however, assume that both the light source and the receiver have dimensions small as compared with L, thus we assume

$$R_i\,, r_k \ll L\,, \tag{4}$$

and therefore we shall neglect in our calculation the dependence of $\boldsymbol{E}_i$ upon the position of the point Q_k. Further we shall use such an approximation that we may assume $\boldsymbol{E}_i$ to be perpendicular to $\boldsymbol{L}$; in order to investigate effects of polarization, we fix two directions « 1 » and « 2 » perpendicular to each other and perpendicular to $\boldsymbol{L}$; the components of $\boldsymbol{E}_i$ in these directions may be denoted by $E_i^{(l)}$, $l = 1, 2$. The corresponding components of the field strength in Q_k at t_k can be written

$$E_i^{(kl)} = E_i^{(l)} e(\gamma t_{ik}) \cos(\omega_i t_{ik} + 2\pi\varphi_i^{(l)}) , \qquad l = 1, 2.$$

If $\varphi_i^{(1)} \neq \varphi_i^{(2)}$, the radiation emitted is elliptically polarized.

Finally, in order to be able to work out *intensities* of the radiation, it is useful to introduce quantities which are proportional to the time derivatives of the field strength. We shall write

$$E_i^{(klm)} = E_i^{(l)} e(\gamma t_{ik}) c_m(\omega_i t_{ik} + 2\pi\varphi_i^{(l)}) , \qquad l = 1, 2, \tag{5}$$

where we suppose

$$c_m(x) = \begin{cases} \cos x , & m = 1 , \\ \sin x , & m = 2 . \end{cases}$$

The four quantities $E_i^{(klm)}$, $l, m = 1, 2$ fully characterize the effect of the emission of the atom in P_i in the point Q_k at t_k. The total field strength in Q_k at t_k is given by the four quantities

$$E^{(klm)} = \sum_i E_i^{(klm)} , \qquad l, m = 1, 2. \tag{6}$$

We shall be interested in the simultaneous values of the field strength in two points say Q_1 and Q_2 at times t_1 and t_2, respectively. The state is described thus by eight quantities $E^{(\alpha)}$, where we suppose that α can take eight values corresponding to the eight values of the triple index klm with $k, l, m = 1, 2$. We shall also denote these eight components by one symbol

$$\mathfrak{E} = E^{(\mathrm{I})}, E^{(\mathrm{II})}, \ldots, E^{(\mathrm{VIII})}, \tag{7}$$

where I, II, ..., VIII stand for 111, 112, ..., 222, respectively.

3. – Presently we determine the simultaneous probability distribution of the eight components of $\mathfrak{E}$. We denote this distribution by $P(\mathfrak{E})$; we shall restrict ourselves to the determination of the logarithmic generating function of $P(\mathfrak{E})$, thus

$$H(\mathfrak{v}) = \ln \int \exp(\mathfrak{E}\mathfrak{v}) P(\mathfrak{E}) \, d\mathfrak{E} , \tag{8}$$

where the integral is an eightfold integral of the eight components of $\mathfrak{E}$ and

$$\mathfrak{E}\mathfrak{v} = \sum_{\alpha=\mathrm{I}}^{\mathrm{VIII}} E^{(\alpha)} v_\alpha ;$$

$$\mathfrak{v} = v_{\mathrm{I}}, v_{\mathrm{II}}, \ldots, v_{\mathrm{VIII}}$$

are the eight transformation parameters corresponding to the components of $\mathfrak{E}$.

So as to determine $P(\mathfrak{E})$ or $H(\mathfrak{v})$ we have to make assumptions about the emissions. Each emission can be characterized by the following parameters

$$\boldsymbol{R}\,,\ \ \boldsymbol{E},\ \ T\,,\ \ \omega\,,\ \ \varphi^{(1)},\ \ \varphi^{(2)},$$

namely, point and time of emission, amplitude and phases of emission. It will be convenient to split $\boldsymbol{R}$ into two components

$$\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B}\,, \tag{9}$$

where $\boldsymbol{A}$ is parallel to $\boldsymbol{L}$, while $\boldsymbol{B}$ is perpendicular to $\boldsymbol{L}$; furthermore, we are interested in the components $E^{(1)}$ and $E^{(2)}$ of $\boldsymbol{E}$. We characterize thus an emission more precisely by a set of parameters , which we denote by a symbol $\mathfrak{A}$, namely

$$\mathfrak{A} = T,\ A,\ \boldsymbol{B},\ E^{(1)},\ E^{(2)},\ \omega,\ \varphi^{(1)},\ \varphi^{(2)}\,. \tag{10}$$

We suppose the probability of an emission inside an interval $\mathfrak{A}, \mathfrak{A} + \mathrm{d}\mathfrak{A}$ to be equal to

$$\overline{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A}\,.$$

The probability density $\overline{p}(\mathfrak{A})$ shall be assumed not to depend explicitly either on the time T or on the average phase

$$\varphi = \tfrac{1}{2}(\varphi^{(1)} + \varphi^{(2)})\,.$$

Thus we may write

$$\overline{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A} = N p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,\mathrm{d}T\,\mathrm{d}\varphi\,, \tag{11}$$

where the symbol $\mathfrak{a}$ stands for the parameters

$$\mathfrak{a} = A,\ \boldsymbol{B},\ E^{(1)},\ E^{(2)},\ \omega,\ \psi \tag{12}$$

only, and where we have introduced

$$\psi = \varphi^{(2)} - \varphi^{(1)}\,.$$

N is the number of impulses emitted per unit time. The quantity

$$n = N/2\gamma \tag{13}$$

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has the dimension of a pure number and can be regarded as a measure of the overlap of the exponentially decreasing wave bands. In most practical cases we may suppose

$$n \gg 1 .$$

So as to determine the generating function $H(\mathfrak{v})$ we have to introduce the functions

$$\mathfrak{E}(\mathfrak{A}) = E^{(\mathrm{I})}(\mathfrak{A}),\ E^{(\mathrm{II})}(\mathfrak{A}),\ \ldots,\ E^{(\mathrm{VIII})}(\mathfrak{A}) ,\tag{14}$$

with

$$E^{(\alpha)}(\mathfrak{A}) = E^{(l)} e(\gamma t_{\mathfrak{A}k}) c_m(\omega t_{\mathfrak{A}k} + 2\pi\varphi^{(l)})\tag{15}$$

and

$$t_{\mathfrak{A}k} = T - t_k + |\boldsymbol{L} - \boldsymbol{R} + \boldsymbol{r}_k|/c .\tag{16}$$

$E^{(\alpha)}(\mathfrak{A})$ gives the eight components of the field strength which arise in Q_1 and Q_2 at the times t_1 and t_2, provided an emission took place with the parameters $\mathfrak{A}$ inside the source.

As it was shown elsewhere ([4]) the generating function $H(\mathfrak{v})$ can be written as

$$H(\mathfrak{v}) = \int (\exp \mathfrak{v}\,\mathfrak{E}(\mathfrak{A}) - 1)\,\bar{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A} ,\tag{17}$$

where we have put

$$\mathfrak{v}\mathfrak{E}(\mathfrak{A}) = \sum_{\alpha=\mathrm{I}}^{\mathrm{VIII}} v_\alpha E^{(\alpha)}(\mathfrak{A}) .$$

4. – With the help of the generating function (17) we can determine the moments of the distribution $P(\mathfrak{E})$. We shall denote the derivatives of $H(\mathfrak{v})$ into $v_\alpha, v_\beta, \ldots, v_\varepsilon$ at the point $\mathfrak{v} = 0$ by H with suitable suffixes. Thus we write

$$\left(\frac{\partial H(\mathfrak{v})}{\partial v_\alpha}\right)_{\mathfrak{v}=0} = H_\alpha , \qquad \left(\frac{\partial^2 H(\mathfrak{v})}{\partial v_\alpha\,\partial v_\beta}\right)_{\mathfrak{v}=0} = H_{\alpha\beta} , \ \ldots .\tag{18}$$

Differentiating (17) into $v_\alpha, v_\beta, \ldots, v_\varepsilon$ we find for $\mathfrak{v} = 0$

$$H_{\alpha\beta\ldots\varepsilon} = \int E^{(\alpha)}(\mathfrak{A}) E^{(\beta)}(\mathfrak{A}) \ldots E^{(\varepsilon)}(\mathfrak{A})\,\bar{p}(\mathfrak{A})\,\mathrm{d}\mathfrak{A} .\tag{19}$$

Introducing the explicit expressions for $E^{(\alpha)}(\mathfrak{A})$ from (15) we find with the help of (11) that on account of the averaging over the phase φ

$$H_\alpha = 0 , \qquad \alpha = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VIII};\tag{20}$$

([4]) G. Graff and L. Jánossy: in press. (*Acta Phys. Hung.* **10**, n. 3).

similarly, all odd order derivatives of H vanish for $\mathfrak{v}=0$, in particular

$$H_{\alpha\beta\gamma}=0\,. \tag{21}$$

Differentiating (8) one to four times, we find with the help of (20) the following expressions, which will be used further below

$$\left\{\begin{aligned} &\langle E^{(\alpha)}\rangle = H_\alpha = 0\,, \qquad \langle E^{(\alpha)}E^{(\beta)}\rangle = H_{\alpha\beta}\,,\\ &\langle E^{(\alpha)^2}E^{(\beta)^2}\rangle - \langle E^{(\alpha)^2}\rangle\langle E^{(\beta)^2}\rangle = H_{\alpha\alpha\beta\beta} + 2H^2_{\alpha\beta}\,,\end{aligned}\right. \tag{22}$$

the meaning of the suffixes α, β is as follows:

$$\alpha = klm\,,\qquad \beta = KLM\,,\qquad k,\,l,\,m,\,K,\,L,\,M = 1,\,2\,. \tag{23}$$

The expressions (22) can be evaluated using (19) and (15). We find with the help of (11) and (13) when we carry out the integration into T

$$\begin{aligned} H_{\alpha\beta} = n\int \exp\,[-\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|]\,c_m(\omega t_{\mathfrak{A}k}+2\pi\varphi^{(l)})\,\cdot&\\ \cdot\, c_M(\omega t_{\mathfrak{A}K}+2\pi\varphi^{(L)})\,E^{(l)}E^{(L)}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\varphi\,.&\end{aligned} \tag{24}$$

The above expression can be simplified if we put $m=M$ and sum over this index; we find

$$\begin{aligned} \sum_{m=M=1}^{2} H_{\alpha\beta} = n\int \exp\,[-\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|]\,\cdot&\\ \cdot\cos\,[\omega(t_{\mathfrak{A}K}-t_{\mathfrak{A}k})+2\pi(\varphi^{(L)}-\varphi^{(l)})]\,E^{(l)}E^{(L)}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,.&\end{aligned} \tag{25}$$

Another important expression derived from (24) is the following:

$$\begin{aligned} 2\sum_{m,\,M=1}^{2} H^2_{\alpha\beta} = n^2\int \exp\,[-\gamma(|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|+|t_{\mathfrak{A}'K}-t_{\mathfrak{A}'k}|)]\,\cdot&\\ \cdot\cos\,[\omega(t_{\mathfrak{A}K}-t_{\mathfrak{A}k})-\omega'(t_{\mathfrak{A}'K}-t_{\mathfrak{A}'k})+2\pi(\varphi^{(L)}-\varphi^{(L)'}-\varphi^{(l)}+\varphi^{(l)'})]\,\cdot&\\ \cdot\, E^{(l)}E^{(l)'}E^{(L)}E^{(L)'}p(\mathfrak{a})\,p(\mathfrak{a}')\,\mathrm{d}\mathfrak{a}\,\mathrm{d}\mathfrak{a}'.&\end{aligned} \tag{26}$$

Finally, we write down an expression containing fourth derivatives which will be needed further below

$$\sum_{m,\,M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2}\,n\int \exp\,[-2\gamma\,|t_{\mathfrak{A}K}-t_{\mathfrak{A}k}|]\,E^{(l)^2}E^{(L)^2}p(\mathfrak{a})\,\mathrm{d}\mathfrak{a}\,. \tag{27}$$

5. – The expressions (25), (26) and (27) can be further simplified if we neglect suitable small terms.

Split $\boldsymbol{R}$ into its longitudinal and transversal part according to (9) and split $\boldsymbol{r}_k$ similarly into

$$\boldsymbol{r}_k = \boldsymbol{a}_k + \boldsymbol{b}_k ,$$

where $\boldsymbol{a}_k$ is parallel, $\boldsymbol{b}_k$ perpendicular to $\boldsymbol{L}$; expanding (16) into powers of $1/L$ we find

$$t_{\mathfrak{A}k} = T - t_k + \frac{1}{c}\left(L + a_k + \frac{(\boldsymbol{b}_k - \boldsymbol{B})^2}{2L} + \text{terms of higher order}\right). \tag{28}$$

Neglecting higher orders we may put

$$t_{\mathfrak{A}K} - t_{\mathfrak{A}k} = t + \tau_{\mathfrak{A}kK} , \tag{29}$$

where

$$t = t_k - t_K + \frac{1}{c}\left(a_K - a_k + \frac{\boldsymbol{b}_K^2 - \boldsymbol{b}_k^2}{2L}\right), \tag{30}$$

and

$$\tau_{\mathfrak{A}kK} = \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{Lc} . \tag{31}$$

We note that

$$\gamma\tau_{\mathfrak{A}kK} = \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\Lambda} \sim \frac{B_0 b_0}{L\Lambda} ,$$

where

$$\Lambda = c/\gamma$$

is the half length of the individual wave trains. If we suppose

$$L,\ \Lambda \gg B_0,\ b_0 , \tag{32}$$

B_0, b_0 are the orders of magnitude of the transversal dimensions of cathode and light source, then we have

$$\gamma\tau_{\mathfrak{A}kK} \sim 0 , \tag{33}$$

and we may put

$$\gamma\,|t_{\mathfrak{A}K} - t_{\mathfrak{A}k}| \sim \gamma t . \tag{34}$$

When evaluating the integrals in Sect. **4**, we cannot neglect, however,

$$\omega\tau_{\mathfrak{A}kK} = 2\pi\,\frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda} \sim \frac{2\pi B_0 b_0}{L\lambda_0} , \tag{35}$$

where we have put $2\pi c/\omega = \lambda$, $2\pi c/\omega_0 = \lambda_0$. The latter quantity has a significance well known in interference optics. If

$$\frac{B_0 b_0}{L\lambda_0} \ll 1\,,$$

then the source P with transversal dimensions B_0 produces on the screen with transversal dimension b_0 a coherent image.

We can, however, neglect

$$(\omega - \omega_0)\tau_{\mathfrak{A}kK} \sim \frac{2\pi B_0 b_0}{L\Lambda'} \sim 0\,, \tag{36'}$$

where

$$\Lambda' = \frac{2\pi c}{\Delta\omega_0} \tag{37}$$

is the coherence length of the beam as determined by the spectral width of the emissions only, when disregarding the effects of damping.

Equations (33) and (34) express that we suppose the transversal dimensions B_0, b_0 of light source and receiver both to be small as compared with

$$\Lambda = \frac{c}{\gamma} \qquad \text{and} \qquad \Lambda' = \frac{2\pi c}{\Delta\omega_0}\,.$$

So as to simplify the integrals in Sect. **4** it is also useful to assume some symmetry properties. If we assume the projection of the light source onto a plane perpendicular to $\boldsymbol{L}$ to have circular symmetry, we can assume the probability of an emission corresponding to a co-ordinate vector $\boldsymbol{B}$ to be equal to that with a co-ordinate vector $-\boldsymbol{B}$; therefore a value of $\tau_{\mathfrak{A}kK} = \tau$ appears with the same probability as $\tau_{\mathfrak{A}kK} = -\tau$ and under the integral we may replace

$$\cos\omega(t + \tau_{\mathfrak{A}kK})$$

by

$$\tfrac{1}{2}(\cos\omega(t + \tau_{\mathfrak{A}kK}) + \cos\omega(t - \tau_{\mathfrak{A}kK})) = \cos\omega t \cos\omega\tau_{\mathfrak{A}kK}\,.$$

Furthermore, we can write because of (36)

$$\omega\tau_{\mathfrak{A}kK} \sim \omega_0\tau_{\mathfrak{A}kK}\,.$$

Finally, if we suppose

$$\int \sin(\omega - \omega_0)t\,p(\alpha)\,\mathrm{d}\omega \sim 0\,,$$

then we can replace under the integral

$$\cos \omega t \qquad \text{by} \qquad \cos \omega_0 t \cdot \cos (\omega - \omega_0) t \, .$$

We thus get the following approximate expressions:

$$\sum_{m=M=1}^{2} H_{\alpha\beta} \approx n \exp [-\gamma |t|] \cos \omega_0 t \int E^{(l)} E^{(L)} \cos (\omega - \omega_0) t \cdot \cdot \cos 2\pi (\varphi^{(L)} - \varphi^{(l)}) \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} p(\mathfrak{a}) \, \mathrm{d}\mathfrak{a} \, , \tag{38}$$

$$2 \sum_{m, M=1}^{2} H_{\alpha\beta}^2 = \left(n \exp [-\gamma |t|] \int E^{(l)} E^{(L)} \cos (\omega - \omega_0) t \cdot \cdot \cos 2\pi (\varphi^{(L)} - \varphi^{(l)}) \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} p(\mathfrak{a}) \, \mathrm{d}\mathfrak{a} \right)^2 . \tag{39}$$

(In obtaining the last expression we also supposed on grounds of symmetry

$$\int \sin 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} \, \mathrm{d}\boldsymbol{B} = 0 \, .)$$

$$\sum_{m, M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2} n \exp [-\gamma |t|] \int E^{(l)^2} E^{(L)^2} p(\mathfrak{a}) \, \mathrm{d}\mathfrak{a} \, . \tag{40}$$

The expressions (38) and (39) can be further simplified if we take the distributions of A, $\boldsymbol{B}$, of ω and of $E^{(1)}$, $E^{(2)}$, $\varphi_2 - \varphi_1$ to be independent of each other

$$p(\mathfrak{a}) = p_1(A, \boldsymbol{B}) p_2(\omega) p_3(E^{(1)}, E^{(2)}, \varphi_2 - \varphi_1) \, ; \tag{41}$$

we have

$$\sum_{m=M=1}^{2} H_{\alpha\beta} = n \exp [-\gamma |t|] \cos \omega_0 t \langle E^{(l)} E^{(L)} \cos 2\pi (\varphi^{(L)} - \varphi^{(l)}) \rangle \cdot \cdot \langle \cos (\omega - \omega_0) t \rangle \left\langle \cos \frac{2\pi \boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} \right\rangle ; \tag{42}$$

$$2 \sum_{m, M=1}^{2} H_{\alpha\beta}^2 = \left(n \exp [-k |t|] \langle E^{(l)} E^{(L)} \cos 2\pi (\varphi^{(L)} - \varphi^{(l)}) \rangle \cdot \cdot \langle \cos (\omega - \omega_0) t \rangle \left\langle \cos \frac{2\pi \boldsymbol{B}(\boldsymbol{b}_K - \boldsymbol{b}_k)}{L\lambda_0} \right\rangle \right)^2 ; \tag{43}$$

$$\sum_{m, M=1}^{2} H_{\alpha\alpha\beta\beta} = \frac{1}{2} n \exp [-2\gamma |t|] \langle E^{(l)^2} E^{(L)^2} \rangle \, . \tag{44}$$

II.

We apply the expressions obtained in the previous Section to the evaluation of certain observables effects.

6. – Two-ray interferometer.

We receive in a point Q of the screen two coherent images of the source P. The effect is the same as if we were to add up the emissions received in two points Q_1 and Q_2 at one and the same instant $t_1 = t_2 = 0$. We may specify the points Q_k by writing $a_1 = \mathbf{b}_1 = 0$, $a_2 = x$, $\mathbf{b}_2 = \mathbf{y}$. If we regard the component polarized into the direction l we find for the intensity in Q

$$J(Q) = \sum_{m=1}^{2} (E^{(1lm)} + E^{(2lm)})^2 . \tag{45}$$

The expected value of $J(Q)$ is obtained with the help of (42) and (22)

$$\langle J(Q)\rangle = 2J_0(1+\varepsilon) ,$$

where

$$J_0 = \langle E^{(l)^2}\rangle ,$$

is the intensity of the single beam, and

$$\varepsilon = \sum_{m,M=1}^{2} H_{\alpha\beta}/J_0 = \exp[-X/\Lambda]\cos(2\pi X/\lambda_0)\cdot \cdot\langle\cos(\omega-\omega_0)X/c\rangle\left\langle\cos 2\pi\frac{\mathbf{B}\mathbf{y}}{L\lambda_0}\right\rangle . \tag{46}$$

and

$$X = x + \frac{y^2}{2L} .$$

The second factor on the right of (46) gives the interference pattern. The first describes its extinction with increasing path difference due to the damping effect, the third term gives the extinction arising from the band width $\Delta\omega_0 = \omega - \omega_0$, the last factor gives the effect of the finite size of the source. The third, respectively fourth factor approach unity if

$$\Delta\omega_0 X/c \to 0 , \qquad \text{resp.}\ \frac{B_0 y}{L\lambda_0} \to 0 .$$

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7. – Fluctuation of intensity in a point.

The mean square fluctuation of intensity in a point Q is given by

$$\langle(\delta J^{(1l)})^2\rangle = \langle J^{(1l)^2}\rangle - \langle J^{(1l)}\rangle^2 .$$

Further

$$J^{(1l)} = E^{(1l1)^2} + E^{(1l2)^2} .$$

Thus

$$J^{(1l)^2} = \sum_{m,M=1}^{2} E^{(1lm)^2} E^{(1lM)^2} ,$$

and therefore according to (22)

$$\langle(\delta J^{(1l)})^2\rangle = \sum_{m,M=1}^{2} H_{\alpha\alpha\beta\beta} + 2\sum_{m,M=1}^{2} H^2_{\alpha\beta} , \qquad \alpha = 1lm,\ \beta = 1lM.$$

Thus if we put $k = K = 1$, $l = L$ we get with the help of (43) and (44)

$$\langle(\delta J^{(1l)})^2\rangle = \tfrac{1}{2} n \langle E^{(l)^4}\rangle + n^2 \langle E^{(l)^2}\rangle^2 .$$

Writing

$$n\langle E^{(l)^2}\rangle = J_0 ,$$

we have

$$\langle(\delta J^{(1l)})^2\rangle / J_0^2 = 1 + \frac{\sigma^2}{2n} = \varkappa_0^2 , \tag{47}$$

where

$$\sigma^2 = \langle E^{(l)^4}\rangle / \langle E^{(l)^2}\rangle^2 \sim 1 . \tag{48}$$

The first term of (47) gives the fluctuation caused by the interference between the independent wave trains, the second term gives the Poisson fluctuation caused by the fluctuation of the number of emission processes per unit time

8. – Fluctuation on an extended cathode.

The current received from a photocathode can be taken to be proportional to the integral of the square of the field strength over the cathode surface; if we consider fluctuations of this current, we have to take into account that the electric recording instrument averages the current intensity over some period τ of time. Thus the intensity fluctuation is characterized by $\varkappa$, where

$$\varkappa^2 = \left[\left\langle\left(\int_0^\tau \mathrm{d}t \int J(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{b}\right)^2\right\rangle - \left\langle \int_0^\tau \mathrm{d}t \int J(\boldsymbol{r}, t)\,\mathrm{d}\boldsymbol{b}\right\rangle^2\right] \frac{1}{J_0^2 S^2 \tau^2} , \tag{49}$$

$J_0 = \langle J^{(l)} \rangle$ being the average value of intensity and $S = \int \mathrm{d}\boldsymbol{b}$ is the illuminated part of the surface of the cathode.

Instead of (49) we may also write

$$\varkappa^2 = \int_0^\tau \int_0^\tau \mathrm{d}t' \mathrm{d}t'' \iint \langle \delta J(\boldsymbol{r}, t')\, \delta J(\boldsymbol{r}'', t'') \rangle \, \mathrm{d}\boldsymbol{b}' \mathrm{d}\boldsymbol{b}'' / S^2 J_0^2 \tau^2 . \tag{50}$$

We note that we may put according to (22)

$$\langle \delta J^{(kl)} \delta J^{(Kl)} \rangle = \sum_{m,\, M=1}^{2} (H_{\varkappa\alpha\beta\beta} + 2H^2_{\alpha\beta}) , \qquad \alpha = klm, \ \beta = KlM.$$

With the help of (44) and (43) we find if we assume $t = t' - t''$, $l = L$

$$\begin{aligned} \langle \delta J(\boldsymbol{r}, t')\, \delta J(\boldsymbol{r}, t'') \rangle &= \frac{1}{2} n \exp[-2\gamma |t' - t''|] \langle E^{(l)^4} \rangle + \\ &+ \left(n \exp[-\gamma |t' - t''| \langle E^{(l)^2} \rangle \langle \cos (\omega - \omega_0)(t' - t'') \rangle \left\langle \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}' - \boldsymbol{b}'')}{L\lambda_0} \right\rangle \right)^2 . \end{aligned} \tag{51}$$

Integrating into $\boldsymbol{b}'$ and $\boldsymbol{b}''$ we get a form factor

$$g^2 = \iint \left\langle \cos 2\pi \frac{\boldsymbol{B}(\boldsymbol{b}' - \boldsymbol{b}'')}{L\lambda_0} \right\rangle^2 \mathrm{d}\boldsymbol{b}' \mathrm{d}\boldsymbol{b}'' / S^2 . \tag{52}$$

If the light spot on the cathode is coherent, *i.e.* if

$$\frac{B_0 b_0}{L\lambda_0} \ll 1 ,$$

then $g \sim 1$.

The integration into t' and t'' can be carried out when we express the square of the expectation value of $\cos (\omega - \omega_0)(t' - t'')$ by a double integral, and carry out the integration into t' and t'' first. We find

$$\varkappa^2 = g^2 f^2 + \frac{\sigma^2}{2n} \theta^2 ,$$

where we have taken

$$\left\{ \begin{aligned} \theta^2 &= \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \exp[-2\gamma |t' - t''|] \mathrm{d}t' \mathrm{d}t'' , \\ f^2 &= \frac{1}{\tau^2} \int \exp[-2\gamma |t' - t''|] \cos (\omega' - \omega_0)(t' - t'') \cos (\omega'' - \omega_0)(t' - t'') \cdot \\ &\qquad \cdot p_2(\omega') p_2(\omega'') \mathrm{d}\omega' \mathrm{d}\omega'' \mathrm{d}t' \mathrm{d}t'' ; \end{aligned} \right. \tag{53}$$

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if $\gamma\tau \gg 1$, we find

$$\theta^2 \sim \begin{cases} 1\,, & \text{if} \quad \gamma\tau \ll 1\,, \\ 1/\gamma\tau\,, & \text{if} \quad \gamma\tau \gg 1 \end{cases}$$

and

$$f^2 \approx \int \frac{\gamma^2(\gamma^2+u^2+v^2)}{(\gamma^2+u^2+v^2)^2-4u^2v^2}\, p_2(\omega_0+u)\,p_2(\omega_0+v)\,\mathrm{d}u\,\mathrm{d}v\,. \tag{54}$$

The latter integrals can be approximately evaluated for certain extreme cases and we find

$$f^2 \sim \begin{cases} 1\,, & \text{if} \quad \gamma \gg \Delta\omega_0\,, \\ \pi p_2(\omega_0)\gamma\,, & \text{if} \quad \gamma \ll \Delta\omega_0\,. \end{cases} \tag{55}$$

Supposing

$$p_2(\omega_0) \approx \sqrt{\frac{2}{\pi}}\,\frac{1}{\Delta\omega_0}$$

(the latter relation holding exactly if $p_2(\omega)$ is a Gaussian distribution), we find

$$\pi p_2(\omega_0) \sim \frac{\sqrt{2\pi}}{\Delta\omega_0}\,, \tag{56}$$

thus

$$\varkappa^2 \approx \begin{cases} 2g^2f^2 - \dfrac{\sigma^2}{2n}\,, & \text{if} \quad \gamma\tau,\ \ \Delta\omega_0\tau \ll 1\,, \\[2ex] \dfrac{g^2}{\gamma\tau} + \dfrac{\sigma^2}{2n\gamma\tau}\,, & \text{if} \quad \gamma\tau \gg \Delta\omega_0\tau\,,\ 1\,, \\[2ex] \dfrac{\sqrt{2\pi}g^2}{\Delta\omega_0} + \dfrac{\sigma^2}{2n\gamma\tau}\,, & \text{if} \quad \Delta\omega_0\tau \gg \gamma\tau\,,\ 1\,. \end{cases} \tag{57}$$

We note that according to (13)

$$2n\gamma\tau = N\tau$$

is the number of emission processes taking place during the collecting time τ; thus the second term in (57) represents the contribution to the fluctuation of the random fluctuations of the number of emission processes taking place during the time τ. The first term gives the fluctuation caused by the interference of the wave trains superposed at random. In general, the first term is more important than the second; however, if either the geometry of the arrangement is such that g^2 becomes small, or if the wave hand is broad so that $\Delta\omega_0 \gg \gamma$, then the first term becomes small as compared with the second one which is independent of the geometry or width of the wave band.

The effect corresponding to the first term was observed recently by BRANNEN, FERGUSON and WEHLAU ([5]) counting individual photons.

9. – Simultaneous fluctuation on two cathodes.

The correlation coefficient of the fluctuations of intensity as observed on two cathodes can be obtained as

$$(58) \qquad \Gamma_{12} = \int \langle \delta J(\boldsymbol{r}_1, t')\, \delta J(\boldsymbol{r}_2, t')\rangle \, \mathrm{d}\boldsymbol{b}_1 \mathrm{d}\boldsymbol{b}_2 \mathrm{d}t' \mathrm{d}t'' / (\langle \delta J_1^2 \rangle \langle \delta J_2^2 \rangle)^{\frac{1}{2}} S^2 \tau^2 ,$$

where the integrations into t' and t'' have to carried out from 0 to τ and the integrations over $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ over the surfaces of the first and the second cathode. If we split a beam into two coherent components and project exactly the same part of the two beams on each cathode, then we find $\Gamma_{12} = 1$. If on the other hand we throw the same image on two similar cathodes but at different distances from the source, *i.e.* if $\boldsymbol{b}_1 = \boldsymbol{b}_2$ but $a_2 = a_1 + x$, we find with the help of (43), if we put $t = x/c$,

$$(59) \qquad \Gamma_{12} = \exp[-2x/\Lambda] \left(\langle \cos 2\pi(\omega - \omega_0)\, x/c^2 \rangle g^2 f^2 + \frac{1}{2n} \sigma^2 \right) \Big/ \left(g^2 f^2 + \frac{\sigma^2}{2n} \right) .$$

For the sake of an example we suppose that the spectral distribution is a Gaussian distribution, *i.e.* that

$$(60) \qquad p_2(\omega) = \frac{\exp[-2(\omega - \omega_0)^2/\Delta\omega_0^2]}{\sqrt{\pi \Delta\omega_0/2}} .$$

We have

$$(61) \qquad \exp[-x/\Lambda] \langle \cos 2\pi(\omega - \omega_0) x/c \rangle = \exp\left[-x/\Lambda - \frac{x^2(\Delta\omega_0)^2}{8c^2} \right] .$$

We see thus that the part of the correlation which depends on the frequency band decreases more rapidly with x than the frequency independent part. For large values of x, we have thus

$$(62) \qquad \Gamma_{12} \approx \exp[-2x/\Lambda] \qquad \text{for} \qquad x \Delta\omega_0/c \gg 1 ,$$

independent of the width of the spectrum. We note, however, that the correlation coefficient Γ_{12} thus obtained follows from a purely classical picture.

([5]) E. BRANNEN, H. I. S. FERGUSON and W. WEHLAU: *Can. Journ. Phys.*, **36**, 871 (1958).

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10. – Stellar interferometer.

It was pointed out by TWISS and HANDBURRY BROWN (1), that the correlations of the fluctuations on two cathodes can be used similarly to a Michelson stellar interferometer for the determination of the angular size of a light source. This effect follows also from our formulae. Calculating the correlation coefficient of the intensity fluctuations for two small cathodes, so that $a_1 = a_2 = 0$; $\boldsymbol{b}_2 = \boldsymbol{b}_1 + \boldsymbol{y}$ we get

$$\Gamma_{12} = \left\langle \cos 2\pi \frac{\boldsymbol{B}\boldsymbol{y}}{L\lambda_0} \right\rangle^2 + \text{terms in } \frac{1}{n}. \tag{63}$$

Supposing the source to be a disc of radius B_0 we find

$$\Gamma_{12} = 4\left(\frac{\sin \alpha B_0}{\alpha B_0} - \frac{1 - \cos \alpha B_0}{\alpha B_0^2}\right)^2, \qquad \alpha = 2\pi y / L\lambda_0 . \tag{64}$$

The correlation decreases with increasing distance y and with increasing angle of vision $2B_0/L$. We have neglected terms in $1/n$; the latter terms become predominant in the region, where the terms we have considered above are small.

11. – Coincidences observed with photon counters.

Suppose two parts of a beam to fall onto the cathodes of photon counters. The expected rate of coincidences registered with an arrangement of resolving time τ is given by

$$\varkappa_{12} = 2p_0^2 \int_0^\tau \langle J(\boldsymbol{r}_1, t) J(\boldsymbol{r}_2, t + t') \rangle \, \mathrm{d}\boldsymbol{b}_1 \, \mathrm{d}\boldsymbol{b}_2 \, \mathrm{d}t' , \tag{65}$$

where p_0 is the expected rate of impulses produced by the unit intensity falling on the cathode.

With the help of (43), (44), (48), (57)

$$\varkappa_{12} = 2N_1^2\tau \left(1 + \frac{\sigma^2}{n\tau} \int_0^\tau \exp[-2\gamma t'] \, \mathrm{d}t' + \frac{g^2}{\tau} \int_0^\tau \exp[-2\gamma t'] \langle \cos(\omega - \omega_0)t' \rangle^2 \, \mathrm{d}t' \right), \tag{66}$$

with

$$N_1 = J_0 p_0 S .$$

J_0 is the average intensity and S the surface of each cathode.

Evaluating the last integral we find for $\gamma\tau \gg 1$

$$\varkappa_{12} = 2N^2\tau\left(1 + \frac{g^2}{\gamma\tau}\left\langle\frac{1}{1+(\omega-\omega_0)^2/\gamma^2}\right\rangle + \frac{\sigma^2}{2n\gamma\tau}\right). \tag{67}$$

Further we may put

$$\left\langle\frac{1}{1+(\omega-\omega_0)^2/\gamma^2}\right\rangle \approx \begin{cases} 1, & \text{if } \Delta\omega_0 \ll \gamma, \\ \pi p_2(\omega_0)/\gamma, & \text{if } \Delta\omega_0 \gg \gamma. \end{cases} \tag{68}$$

The rate of accidental coincidences, in case of a constant intensity would be

$$\varkappa_{12}^{(0)} = 2N^2\tau. \tag{69}$$

The excess $\varkappa_{12} - \varkappa_{12}^{(0)}$ consists of two components. The first one, which is in general the more important, is sensitive to the degree of coherence of the light spot on the cathode and in addition to the spectral width of the band of emissions. The second term (which represents a kind of Poisson fluctuation) depends only on the rate of emission processes of the source. The latter term should become preponderant if the first term becomes small on account of geometry or band width.

We hope to return to the problem as to what modifications in the above formulae are to be expected if the light waves are subjected to quantization.

RIASSUNTO (*)

Estendendo calcoli eseguiti prima della (3) il presente lavoro si occupa della statistica esatta della fluttuazione dei treni d'onde emessi da una sorgente di estensione non trascurabile. Come precedentemente, si assume che i singoli treni d'onde decadano esponenzialmente; tuttavia qui si assume che i singoli treni abbiano differenti frequenze, intensità e polarizzazioni. Vari effetti osservati sperimentalmente, fra cui quello ottenuto coll'interferometro stellare da HANBURY BROWN e TWISS, si possono giustificare quantitativamente.

(*) *Traduzione a cura della Redazione.*