Connection Between Generalized Glauber Operators and Two-Photon Coherent States of the Radiation Field.

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(ricevuto il 30 Luglio 1980).

According to the definition of the lowering and raising operators a and $a^\dagger$ of the harmonic oscillator (¹), we can define new operators which depend on two complex parameters as follows:

$$A = \frac{1}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\left(\alpha\hat{q} + \frac{i}{\hbar}\beta\hat{p}\right), \qquad A^\dagger = \frac{1}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\left(\alpha^*\hat{q} - \frac{i}{\hbar}\beta^*\hat{p}\right), \tag{1}$$

where

$$[\hat{p}, \hat{q}] = -i\hbar, \qquad [A, A^\dagger] = 1 \quad \text{and} \quad \alpha^*\beta + \beta^*\alpha > 0. \tag{2}$$

If we set $\alpha = \beta\gamma$, the above operators in the q-representation take the following form:

$$A = \frac{\beta}{|\beta|}\frac{1}{\sqrt{2\gamma_1}}\left(\gamma q + \frac{\partial}{\partial q}\right), \qquad A^\dagger = \frac{\beta^*}{|\beta|}\frac{1}{\sqrt{2\gamma_1}}\left(\gamma^* q - \frac{\partial}{\partial q}\right), \tag{3}$$

which with the help of the definition of the operators (²) $a(\gamma)$ and $a^\dagger(\gamma)$ can be written as

$$A = \frac{\beta}{|\beta|}a(\gamma), \qquad A^\dagger = \frac{\beta^*}{|\beta|}a^\dagger(\gamma), \tag{4}$$

where

$$a(\gamma) = \frac{1}{\sqrt{2\gamma_1}}\left(\gamma q + \frac{\partial}{\partial q}\right), \qquad a^\dagger(\gamma) = \frac{1}{\sqrt{2\gamma_1}}\left(\gamma^* q - \frac{\partial}{\partial q}\right) \tag{5}$$

and $\gamma = \gamma_1 + i\gamma_2$, $\gamma_1 > 0$.

(¹) A. JANNUSSIS: *General Glauber operators and their coherent states*, in *Seminar Group Theoretical Methods in Physics* (Moscow, 1979).
(²) A. JANNUSSIS, N. PATARGIAS and L. PAPALOUCAS: *J. Phys. Soc. Jpn.*, **47**, 1003 (1979).

So instead of operators (1) it is sufficient to study operators (5).

In what follows we shall call operators (1) or operators (5) generalized Glauber operators. We can extend these operators in a n-dimensional space, that is

$$a(\gamma_l) = \frac{1}{\sqrt{2\,\mathrm{Re}\,\gamma_l}}\left(\gamma_l q_l + \frac{\partial}{\partial q_l}\right), \qquad a^\dagger(\gamma_l) = \frac{1}{\sqrt{2\,\mathrm{Re}\,\gamma_l}}\left(\gamma_l^* q_l - \frac{\partial}{\partial q_l}\right). \tag{6}$$

Operators (1) or, in general, the operators $A+\lambda I$, $A^\dagger + \lambda^* I$ (I is the unit operator) are found in several physical problems [3,4] and especially in quantum-mechanical nonstationary systems by the group of Lebedev Physics Institute with the help of the integral of motion [5]. They are also studied by DEKKER [6] in the phase-space quantization of the linearly damped harmonic oscillator.

All the known operators of Jannussis *et al.* [7], Canivell and Seglar [8], Dekker and even those of Yuen [9], which describe the two-photon coherent states, are particular cases of operators (1).

For a fixed radiation mode with photon annihilation operators (1)

$$\begin{cases} a = \dfrac{1}{\sqrt{2}}\left(\sqrt{\dfrac{\omega}{\hbar}}\,q + i\sqrt{\dfrac{\hbar}{\omega}}\,p\right), \\[2ex] a^\dagger = \dfrac{1}{\sqrt{2}}\left(\sqrt{\dfrac{\omega}{\hbar}}\,q - i\sqrt{\dfrac{\hbar}{\omega}}\,p\right), \end{cases} \tag{7}$$

YUEN [8] defines the following operators:

$$b = \mu a + \nu a^\dagger\,, \qquad b^\dagger = \mu^* a^\dagger + \nu^* a\,, \qquad [a, a^\dagger] = 1 \tag{8}$$

for a pair of c numbers μ, ν obeying

$$[b, b^\dagger] = |\mu|^2 - |\nu|^2 = 1\,. \tag{9}$$

The generalized Glauber operators (1) with the help of the annihilation and creation a and $a^\dagger$ can be written as

$$A = \frac{\alpha\sqrt{\hbar/2\omega} + \beta\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\,a + \frac{\alpha\sqrt{\hbar/2\omega} - \beta\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\,a^\dagger\,, \tag{10}$$

$$A^\dagger = \frac{\alpha^*\sqrt{\hbar/2\omega} - \beta^*\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\,a + \frac{\alpha^*\sqrt{\hbar/2\omega} + \beta^*\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\,a^\dagger\,. \tag{11}$$

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(3) V. DODONOV and V. MAN'KO: *Nuovo Cimento B*, **44**, 265 (1978); *Phys. Rev. A*, **20**, 550 (1979) and references therein.

(4) V. DODONOV, L. MALKIN and V. MAN'KO: *Physica (The Hague)*, **59**, 403 (1972).

(5) V. DODONOV and V. MAN'KO: *Physica A (The Hague)*, **94**, 403 (1978) and references therein.

(6) H. DEKKER: *Physica A (The Hague)*, **95**, 311 (1979); *Phys. Lett. A*, **76**, 362 (1980) and references therein.

(7) A. JANNUSSIS, N. PATARGIAS and G. BRODIMAS: *J. Phys. Soc. Jpn.*, **45**, 336 (1978).

(8) V. CANIVELL and P. SOGLAR: *Phys. Rev. D*, **15**, 1050 (1977); **18**, 1082 (1978); *Physica A (The Hague)*, **94**, 254 (1978).

(9) H. YUEN: *Phys. Rev. A*, **13**, 2226 (1976).

Comparing the above operators with the operator of Yuen [9], we obtain the following relations between the complex numbers (α, β) and (μ, ν):

$$\mu = \frac{\alpha\sqrt{\hbar/2\omega} + \beta\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}}\,, \qquad \nu = \frac{\alpha\sqrt{\hbar/2\omega} - \beta\sqrt{\omega/2\hbar}}{\sqrt{\alpha\beta^* + \alpha^*\beta}} \tag{12}$$

and inversely

$$\frac{\alpha}{\sqrt{\alpha\beta^* + \alpha^*\beta}} = \sqrt{\frac{\omega}{2\hbar}}(\mu + \nu)\,, \qquad \frac{\beta}{\sqrt{\alpha\beta^* + \alpha^*\beta}} = \sqrt{\frac{\hbar}{2\omega}}(\mu - \nu)\,. \tag{13}$$

As we can see by relations (12) and (13), Yuen's operators are some particular cases of the generalized Glauber operators and the generalized Glauber operators (1) are a canonical linear transformation of the simple Glauber operators a and $a^\dagger$. More details about the canonical transformation can be found in Yuen's [9] paper when it is combined with the Von Neumann [10] theorem which asserts that every canonical transformation can be represented as a unitary one.

We can generalize operators (1) in the phase space with the help of the Wigner operators

$$\hat{q} = q \pm \frac{i\hbar}{2}\frac{\partial}{\partial p}\,, \qquad \hat{p} = p \pm \frac{i\hbar}{2}\frac{\partial}{\partial q}\,. \tag{14}$$

Now it is possible to define the generalized operators in the phase space for arbitrary complex constants α, β, γ and δ:

$$A_\pm = \frac{1}{\sqrt{2}}\left(\frac{\beta}{|\beta|}\,a(\gamma) \pm i\,\frac{\alpha}{|\alpha|}\,a(\delta)\right), \tag{15}$$

$$A_\pm^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\beta^*}{|\beta|}\,a^\dagger(\gamma) \mp i\,\frac{\alpha^*}{|\alpha|}\,a^\dagger(\delta)\right). \tag{16}$$

One readily proves the following commutation relations:

$$\begin{cases} [A_+, A_+^\dagger] = 1\,, & [A_-, A_-^\dagger] = 1\,, \quad [A_+, A_-^\dagger] = 0\,, \\ [A_-, A_+^\dagger] = 0\,, & [A_+, A_-] = 0\,, \quad [A_+^\dagger, A_-^\dagger] = 0\,. \end{cases} \tag{17}$$

The eigenfunctions and the eigenvalues of the number operator $A^\dagger A$ and the coherent states of the lowering operator A are found in ref. [1] and in q-representation are

$$\langle n|q\rangle = \varphi_n(q) = \sqrt[4]{\frac{\alpha\beta^* + \alpha^*\beta}{2\pi|\beta|^2}}\,\frac{1}{\sqrt{2^n n!}}\exp\left[-\frac{\alpha\beta^*}{2|\beta|^2}\right] H_n\left(\sqrt{\frac{\alpha\beta^* + \alpha^*\beta}{2|\beta|^2}}\,q\right) \tag{18}$$

with corresponding eigenvalues $n = 0, 1, 2, \ldots$, $H_n(x)$ are the Hermite polynomials. So

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[10] J. VON NEUMANN: *Math. Ann.*, **104**, 570 (1931).

there is an orthonormal basis $|n\rangle$ for which the following relations hold:

$$(19)\qquad A^\dagger A|n\rangle = n|n\rangle\,,\qquad A^\dagger|n\rangle = \frac{\beta^*}{|\beta|}\sqrt{n+1}|n+1\rangle\,,\qquad A|n\rangle = \frac{\beta}{|\beta|}\sqrt{n}|n-1\rangle\,.$$

The coherent states of operator A are given as follows:

$$A|\lambda\rangle = \lambda|\lambda\rangle\,,$$

$$(20)\qquad |\lambda\rangle = \sqrt[4]{\frac{\alpha\beta^*+\alpha^*\beta}{2\pi|\beta|^2}}\exp\left[-\frac{\alpha\beta^*}{2|\beta|^2}q^2-\frac{\beta^{*2}}{|\beta|^2}\frac{\lambda^2}{2}-\frac{1}{2}|\lambda|^2+\sqrt{2}\lambda\frac{\beta^*}{|\beta|}\sqrt{\frac{\alpha\beta^*+\alpha^*\beta}{2|\beta|^2}}\,q\right]=$$
$$=\exp\left[-\frac{1}{2}|\lambda|^2\right]\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle = \exp\left[-\frac{1}{2}|\lambda|^2\right]\sum_{n=0}^{\infty}\frac{(\lambda A^\dagger)^n}{n!}|0\rangle = D(\lambda)|0\rangle\,,$$

where $D(\lambda)$ is the displacement Weyl operator.

The pairs of operators (12) and (13) in the phase space have physical meaning, since we can find the Casimir invariant

$$(21)\qquad N = \tfrac{1}{2}(A_+^\dagger A_+ + A_+A_+^\dagger + A_-^\dagger A_- + A_-A_-^\dagger) = 1 + a^\dagger(\gamma)a(\gamma) + a^\dagger(\delta)a(\delta)$$

as well as the charge operator Q, which, according to SKAGERSTAM [11], has the form

$$(22)\qquad Q = A_+^\dagger A_+ - A_-^\dagger A_- = i\left[\frac{\alpha\beta^*}{|\alpha||\beta|}a^\dagger(\gamma)\,a(\delta)-\frac{\alpha^*\beta}{|\alpha||\beta|}a(\gamma)\,a^\dagger(\delta)\right].$$

Now we find the generalized coherent states of the operators Q and A_+A_- which have zero commutator

$$(23)\qquad [Q, A_+A_-] = 0\,.$$

The coherent states satisfy the following equations:

$$(24)\qquad A_+A_-|\xi,\nu\rangle = \xi|\xi,\nu\rangle\,,$$

$$(25)\qquad Q|\xi,\nu\rangle = \nu|\xi,\nu\rangle\,.$$

According to ref. [1] the coherent states are

$$(26)\qquad |\xi,\nu\rangle = \frac{\sqrt[4]{2\gamma_1}\sqrt[4]{2\delta_1}}{2\pi}\exp\left[-\frac{|\xi|}{2}-\frac{1}{2}(\gamma q^2+\delta p^2)\right]\cdot$$
$$\cdot\int_0^{2\pi}\exp\left[\sqrt{\xi}\left(\frac{\sqrt{2\gamma_1}|\beta|}{\beta}q\cos\theta+\frac{\sqrt{2\delta_1}|\alpha|}{\alpha}p\sin\theta\right)-\right.$$
$$\left.-\frac{\xi}{2}\left(\frac{|\beta|^2}{\beta^2}\cos^2\theta+\frac{|\alpha|^2}{\alpha^2}\sin^2\theta\right)+i\nu\theta\right]d\theta\,,$$

where ν is an integer.

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[11] B. SKAGERSTAM: *Phys. Lett. A*, **69**, 76 (1978); *Phys. Rev.*, **190**, 2471 (1979).

According to ref. [11] we can construct the following self-adjoint operators:

$$C = A_+ A_- + A_+^\dagger A_-^\dagger \,, \qquad D = \frac{1}{i}(A_+ A_- - A_+^\dagger A_-^\dagger) \tag{27}$$

and the commutator of these operators is

$$[C, D] = -iN \,. \tag{28}$$

The Casimir invariant N corresponds to the Hamilton operator of two uncoupled harmonic oscillators with friction as can be seen by comparing our results with the paper of Kerner [12]. With the help of the Schwarz inequality we obtain the product of the variances

$$\Delta C \cdot \Delta D \geqslant \frac{\langle N \rangle}{2} \,, \tag{29}$$

where ΔC and ΔD are the dispersions of the operators C and D in the state considered. We call the operators of relation (1) generalized Glauber operators since by their nature they discribe friction phenomena as is proved in ref. [1,2].

In addition, the operators on the Gauss plane which have been studied recently by Jannussis *et al.* [13] are also special cases of operators (15) and (16). These operators are of great physical importance because of the complex parameter $\gamma = \gamma_1 + i\gamma_2$, since its real part γ_1 represents the rotation rate, while its imaginary part γ_2 the dissipation rate.

The generalized Glauber operators or that of Yuen's have been used recently by Helström [14] for the study of the linear quantum channel with termal noise.

We think that the generalized Glauber operators will find many applications in physics and particularly in the problem of the dissipate quantum systems as well as in the problem of quantum noise.

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(12) E. Kerner: *Com. J. Phys.*, **36**, 371 (1958).

(13) A. Jannussis, G. Brodimas and L. Papaloucas: *Phys. Lett. A*, **71**, 301 (1979).

(14) C. Helström: *J. Math. Phys.* (*N. Y.*), **20**, 2063 (1979).