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Gravitational Fields and Quantum Mechanics.

T. Regge

Istituto di Fisica dell'Università - Torino
Istituto Nazionale di Fisica Nucleare - Sezione di Torino

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Summary. — Several kinds of measurements of gravitational fields are here qualitatively discussed. It is shown that no one of them yields satisfactory results if the region in which the field is measured has linear dimensions smaller than $L = Gh/c^2$.

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1. – The problem of the measurability of gravitational fields or more properly of their averages on a given space-time domain (here shortly GFA) from the point of view of quantum mechanics seems to be a hard and a premature one and no satisfactory theory of this subject has been so far constructed.

Most difficulties arise from the non-linearity of Einstein's equations since it is clear that one cannot go very far in studying a linear approximation of these general equations.

In the present note we shall attempt a discussion of some features of the theory of gravitation which can be deduced from simple qualitative arguments like ideal experiments.

No claim of rigor and of completeness is here advocated to our conclusions, we think that they should be more valued as a hint to a further investigation than as a clear cut result.

Our discussion essentially shows that Einstein's field dynamics (following Wheeler here called Geometrodynamics) cannot be a true parallel theory of Electrodynamics and that important differences arise whenever high accuracy is desired and whenever measurements are carried out in small regions of linear dimension D of the order of $L = \sqrt{hG/c^3} \approx 4 \cdot 10^{-33}$ cm, where $G = 6.67\ 10^{-8}\ \text{cm}^3\,\text{s}^{-2}\,\text{g}^{-1}$ is Newton's constant.

Quite an amount of interesting arguments have been proposed by Wheeler and others ([1-3]) supporting the view that important changes should be expected on the geometrical structure of space at such small distances.

We refer to the original papers by WHEELER and co-workers for a complete discussion of which we wish to give here short account.

1) The field fluctuations over regions of space-time dimensions of the order of L^4 are of the order of magnitude of one in the potentials.

2) Consequently a microcurvature of space over such small distances should be expected.

3) Most likely space shows multiple connectedness if observed on such a detailed scale. There are no *a priori* reasons why a multiple connected space should be rejected as unphysical.

Quantization of geometrodynamics can be carried out in principle with the help of Feynman's method of sum over histories. On this difficult task Misner has made quite important progresses ([4]). In the following the gravitational field will be measured in several different ways. In spite of these differences they all suffer the same limitation thus supporting the view that this may be a general feature of a possible theory.

2. – Before beginning the main problem of the measurement of GFA we would like here to argue about the meaning of « point mass » in general relativity. In the classical physics a point mass is usually meant to be a body satisfying the following properties:

i) It has a given mass $M \neq 0$.

ii) Its acceleration depends on the value which the field of force has in the point where the body is located and it depends only on this value.

In the Newtonian approximation nothing prevents us to think that these two conditions can be realized as closely as wanted.

Elementary particles have been since long treated as locally interacting. Difficulties arise only in connection with the well known self-energy infinities.

In general relativity, once the mass of the body is fixed, condition ii) is hardly realizable in virtue of the non-linearity of Einstein field equations.

([1]) J. A. WHEELER: *Phys. Rev.*, **97**, 520 (1955).

([2]) C. MISNER abd J. A. WHEELER: *Classical Physics as Geometry*, in *Rev. Mod. Phys.*, reference in proof.

([3]) J. A. WHEELER: *On the Nature of Quantum Geometrodynamics*, in *Rev. Mod. Phys.* (reference in proof).

([4]) C. MISNER: *Rev. Mod. Phys.* (reference in proof).

This is best seen by an analysis of a paper by INFELD and SCHILD ([5]) on the motion of test particles. These authors give a simple derivation of the geodesic postulate from Einstein equation. They were able to show that a test particle, in the limit in which its mass vanishes, will move along a geodesic. If the mass of the particles has instead a fixed finite value it is impossible to define exactly a geodesic because the body itself will perturbe the metric of the surrounding space in a region of size $\approx MG/c^2$. Moreover it is very unlikely that even if the geodesic can be defined the body will actually follow it. The derivation of the geodesic postulate is there valid only if the background space (defined as the one we would get putting $M = 0$) is with good approximation euclidean in the mentioned region of size MG/c^2 (*). Any departure from this condition will cause the test body to undergo a different motion independently from its internal structure.

The motion of our test body is therefore determined by the structure of the background field over the mentioned region.

In other words the acceleration of the body depends on the values of the field spread over a domain of size $R = MG/c^2$. No particle of this kind will ever behave as a point mass and satisfy ii). R can be regarded as the minimal possible size for a body of mass M. It is worth pointing out that the non-linearity of the field equations plays an essential role in the derivation of this result.

3. – We shall discuss here several methods of measurement of GFA. For the sake of simplicity we shall suppose that the field to be measured can be treated in the linear approximation. Non-linear phenomena will appear only near the test bodies. Furthermore the speed of these bodies will be assumed to be much smaller than c. Under these conditions an obvious experiment could be carried out by determining the momentum change $\Delta \boldsymbol{p}$ of a body of mass M during a given time. We shall call the quantity

$$\Gamma = \frac{\Delta \boldsymbol{p}}{c^2 M \cdot T},$$

GFA over the region of size D covered by the body and during the time T. If the body is localized within the region of size D, $\Delta \boldsymbol{p}$ will be uncertain by the amount h/D. Moreover M cannot exceed Dc^2/G for what said before.

([5]) L. INFELD and A. SCHILD: *Rev. Mod. Phys.*, **21**, 408 (1949).

(*) Mathematically this is expressed by the condition that $a_{\mu\nu}$ should be small sompared with $\eta_{\mu\nu}$ wherever it is not possible to retain the first term only in the expansion (2.05). It is easy to see that all the following derivations break down in absence of this condition.

Γ is therefore affected by the error

$$\Delta\Gamma \approx \frac{h}{M\cdot c^2 T\cdot D} > \frac{L^2}{D^2\cdot cT}\,.$$

This error depends only on the universal constants h, c, G and on the space-time structure of the measurement. Actually there are other sources of error. One is that the body may radiate gravitational waves during the process. The true nature of gravitational waves is still subjected to some discussion. As a provisional hypothesis we shall assume for them properties strictly similar to those of the electromagnetic waves. We can then follow the discussion of BOHR and ROSENFELD [6] with slight changes in the coupling constants. We should expect then that the uncertainty due to the radiated field is of the kind

$$\Delta\Gamma_R \approx MTDA$$

where $A = cL^2/hTD^2Z$; Z is the largest between D and cT.

If the body is bound to a spring of elastic constant c^2M^2TA the correction entirely cancels out. This method works only if the period of the oscillator body+spring is much larger than T. This leads to the inequality

$$\frac{1}{\sqrt{MTA}} \gg cT\,. \tag{7}$$

This leads again to $M \ll Dc^2/G$ if $D < cT$ and to $M < D^3c^2/(cT)^2\cdot G$ if $D > cT$, thus confirming the adopted limitation for the mass.

We complete the discussion by pointing out that the field induced by a quantum whose frequency may range from 0 to $\frac{1}{2}T$ at the distance D may be as high as $L^2/D^2\cdot cT$. Since such a quantum is needed in order to define a pulse of duration T in the observing radiation an additional uncertainty in Γ is introduced of the same order of magnitude of $\Delta\Gamma$.

4. – The arguments above lead to the conclusion that on a space time region of size D the uncertainty in the Christoffel symbols should be about L^2/D^3 and in the metric tensor (*) correspondingly of the order of L^2/D^2. If D is a macroscopic length the sedeviations are phantastically small and even on the atomic scale they are still negligible. However when D becomes comparable with L it becomes increasingly difficult to mantain the usual notions of space and the effect of the microcurvature becomes evident.

Quite in agreement with these conclusions is the result of a close analysis of the Heisenberg microscope experiment. Apart from the implicit enormous

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(6) N. BOHR and D. ROSENFELD: *Danske Vid. Selskab. Math. Phys. Medd.*, **12**, 8 (1933).

difficulties arising from the generalized Carnot principle (7) we meet a fundamental obstacle when high accuracies are desired.

In the Brillouin apparatus the position of the body is measured by sending a suitable superposition of waves through a wave guide. If we wish to narrow down the position of the body to an uncertainty of less than D then forcedly we must use waves of frequency up to hc/D. The metric field inside the region where we know to be the body is then uncertain by the amount $\Delta\Gamma \approx L^2/D^2$ introduced by the observing radiation. There is no use trying to let $D \to L$ because the distance D' of the body from any point is uncertain by an amount $\simeq L^2/D$.

From this point of view bodies are never localized. In the non local quantum field theory a question of primary importance was to decide whether or not field operators commute at space like distances or, in other words, if we have local or non-local commutativity. From our point of view before asking this question one should try to give a well defined meaning to the space like or time like attribute when the considered interval is comparable with L.

5. – It may seem premature to base the deduction in Sect. **4** on a single experiment. Therefore we discuss quite different procedures showing that they yield the same qualitative results.

One could, for instance, consider a pendulum. For « pendulum » we mean here a body without internal degrees of freedom hung to a wire (or equivalent constraint). We allow a good deal of ideal properties to our apparatus and suppose that the wire is inextensible. The length of it is Q, the mass of the pendulum is M, its radius R. We measure the period τ of the pendulum and then we deduce $g = c^2\Gamma$ from

$$\Gamma = \frac{4\pi^2 Q}{c^2\tau^2} \quad \text{if} \quad R \ll Q . \tag{5}$$

If the oscillations are observed during the time T we expect to make an error $1/T$ in $1/\tau$. We shall have correspondingly

$$\Delta\Gamma \approx 4\pi^2 Q \frac{2}{c^2 T\tau} \gg \frac{8\pi^2 R}{c^2 T\tau} .$$

Furthermore if we want a GFA over a region of size D the pendulum should not swing farther than D. In the ground state it still swings as far as $\sqrt{hQ/gM\tau}$. We are forced to require that

$$D^2 > \frac{hQ}{gM\tau} . \tag{7}$$

(7) L. Brillouin: *Journ. Appl. Phys.*, **25**, 887 (1954).

From (5) and (7) we get then

(8
$$\frac{1}{\tau} < \frac{D^2 g M}{hQ} \approx \frac{4\pi^2 D^2 M}{\tau^2 h} \quad \text{or} \quad \tau < \frac{4\pi^2 D^2 M}{h} .$$

(6) is then equivalent to (remembering $M < Rc^2/G$)

(9)
$$\Delta\Gamma \gg \frac{8\pi^2 R}{c^2 T\tau} \gg \frac{2L^2}{cT\cdot D^2} .$$

The pendulum does not measure Γ with any more accuracy than the first method. Another way to measure the GFA is provided by the dynamometer. An ideal apparatus of this kind consists of a body of mass M hung to a perfectly elastic spring of constant k. Under the pull of the body the spring is lengthened by the amount ξ.

The acceleration of gravity is then given by

(10)
$$g = \frac{K\xi}{M} .$$

The hanging body however is not usually at rest and we need to measure its position twice, at the times t at the time $T+t$ (T being a half integer number of periods) and to average the positions then obtained in order to deduce ξ. The energy of the oscillator is therefore uncertain by the amount h/T

If the GFA is to be determined over a region smaller than D the amplitude of the oscillations should not exceed this value. Consequently

(11)
$$D^2 > \frac{h}{kT} \quad \text{or} \quad k > \frac{h}{TD^2} .$$

The error in the measure of g arises from the uncertainty in ξ which s larger than MG/c^2. It follows

$$\Delta g = \frac{g\Delta\xi}{M} > \frac{h\Delta\xi}{MTD^2} > \frac{hG}{c^2 T\cdot D^2} = \frac{L^2 c^2}{cT\cdot D^2} .$$

An alternative method consists in placing an atom of radius D, decaying with the emission of a photon of frequency ν, on the region where we wish to measure the GFA. If the gravitational potential γ is not too large we can deduce it from the observed red shift $\Delta\nu$ by the formula

$$\gamma = \frac{\Delta\nu}{\nu} - \gamma_A ;$$

γ_A is the potential induced by the atom itself which must be subtracted in order to measure the background field only. T is the length of the experiment. We suppose $T\nu \gg 1$.

However the larger is ν the larger is the indeterminacy $\Delta\gamma$ in γ_A because the mass of the atom during the transition is only known with the error $h\nu/c^2$.

We have within the atom $\Delta\gamma_A = h\nu G/Dc^4$, this is also the indeterminacy in γ and agrees with the previous experiments (*):

$$\Delta\gamma = \frac{h\nu G}{Dc^4} > \frac{L^2}{cT\cdot D}\,.$$

6. – We hope that our discussion has helped to point out some of the features and difficulties of a quantum theory of gravitation.

Most probably a satisfactory theory of this kind will proceed along quite different patterns of thought from those to which we have been used in dealing with the classical Riemannian Geometry.

We must expect that the usual space-time continuum will show upsetting features if observed on a scale of distances comparable with L. However, as clearly stressed by WHEELER, classcal Geometrodynamics is already so distinctively well founded from the logical point of view that it is certainly worth giving a close look to quantum Geometrodynamics.

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(*) It may not be obvious that if the average of a function F over the space-time region Σ is known with an error F then the average grad F over a region of comparable size D can be measured only with the error $\Delta F/D$. Actually a measurement of grad F could be performed by measuring F on two identical regions Σ_1 and Σ_2 of size D about and shifted apart at the distance D and taking the ratio: $(F_1 - F_2)/D$. Such a ratio can be easily expressed as an average grad F over the region sum of Σ_1 and Σ_2 and the intermediate points. The error in the above measured grad F is clearly $\Delta F/D$ apart from an insignificant numerical factor.

([8]) O. KLEIN: *Suppl. Nuovo Cimento*, **6**, 344 (1957).

RIASSUNTO

In questo lavoro vengono discussi alcuni metodi di misura del campo gravitazionale. Semplici considerazioni di carattere fisico permettono di dedurre che nessuno di essi può considerarsi soddisfacente se la regione in cui il valore medio del campo è determinato ha dimensioni lineari più piccole di $L = Gh/c^2$.