A Causal Interpretation of the Pauli Equation (B).

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1. – Introduction.

In a previous paper [1], to which we shall hereafter refer as (A), we have developed a causal interpretation of the Pauli equation, in terms of the model of a fluid composed of spinning bodies, in which the spin angular momentum of each body is directed along its first principal axis (i.e., its axis of symmetry). The object of the present paper is two-fold, first, to generalize the above model to show how the Pauli equation results as a special case of a theory in which the angular momentum may point in an arbitrary direction, and secondly, to illustrate our model of the Pauli theory in terms of the simple example of a stationary state of an electron in an atom, and to apply this example in the theory of measurements.

In Secs. **2** and **3** we shall develop our more general model and show that in terms of it, the Pauli equation follows as a consistent subsidiary condition. In other words, if the angular momentum points along the first principal axis at any instant, say $t = 0$, then the torques will be such that this condition will be maintained for all time. Moreover, small deviations from this condition lead to a rapid oscillation of the motion about a mean, in which the

[1] D. BOHM, R. SCHILLER and R. TIOMNO: *Nuovo Cimento*, **1**, 48 (1955).

condition is satisfied. Physically, such an oscillatory motion corresponds to a precession of the angular momentum about the direction of the axis of symmetry, in which the component of the angular momentum perpendicular to this axis averages out to zero. Thus, even in the case of a general orientation of the angular momentum vector, the Pauli equation would still be a good approximation for processes that are slow compared with the rate of precession described above, while for more rapid processes, it would cease to be a good approximation. Instead, a more general equation (which will be seen to be non-linear) would have to be used. Thus, we are led to a specific example in which the causal interpretation of the quantum theory implies new kinds of equations in connection with high frequencies (and therefore high energies), equations which would reduce approximately to those of the usual quantum theory only at those relatively low energies for which the usual theory is known to be valid.

In Sec. **4**, we then illustrate our model of the Pauli equation in terms of a stationary state of an electron in an atom. This illustration serves not only to bring out important topological properties of a field of body orientations, such as is implied by our model, but it also shows how a continuous distribution of spin angular momentum throughout the fluid can lead to discrete or « quantized » possible values for the angular momentum of the fluid as a whole. Quantization of angular momentum (and of other variables) is therefore seen to arise as an over-all property of the fluid, on the basis of a lower level of continuous motion, much as discrete frequencies of stationary modes of vibration arise in connection with the motions of continuous fields in classical mechanics.

In Sec. **5**, we indicate how the process of measurement is to be treated in terms of our model. What is observed in a measurement of the angular momentum « observable » will be seen to be just one of the discrete possible stationary values for the total angular momentum of the system, and not the continuously distributed angular momenta existing in the various parts of the fluid.

Finally, in Sec. **6**, we summarize the essential features of the model, and suggest possible further directions of research.

2. – Treatment of Arbitrary Direction of Spin Motion for a Fixed Rigid Body.

Our first step will be to extend to the case of arbitrary spin motions the hamiltonian formalism in terms of spinor variables developed in paper (A).

To do this, we begin with the case of a rigid body at rest. In the next section, we shall then extend the theory to the more general case of a field of rigid spinning bodies with translational motions.

The lagrangian of a rigid body with no translational motion is just the kinetic energy due to its spin, which is ([2])

(1) $$L = \tfrac{1}{2}\sum_{i,k} I_{ik}\omega_i\omega_k\,,$$

where the ω_i represent the components of the angular velocity, and I_{ik} is the tensor for the moment of inertia of the body. The components of the angular momentum are then given by

(2) $$s_i = \sum_k I_{ik}\omega_k$$

so that

(3) $$L = \tfrac{1}{2}\sum_i s_i\omega_i\,.$$

The next problem is to express the ω_i in terms of spinors. Now, the relationship between the ω_i and the Euler angles is ([3])

(4) $$\begin{cases} \omega_z = \dot{\psi}\cos\theta + \dot{\varphi}\,, \\ \omega_x = \dot{\psi}\sin\theta\sin\psi + \dot{\theta}\cos\psi\,, \\ \omega_z = \dot{\psi}\sin\theta\cos\psi - \theta\sin\psi\,. \end{cases}$$

We now return to our spinor $\beta = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$, defined in paper (A), eq. (7). We have

$$\dot{\beta}_1 = \exp\left[\frac{i(\psi+\varphi)}{2}\right]\left(\frac{i}{2}(\dot{\psi}+\dot{\varphi})\cos\frac{\theta}{2} - \frac{\dot{\theta}}{2}\sin\frac{\theta}{2}\right)$$

$$\dot{\beta}_2 = i\exp\left[\frac{i(\psi-\varphi)}{2}\right]\left(\frac{i}{2}(\dot{\psi}-\dot{\varphi})\sin\frac{\theta}{2} + \frac{\dot{\theta}}{2}\cos\frac{\theta}{2}\right).$$

Now let us consider the quantity

$$\beta^*\sigma_z\dot{\beta} - \dot{\beta}^*\sigma_z\beta = i\left(\cos^2\frac{\theta}{2}(\dot{\psi}+\dot{\varphi}) - \sin^2\frac{\theta}{2}(\dot{\psi}-\dot{\varphi})\right) = i(\dot{\psi}\cos\theta + \dot{\varphi})\,.$$

Hence we obtain

$$\omega_z = i(\dot{\beta}^*\sigma_z\beta - \beta^*\sigma_z\dot{\beta})\,.$$

([2]) In the present paper, we leave out the electromagnetic field, which makes no essential change in the formulation of the theory.

([3]) H. GOLDSTEIN: *Classical Mechanics* (Cambridge Mass., 1950). See Chap. 4.

And more generally, because of rotational symmetry

$$\boldsymbol{\omega} = i(\dot{\beta}^*\boldsymbol{\sigma}\beta - \beta^*\boldsymbol{\sigma}\dot{\beta}) . \tag{5}$$

The kinetic energy then takes the form

$$L = T = \frac{i}{2}\,(\dot{\beta}^*(\boldsymbol{\sigma}\cdot\mathbf{s})\beta - \beta^*(\boldsymbol{\sigma}\cdot\mathbf{s})\dot{\beta}) . \tag{6}$$

We now define the momenta canonically conjugate to the components of the spinors β and β^*, which are

$$p_1 = \frac{\partial L}{\partial\dot{\beta}_1}, \quad p_2 = \frac{\partial L}{\partial\dot{\beta}_2}, \quad p_1^* = \frac{\partial L}{\partial\dot{\beta}_1^*}, \quad p_2 = \frac{\partial L}{\partial\dot{\beta}_2^*} . \tag{7}$$

We also define the spinors

$$\pi = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \pi^* = \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix}, \tag{8-a}$$

so that formally speaking

$$\pi = \frac{\partial L}{\partial\dot{\beta}}, \quad \pi^* = \frac{\partial L}{\partial\dot{\beta}^*} . \tag{8-b}$$

We then obtain from (6)

$$\pi = -i(\sigma\cdot\mathbf{s})\beta^* , \tag{9-a}$$

$$\pi^* = i(\boldsymbol{\sigma}\cdot\mathbf{s})\beta . \tag{9-b}$$

Note that in eq. (6), $\mathbf{s}$ is a function of $\dot{\beta}$ given by eqs. (2) and (5). As can easilly be verified, however, this dependence merely leads to a multiplication by a factor of two of the result that would be obtained if $\mathbf{s}$ were not a function of $\dot{\beta}$.

We readilly see that under rotations, π transforms contragrediently to β. To prove this, we form the quantity $\pi\beta$. From eq. (9-*a*), we obtain

$$\pi\beta = -\,i(\beta^*(\boldsymbol{\sigma}\cdot\mathbf{s})\beta) ,$$

which is evidently a scalar, so that π must transform in the same way as β^*, or in other words, contragrediently to β.

To solve for s_i, we multiply (9-a) by $\beta\sigma_i$, and (9-b) by $\beta^*\sigma_i$. We obtain

$$\mathbf{s} = \frac{i}{2}(\pi\boldsymbol{\sigma}\beta - \pi^*\boldsymbol{\sigma}\beta^*). \tag{10}$$

We shall now find it convenient to obtain the expressions for the projections of the angular momentum vector on the principal axes of the body. It is evident that each of these projections is a scalar with regard to rotations of the space axes, since by definition, these components will not depend on which space axes we choose for the expression of $\mathbf{s}$. We can therefore evaluate the components of the angular momentum along the body axes by choosing our space-axes to agree with the body axes. In this case, according to paper (A), eq. (7), we have $\beta = \begin{pmatrix}1\\0\end{pmatrix}$, and for the components of the angular momenta, we get

$$T_1 = i\,\frac{(\pi_1 - \pi_1^*)}{2}, \quad T_2 = i\,\frac{(\pi_2 - \pi_2^*)}{2}, \quad T_3 = \frac{(\pi + \pi^*)}{2}. \tag{11-a}$$

We also introduce for convenience, the quantity, T_0, which as we shall see presently is zero in our problem,

$$T_0 = \frac{\pi_1 + \pi_1^*}{2}. \tag{11-b}$$

We wish now to express the T_i in terms of the spinor quantities, taken in an arbitrary frame of reference. To do this, we seek a set of scalar functions of the spinors π and β which reduce to the T_i when we chose the space frame to be the same as the body frame.

Let us first consider the scalar $\pi\beta$. In the frame in which $\beta = \begin{pmatrix}1\\0\end{pmatrix}$, this reduces to

$$\pi\beta = \pi_1 = T_0 - iT_1,$$

so that

$$T_1 = 2\,\frac{(\pi\beta - \pi^*\beta^*)}{2}, \tag{12-a}$$

$$T_0 = \frac{\pi\beta + \pi^*\beta^*}{2}. \tag{12-b}$$

Now from eq. (9-b), we see (since $\mathbf{s}$ is real) that $T_0 = 0$. This relationship is a subsidiary condition whose meaning can be seen by writing β and π in extended form

$$\beta = \begin{pmatrix} b_1 + ib_2 \\ b_3 + ib_4 \end{pmatrix}, \quad \pi = \begin{pmatrix} p_1 - ip_2 \\ p_3 - ip_4 \end{pmatrix},$$

where p_i is canonically conjugate to b_i. We then get

$$\pi\beta + \pi^*\beta^* = b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 . \tag{1)3}$$

If we consider the space of the Cayley-Klein parameters introduced in paper (A), eq. (8-*b*), we see that from the relation $b_1^2+b_2^2+b_3^2+b_4^2=1$, it follows that we must remain on a unit hyper-sphere in this four dimensional space. Now eq. (13) then expresses the fact that the « radial momentum » in this hyper-space is zero. Such a result is to be expected, since the kinetic energy is a function only of the angular velocities, which do not involve the « radial » component of the velocity in the C.K. space, so that the corresponding momentum must be zero.

From (11-*a*) and (11-*b*), we readilly obtain for the total spin

$$S^2 = T^2 = (\pi^*\pi)^2 - T_0^2 = (\pi^*\pi)^2 \qquad (\text{since } T_0 = 0). \tag{14}$$

We now wish to obtain expressions corresponding to (12-*a*) and (12-*b*) for T_2 and T_3. To do this, we first consider the spinor

$$\widetilde{\beta} = \begin{pmatrix} -\beta_2^* \\ \beta_1^* \end{pmatrix}. \tag{15}$$

The spinor $\widetilde{\beta}$ has the same transformation properties under rotation as does β itself, as can be readilly verified by direct computation with the transformation matrix $\exp[i(\boldsymbol{R}\cdot\boldsymbol{\sigma})/2]$. Indeed, we have $\widetilde{\beta} = i\sigma_y\beta^*$, which is just the non-relativistic form of the so-called charge conjugate spinor [4] which, as is well-known, has the same transformation properties as the spinor β itself.

We now consider the invariant $\pi\widetilde{\beta}$. If we evaluate this in a frame in which the space and body axes are the same, so that $\widetilde{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we obtain (using eq. (11-*a*))

$$\pi\widetilde{\beta} = \pi_2 = T_3 - iT_2 , \tag{16-a}$$

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[4] W. Pauli: *Rev. Mod. Phys.*, **13**, 203 (1941).

or

$$T_3 = \frac{\pi\tilde{\beta} + \pi^*\tilde{\beta}^*}{2}, \qquad T_2 = i\,\frac{(\pi\tilde{\beta} - \pi^*\beta^*)}{2}. \tag{16-b}$$

It is readilly verified by direct computation that (T_1, T_2, T_3) satisfy the characteristic P.B. relationships of angular momenta, $(T_1, T_2) \doteq T_3$, etc.. One also readily verifies that the P.B. of any component of T, with any component of $\boldsymbol{S}$ are zero, as they have to be, since $\boldsymbol{S}$ is the generator of the infinitesimal rotation of the space axes and since the T_i are scalars with respect to such rotations.

We shall now go to the hamiltonian formalism. The hamiltonian is

$$H = \sum_i s_i w_i - L = \sum \frac{s_i w_i}{2} = L. \tag{17}$$

We must now eliminate the ω_i from the hamiltonian. This is most conveniently done by expressing the angular momentum in terms of its projection T_i on the principal axes of the body. This gives

$$H = \frac{T_1^2}{2I_1} + \frac{T_2^2}{2I_2} + \frac{T_3^2}{2I_3}, \tag{18}$$

where I_1, I_2, I_3, are the moments of inertia relative to the principal axes. For the case of a symmetrical top, which interests us here, $I_2 = I_3 = I$. Let us write $I_1 = I/\varepsilon$. Eq. (18) then becomes

$$H = \frac{1}{2I}(\varepsilon T_1^2 + T_2^2 + T_3^2) = \frac{1}{2I}(T^2 + (\varepsilon - 1)T_1^2). \tag{19}$$

To obtain the equations of motion for $\boldsymbol{S}$, we note that the P.B.'s of $\boldsymbol{S}$ with the T_i are zero. Thus the components of the angular momentum, taken with respect to axes fixed in space, are constants of the motion. As for the T_i, we note that T_1 is also a constant of the motion. For T_2 and T_3, we readilly obtain the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_3 - iT_2) = i\,\frac{(\varepsilon - 1)}{I}\,T_1(T_3 - iT_2). \tag{20}$$

Thus, the component of the angular momentum normal to the first principal axis rotates with angular velocity, $\omega = ((\varepsilon - 1)/I)T_1$. This means that the first principal axis rotates around the direction of the vector, $\boldsymbol{s}$, with the angular velocity ω. Thus, on the average the first principal axis points in the direction of $\boldsymbol{s}$; and insofar as phenomena that do not change much in the time $\tau = 1/\omega$ are concerned, the body acts as if its angular momentum were parallel to this principal axis.

We note that when $\varepsilon < 1$ (disc-shaped object), the energy is a minimum (for a given total angular momentum $|T|$) when $T_1^2 = T^2$; i.e., when the principal axis points along the direction of $\boldsymbol{s}$. Thus, for this case, the motion in which the principal axis is parallel to $\boldsymbol{s}$ will be *stable*. If $\varepsilon > 1$, (cigar-shaped object), such motion becomes unstable.

Finally, we shall express in terms of spinors the condition that the motion reduces to the type implied by the Pauli equation, in which the spin points along the principal axis. If this condition is satisfied, we have $T_2 = T_3 = 0$; or $\pi\tilde{\beta} = 0$. This leads to

$$\pi_1\beta_2^* = \pi_2\beta_1^* \; ; \qquad \text{or} \qquad \pi_1/\pi_2 = \beta_1^*/\beta_2^* \, ,$$

so that

$$\pi = ik^2\beta^* \, , \tag{21}$$

where k is a constant, and where we have chosen the factor of i to simplify subsequent expressions. But in order that $T_0 = (\pi\beta + \pi^*\beta^*)/2 = 0$, k^2 must be a real number. If this is the case, however, then we see that $i\beta^*$ is canonically conjugate to β, which is what is essentially the same relation as paper (A) eq. (14), which holds in the Pauli theory. To demonstrate in more detail the relationship between π, β, and the Pauli spinor, we make the following canonical transformation to the spinors A and B

$$\begin{cases} A = (k\beta + i\pi^*/k)1/\sqrt{2} \, , \\ B = (k\beta - i\pi^*/k)1/\sqrt{2} \, . \end{cases} \tag{22}$$

From the above, we readilly prove the Poisson-bracket relations

$$[A, B] = [A^*, B^*] = 0 \; ; \qquad [A^*, A] = [B^*, B] = i \, . \tag{23}$$

To satisfy eq. (21), we choose $B = 0$. Then the spinor A^* has the same P.B.'s with A as does the Pauli spinor (see paper (A), eq. (14)). Moreover, we also have $A = k\beta$, so that β is proportional to the Pauli spinor. To express the angular momentum, we eliminate π and β from eq. (10), obtaining

$$\boldsymbol{s} = k(A^*\boldsymbol{\sigma}A) \, . \tag{24}$$

Thus, the theory is able to treat an arbitrary value of the spin angular momentum. To obtain the same value as in the Pauli theory we must set $k=\hbar/2$. In this way, we complete our demonstration showing how the canonical formalism for an arbitrary direction of spin reduces to that of the Pauli theory when the spin points along the principal axis.

3. – Extension to Rigid Bodies with Translational Motion.

In the previous section, we developed a hamiltonian formalism that permits a treatment in terms of spinors of arbitrary motions of a rigid body fixed in space. We shall now extend this formalism to permit the treatment of a field of such rigid bodies, undergoing translation through space.

To do this, we shall first write the kinetic energy of translation of the bodies in terms of Clebsch parameters as we did in the case of the Pauli theory. Now, in the Pauli theory, we found that one rather naturally obtained the expression given in paper (A), eq. (25) for the velocity, $\boldsymbol{v} = \hbar/2(\nabla\psi + \cos\theta\,\nabla\varphi)$, which involved only the pair of Clebsch parameters, $\cos\theta$ and $\varphi/2$, along with the velocity potential, $\psi/2$. It is possible, however, to introduce as many additional pairs of Clebsch parameters as we please in the definition of the velocity. Thus we may write

$$\boldsymbol{v} = \nabla\lambda + \sum_i \xi_i \nabla\eta_i \,. \tag{25}$$

Now, it is true that the most general velocity field can in principle always be expressed without these additional pairs of Clebsch parameters. Nevertheless, the physical conditions of the problem may often make it convenient to introduce such additional pairs. Indeed, as we shall see presently, the treatment of the problem of the motion of a body with a general orientation of its angular momentum relative to its principal axes is an example of just such a problem.

In terms of the expression (25) for the velocity, the kinetic energy of translation of the bodies then becomes

$$T = \frac{\varrho}{2m}\,(\nabla\lambda + \sum_i \xi_i\,\nabla\eta_i)^2\,\mathrm{d}\boldsymbol{x}\,. \tag{26}$$

Now, introducing the P.B. relations

$$[\varrho(\boldsymbol{x}), \lambda(\boldsymbol{x}')] = \big[(\varrho(\boldsymbol{x})\xi_i(\boldsymbol{x})),\, \eta_i(\boldsymbol{x}')\big] = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{27}$$

with all other P.B.'s zero, we get for the equations of motion

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \boldsymbol{v}) = 0\,, \tag{28-a}$$

$$\frac{\partial \lambda}{\partial t} - \sum_i \xi_i (\boldsymbol{v}\cdot\nabla)\frac{\eta_i}{m} + \frac{v^2}{2m} = 0\,, \tag{28-b}$$

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = \frac{\mathrm{d}\eta_i}{\mathrm{d}t} = 0\,. \tag{28-c}$$

Eq. (28-*a*) expresses just the conservation of bodies, while (28-*b*) is an extension of the Hamilton-Jacobi equation. Eq. (28-*c*) expresses the constancy of the Clebsch parameters as we follow a moving body. It is this property of the Clebsch parameters that is most interesting to us here; for eq. (28-*c*) means that the ξ_i and the η_i may represent internal properties of bodies (such as angle or spin variables). While these variables remain constant in eq. (28-*c*), if we add terms to the hamiltonian involving the ξ_i and the η_i, then the equation that takes the place of (28-*c*) will tell how these properties change *as we follow the moving body.*

The Clebsch parameters therefore provide a natural canonical treatment of the motions of a field of parameters that represent internal properties of moving bodies.

Now, in the Pauli theory, the current vector was expressed as

$$\boldsymbol{j} = \varrho\,\nabla\lambda + (\varrho\xi)\,\nabla\varphi\,.$$

with

$$\xi = \cos\theta/2\,, \qquad \eta = -\,\varphi/2\,, \qquad \lambda = -\,\psi/2\,.$$

Let us recall that $\hbar\varrho$ was canonically conjugate to $-\,\psi/2$, $\hbar\varrho\xi$ to $-\,\varphi/2$. Thus writing $\hbar\varrho/2 = J_\psi$, $\hbar\varrho\cos\theta/2 = J_\varphi$, we have (remembering that J_ψ is density of the component of the angular momentum in the direction of the principal axis, and J_φ the density of the Z component of the angular momentum)

$$\boldsymbol{j} = J_\psi\,\nabla\psi + J_\varphi\,\nabla\varphi\,.$$

But now we are going to give up the restriction that S is directed along the principal axis. To complete the description of the angular momentum, we shall introduce, J_θ, the momentum canonically conjugate to the Euler angle, θ. This is equal to the density of angular momentum about an axis perpendicular to the plane containing the Z axis and the principal axis (1) of the body. In order that the P.B.'s of the kinetic energy of translation with J_θ and θ

shall give $\mathrm{d}J_\theta/\mathrm{d}t$ and $\mathrm{d}\theta/\mathrm{d}t$ respectively, it is clearly necessary, however, that $\boldsymbol{j}$ shall be the same functions of J_θ and θ as it is of J_φ and φ, and of J_ψ and ψ. This, we write

$$\boldsymbol{j} = J_\psi \nabla\psi + J_\varphi \nabla\varphi + J_\theta \nabla\theta\,.$$

Now, in order to obtain a simple model that approaches the Pauli case, we set the density of bodies equal to $2J_\psi/\hbar$, which is proportional to the density of the component of the angular momentum along the first principal axis. This implies that all bodies have the same value of this component of the angular momentum. We shall see presently that this assumption is consistent with the equations of motion that we are going to adopt. We then get

$$\boldsymbol{v} = \frac{\boldsymbol{j}}{\varrho} = \frac{\hbar}{2}\,(\nabla\psi) + \frac{J_\varphi}{\varrho}\,\nabla\varphi + \frac{J_\theta}{\varrho}\nabla\theta\,. \tag{27}$$

The next step is to add the kinetic energy of spin to the hamiltonian. To do this, we introduce the quantities $Q_i = \varrho T_i$, which are the densities of the components of the angular momenta taken along the principal axes. From the results of the previous section, this is

$$H_1 = \int \frac{1}{2\varrho T}\,(Q^2 + (\varepsilon - 1)Q_1^2)\,\mathrm{d}\boldsymbol{x} \tag{28}$$

(note that by definition, $Q_1 = J_\psi$).

Finally, we must add an appropriate generalization of the quantum mechanical part of the energy (« quantum potential » plus spin potential), which, for the Pauli case is expressed in paper (A), eq. (35-*b*). We write for the term

$$H_2 = \frac{\hbar^2}{2m}\int \sum_{ij} \left(\frac{\partial S_i}{\partial x_j}\right)^2 \mathrm{d}\boldsymbol{x}\,. \tag{29}$$

The complete hamiltonian is then

$$H = H_x + H_1 + H_2\,. \tag{30}$$

Let us now obtain some of the equations of motion. First of all, since Q_1 has zero P.B.'s with S_j, we get for $Q_1 = J_\psi$,

$$\frac{\partial Q_1}{\partial t} + \operatorname{div}\,(Q_1\,\boldsymbol{v}) = 0\,. \tag{31}$$

This is just a conservation equation, for the component of the angular momentum along the first principal axis. Since the density of bodies, ϱ, satisfies the same equation, we see that if we choose $\hbar\varrho/2 = Q_1$ at any instant, say $t = 0$, they will remain equal for all time. But $\hbar\varrho/2 = Q_1$ implies that all of the bodies have the same value of the component of the angular momentum along the principal axis. Since this component is a constant of the motion, the condition that it has the same value for all the bodies will evidently likewise be maintained for all time, if it is satisfied at any time.

As for Q_2 and Q_3, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{Q_3 - iQ_2}{\varrho}\right] = \frac{(\varepsilon - 1)Q_1}{I\varrho}\,\frac{Q_3 - iQ_2}{\varrho} \tag{32}$$

Thus, as in the case of the freely rotating body, the component of the angular momentum perpendicular to the first principal axis rotates with angular velocity $\omega = ((\varepsilon - 1)Q_1)/I\varrho$. We may evidently then adopt $Q_2 = Q_3 = 0$ as a consistent subsidiary condition, so that if the body is set with $\boldsymbol{S}$ originally in the direction of its first principal axis, then the torques will be such that this condition is maintained for all time. Moreover, if the principal axis is nearly in the direction of the angular momentum, then Q_2 and Q_3 will rotate with angular velocity, ω, so that for phenomena involving characteristic times τ, which are much less than $1/\omega$, the body will act as if $\boldsymbol{S}$ were parallel to the first principal axis.

Since $Q_2 = Q_3 = 0$ is a consistent subsidiary condition, we may obtain the equations of motion of the remainder of the variables by setting $Q_2 = Q_3 = 0$ in the hamiltonian. The parts, $H_T + H_2$, then reduce to the Pauli hamiltonian (see paper (A) eqs. (31) and (35)). The term, H_1 becomes

$$H_1 = \int \frac{\varepsilon Q_1^2}{2\varrho I}\,\mathrm{d}\boldsymbol{x} = \frac{\hbar^2 \varepsilon}{8I}\int \varrho\,\mathrm{d}\boldsymbol{x}\,,$$

where we have written $Q_1 = \hbar\varrho/2$. This part of the hamiltonian will have the same effect as adding to the Hamilton-Jacoby equation (paper (A), eq. (34-b)) a «rest mass» term such as would be obtained by deducing the Pauli equation as the non-relativistic limit of the Dirac equation. To bring out this property of H_1 more directly, we may make the canonical transformation to the Pauli spinor, A, given in eq. (22), which yields the «rest mass» term,

$$H_1 = \frac{\hbar^2 \varepsilon}{8I}\int (A^* A)\,\mathrm{d}\boldsymbol{x}\,. \tag{33}$$

We conclude then that if the subsidiary condition, $Q_2 = Q_3 = 0$ is satisfied, our set of equations representing the motion of a field of spinning bodies will reduce essentially to the Pauli equation, which is a *linear* equation for a spinor, Ψ. Moreover, even if the spin does not point along the principal axis, the components Q_2 and Q_3 will in general turn with some angular velocity, ω, so that for processes involving characteristic times, $\tau \gg 1/\omega$, the Pauli equation will still serve as an adequate approximation, since the effects of the components Q_2 and Q_3 will average out to zero in such processes. But in processes so rapid that $\tau \cong 1/\omega$, the Pauli equation would in such cases cease to be a good approximation, and one would have to go back to our basically non-linear set of equations for the various spin and angle variables. Of course, in the present theory, there is nothing to determine the value of ω. We may, however, consistently suppose that $1/\omega$ is of the order of the times involved in the high energy processes connected with the « creation », « destruction » and « transformation » of so-called « elementary » particles. In this case, the linear Pauli equation would be a good approximation for atomic processes, but would break down completely with regard to the high energy processes described above. Of course, to treat such processes correctly, we should need a relativistic theory, and this will require a causal interpretation of the Dirac equation. But we can already see that the generalization of the Pauli equation adopted here leads in a natural way to a breakdown in connection with high energy processes of the hypothesis of linear superposition, which is one of the basic postulates underlying the present form of the quantum theory. When this postulate fails, then the usual interpretation of the quantum theory cannot consistently be applied. This is an example of how a causal interpretation of the quantum theory permits the consideration of new kinds of theories, not permitted if one restricts oneself to the theories that are consistent with the usual interpretation.

Finally, we note that in the present theory, the basic meaning of $\hbar$ is that $\hbar/2$ is the angular momentum per body in the fluid. Thus, in the theory proposed here, the statement that $\hbar$ is a universal constant of nature implies that all bodies have the same spin angular momentum. This requirement is, as we have seen, a subsidiary condition that is consistent with the equations of motion that we have adopted. Of course, we have not yet explained why the subsidiary condition should be universally satisfied but this may perhaps be done later in connection with more extensive theories, such as, for example, those concerned with a causal interpretation of the Dirac equation and second quantization.

4. – An Illustration in Terms of Orbits in a Hydrogen Atom.

We shall now illustrate the causal interpretation of the Pauli equation in terms of the example of an electron in a hydrogen atom. This illustration will also permit us to draw some interesting conclusions concerning the meaning of the quantization conditions in the hydrodynamic model with spin.

We begin by considering a stationary state, in which the spinor wave function α, is given by

$$\alpha = R(\boldsymbol{x}) \exp[-iEt/\hbar] \begin{pmatrix} \beta_1(\boldsymbol{x}) \\ \beta_2(\boldsymbol{x}) \end{pmatrix} = \tag{34}$$

$$= R(\boldsymbol{x}) \exp[i\psi(\boldsymbol{x}, t)/2] \begin{pmatrix} \cos\theta/2 & \exp[i\varphi/2] \\ i\sin\theta/2 & \exp[-i\varphi/2] \end{pmatrix},$$

where $R(\boldsymbol{x}) = \sqrt{|\alpha_1|^2 + |\alpha_2|^2}$ is the normalized spinor of eq. (7) and

$$\psi(\boldsymbol{x}, t) = -\frac{2Et}{\hbar} + \psi_0(\boldsymbol{x}), \qquad \text{with} \qquad \psi_0(\boldsymbol{x}) = [\psi(\boldsymbol{x}, t)]. \tag{35}$$

(Note that here $\psi(\boldsymbol{x}, t)$ refers to the *Euler angles* of the body which is a function of position, while α is the wave-function spinor).

We see then that in a stationary state the orientation angles θ and φ are in general functions of position, but are constants in time. However, the angle ψ of rotation about the first principal axis increases linearly with the time, when evaluated at a given position, $\boldsymbol{x}$.

If, however, we look at a particular moving body, then it will follow some orbit in space, in which the various Euler angles and spin directions may change with time. But the motion must be periodic, in the sense that after the body returns to its original position in space, the Euler angles θ and φ must return to their original values (plus perhaps some multiple of 2π for the Euler angle φ) while ψ must have changed by $-2E\Delta t/\hbar$ (plus a suitable multiple of 2π). If these conditions were not satisfied, then we could not have a stationary state in which the only change with time of the state of the fluid at a given point is at most a rotation of each body around its axis of symmetry.

To show how this wave function leads to quantization of angular momentum, we shall now evaluate the action integral, $T = \oint \boldsymbol{p}\cdot d\boldsymbol{x}$, where $\boldsymbol{p}$ is the momentum of the motion of the center of mass of the body in its periodic orbit. Expressing $\boldsymbol{p}$ with the aid of paper (A) eq. (23), we obtain

$$\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}(\nabla\psi + \cos\theta\,\nabla\varphi)\cdot \mathrm{d}\boldsymbol{x}\,.$$

Now $\mathrm{d}\boldsymbol{x}$ represents the change of $\boldsymbol{x}$ as we follow the moving body in its orbit. Let us then introduce the differential, $\delta\boldsymbol{x}$, which represents the change of $\boldsymbol{x}$ at a fixed instant of time. Now, from eq. (35), we have $\nabla\psi = \nabla\psi_0(\boldsymbol{x})$. Since neither θ, φ nor ψ_0 are functions of t, we can then replace $\mathrm{d}\boldsymbol{x}$ in the action integral by $\delta \boldsymbol{x}$, obtaining

$$\boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}(\nabla\psi + \cos\theta\,\nabla\varphi)\cdot\delta\boldsymbol{x} = \frac{\hbar}{2}(\delta\psi_0 + \cos\theta\,\delta\varphi) \tag{36-a}$$

and

$$\oint \boldsymbol{p}\cdot \mathrm{d}\boldsymbol{x} = \frac{\hbar}{2}\oint(\delta\psi + \cos\theta\,\delta\varphi)\,, \tag{36-b}$$

where on the right hand side of (36-*b*), the integration is carried out around a circuit in space taken at a particular time, but the same circuit as is made by the actual orbit of the body.

We must now find out how ψ_0 and φ change as we go a around the circuit. Obviously we must have $\Delta\psi_0 = 2n_1\pi$ and $\Delta\varphi = 2n_2\pi$ where n_1 and n_2 are integers, in general different from each other. But now we shall show that n_1 and n_2 must satisfy the further conditions that n_1+n_2 and n_1-n_2 shall be *even* numbers.

To prove this, we must first briefly review some of the topological properties of the rotation group ([5,6]). Consider, for example, a series of infinitesimal space displacements in our field of body orientations, which add up to give some finite displacement. Such a series of space displacements will lead to a corresponding series of infinitesimal rotations, which add up to give the total rotation needed to bring the body from the orientation that it had at the beginning of the series to the one that it has at the end. Now, if the series takes the form of a closed circuit, it is evidently necessary that the body finally return to its original orientation, so that, as we have already pointed out, an integral number of rotations of 2π must have taken place along the circuit. But now, let us suppose that we make the circuit smaller and smaller, permitting it to shrink continuously down to a certain point. The total rotation that takes place as we follow the circuit must then decrease continuously to zero. Otherwise, we should have a finite rotation connected with an infinitesimal displacement in space, and this would not be consistent with the requirement that the field of body orientations be continuous.

In the mathematical expression of the above requirement of continuity in the field of body orientations, the Euler angles do not provide a convenient parametrization, because when $\theta = 0$, singularities can appear in the para-

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([5]) H. WEYL: *Theory of Groups and Quantum Mechanics* (New York, 1928). See p. 180.

([6]) F. D. MURNAGHAM: *The Theory of Group Representations* (Baltimore, 1938). See p. 318.

metrization that do not correspond to any real physical singularity in the field of orientations. Indeed, when $\theta = 0$, the only quantity defined by the actual rotation of the body is the sum of the angles, $\varphi + \psi$, while the difference, $\varphi - \psi$ is completely arbitrary. Thus, if we deform our circuit past a point where $\theta = 0$, it is possible for φ and ψ separately to undergo large changes, even when the actual rotation (and therefore along with it the sum, $\varphi + \psi$) changes only by infinitesimal amounts. Hence continuous changes in the orientations of the bodies need not always be reflected as continuous changes in the Euler angles; and for this reason, there is no guarantee that as we shrink a circuit in a field of rotations down to a point, the corresponding circuit in the space of the Euler angles will also shrink down to a point. As a result, the expression of the requirement that the field of body orientations be everywhere continuous would be rather complicated if done in terms of Euler angles.

To avoid the kind of difficulty described above, we may parametrize the rotation group, not in terms of Euler angles, but rather, in terms of the Cayley-Klein parameters, which are (according to paper (A), eq. (8-b)):

$$b_1 = \cos\theta/2\cos(\psi + \varphi)/2\,, \qquad b_2 = \cos\theta/2\sin(\psi + \varphi)/2\,,$$
$$b_2 = -\sin\theta/2\sin(\psi - \varphi)/2\,, \qquad b_4 = \sin\theta/2\cos(\psi - \varphi)/2\,.$$

For the Cayley-Klein parameters have the advantage that they provide everywhere a *locally* unique and continuous representation of the rotation group, in the sense that an infinitesimal rotation always implies a corresponding infinitesimal change in these parameters. To demonstrate this property of the representation, we note that the unit spinor (which is in a one-to-one correspondence with the Cayley-Klein parameters) transforms under the infinitesimal rotation, $\boldsymbol{\omega}\,\mathrm{d}t$, as $\delta\beta = i(\mathrm{d}t/2)(\boldsymbol{\omega}\cdot\boldsymbol{\sigma})\beta$. Thus, an infinitesimal rotation must *always* produce a corresponding infinitesimal change in β (and therefore in the C.K. parameters).

It must be remembered, however, that the unique character of the C.K. parameter representation applies only to *small* rotations. Indeed, with regard to the properties of the rotation group «in the large», this representation is two-valued. For corresponding to any given rotation, there always exist *two* sets of the b_i, one of which is the negative of the other. But even the two-valued character of the C.K. parameters in the large is significant; for it provides a mathematical description of the fact that the rotation-group is *doubly-connected*, a fact that is, as we shall see presently, very important in the determination of the changes of angle that are permissible in going around a circuit.

To see the meaning of the two-valued character of the representation of the rotation group in terms of the C.K. parameters (and therefore in terms of the spinors), let us consider a circuit in space that leads to a continuous series of infinitesimal rotations of the body orientations. There is of course an ambiguity in the point in the C.K. space at which we must start, because, corresponding to the initial orientation of the body, there are two possible sets of values of the C.K. parameters. But after we have chosen one of these sets, then the path in C.K. space that corresponds to our series of rotations is determined uniquely because of the locally continuous character of the representation of the rotation group in terms of the C.K. parameters. Let us now consider a circuit leading to a rotation of 2π about some axis. It is clear from the definition of the b_i that such a rotation carries us from a given set, (b_1, b_2, b_3, b_4) of C.K. parameters to the corresponding set $(-b_1, -b_2, -b_3, -b_4)$. Thus, even though the body has come back to its original orientation, the C.K. parameters have not come back to their original values. Indeed, geometrically speaking, this rotation carries us only half way around the hyperspherical surface on which we must remain in the four dimensional space of the b_i. As in the case of a great circle on a sphere in three dimensional space, there is no way to *continuously* deform such a curve to a point *while keeping the endpoints fixed* (7). As a result, when we make a closed circuit in the field of body orientations, if the body rotates through 2π along this circuit, then as the circuit is shrunk down to a point, the total rotation that takes place along this curve will have to remain equal to 2π, no matter how small the circuit becomes, so that the field of body orientations cannot be continuous. On the other hand, if the circuit carries us through *two* rotations of 2π, we are brought back to the original values of the C.K. parameters; and in the space of these parameters, the corresponding path goes all the way around the hypersphere. Then, as in the case of a complete great circle on a sphere in three dimensional space, this curve can be deformed continuously to a point, while keeping the endpoints fixed. From this, we conclude that a circuit involving two rotations of 2π can be shrunk continuously to one having no rotation at all, while a circuit involving a single rotation of 2π cannot. More generally, this will happen wherever $\varphi+\psi$ and $\varphi-\psi$ change by even multiples of 2π on going around a circuit; or whenever n_1+n_2 and n_1-n_2 are *even* integers. Thus, we justify the conditions which we gave earlier, regarding the permissible changes of value that the wave function may suffer on going around a circuit.

We may illustrate the conclusions of the preceeding paragraph in terms of an example in which a given circuit carries us through two rotations of 2π

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(7) The fixing of the end points corresponds to the fact that we are choosing a circuit that passes through a certain point in space. Since the orientation of the body is fixed at this point, the C.K. parameters cannot change at the endpoints of the circuit, no matter how this circuit is shrunk or otherwise changed.

in any specified direction. Now, as the circuit is shrunk, it would be possible for the rotation to become equivalent to two separate rotations of 2π, each carried out in a different direction; for in this case, the initial and final orientation of the body would still remain unchanged when the circuit was altered. Thus, a continuous deformation of the path in the C.K. space would take place, with the endpoints of the path held fixed; for the direction of one of the rotations of 2π could change continuously relatively to that of the other. But when we arrived at a path in which each rotation had an opposite sense, to that of the other then the two rotations would cancel each other; and we would have no rotation at all. Thus, we have given an example of how a circuit involving two rotations of 2π can be deformed *continuously* into one involving no rotation at all. As we have shown, however, this cannot be done, with a circuit involving a single rotation of 2π.

We see then how the two-valued character of the C.K. parameters (and therefore of the spinors) reflects the topological connectivity properties of the rotation group. Of course, if we had to deal only with a single body, these topological properties would not be relevant, because a single rotation of 2π would, after all, bring the body to an equivalent orientation in space, so that at a given time, no physical difference could exist between a body that had suffered only a single rotation of 2π, and one that had suffered two such rotations. On the other hand, because we are dealing with a *field* of body orientations, the relative amounts of rotation suffered by bodies in different parts of the field become relevant; in the sense that if the orientation changes by an odd number of rotations of 2π along a circuit, then this circuit cannot continuously be shrunk to one with no rotation at all, while if it changes by an even number of rotations of 2π, it can so be shrunk. Thus, the two-valued character of the spinors (and of the C.K. parameters) describe physically significant properties of a field of body orientations. In this way, we obtain a reason why the basic quantum-mechanical theory of the electron which concerns itself with such a field of body orientations, should be expressed in terms of spinors.

We now apply the above conditions on the continuity of the angles of rotation to a stationary state of a hydrogen atom, which in the usual interpretation is described as having an orbital angular momentum, l, and a Z component of the total angular momentum (spin plus orbital) of $k = (l + \frac{1}{2})$. For this state, the wave function is [8]

$$\alpha = \frac{R(\boldsymbol{x})}{\sqrt{2l+1}} \begin{pmatrix} \sqrt{l+k}\, P_l^{(k-\frac{1}{2})}(\theta') & \exp\left[i(k-\frac{1}{2})\varphi'\right] \\ \sqrt{l-k+1}\, P_l^{(k+\frac{1}{2})}(\theta') & \exp\left[i(k+\frac{1}{2})\varphi'\right] \end{pmatrix}, \tag{37}$$

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[8] See D. BOHM: *Quantum Theory* (New York, 1951); Chap. 17, eq. (73).

where θ' and φ' are the polar angles of the position vector, $\boldsymbol{x}$, of *the center of the body*, while $P_l^{(k-\frac{1}{2})}(\theta')$ is a suitably normalized associated Legendre polynomial.

Comparison with eq. (34) and (35) then indicates that

$$\frac{\psi_0}{2} = k\varphi' , \quad \varphi = -\varphi', \quad \operatorname{tg}\frac{\theta}{2} = \frac{\sqrt{l+k}}{\sqrt{l-k+1}} \frac{P_l^{(k-\frac{1}{2})}(\theta')}{P_l^{(k+\frac{1}{2})}(\theta')} . \tag{38}$$

Since $k = l + \frac{1}{2}$, and since φ' changes by 2π as we go around a circuit, we see that $\Delta\psi_0 = (2l+1)2\pi$ and $\Delta\varphi = -2\pi$. Thus, the topological relationship that $\varphi + \psi$ and $\varphi - \psi$ change by *even* multiples of 2π on going around a circuit is satisfied.

We are now ready to compute the action integral (75) for this case. To do this, we note that the velocity, $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\nabla\varphi)$, has only a component in the direction of φ'. Thus, the body moves in a circle, the plane of which is normal to the Z axis. From (77), we see that θ is constant along this circle. Thus we obtain

$$T = h\left(k - \frac{\cos\theta}{2}\right) = h\left(n + \frac{1}{2}(1 - \cos\theta)\right), \tag{39}$$

where n is an integer lying between 1 and -1.

The Z component of the *orbital* angular momentum of the body is

$$p_\varphi = \frac{T}{2\pi} = \hbar\left(k - \frac{\cos\theta}{2}\right). \tag{40}$$

The combined Z component of spin and orbital angular momentum is

$$p_\varphi + \frac{\hbar\cos\theta}{2} = \hbar k . \tag{41}$$

Thus, we find that the total angular momentum is just equal to that predicted in the usual theory. This angular momentum divides itself, however, between the spin and orbital angular momentum in accordance with the colatitude angle θ' of the radius vector of the body relative to the center of the atom (which determines θ through eq. (38)).

The result (39) for the action variable is analogous to that obtained from the Bohr-Sommerfeld quantum condition (of the « classical » quantum theory) in which even the feature of fractional quantum numbers is present, because these are often found to give better agreement with experiment than can be obtained with integral quantum numbers. There are, however, three important differences from the Bohr-Sommerfeld theory. First, eq. (39) is an *exact* equation, and not an approximate one. Secondly, in the evaluation of J, the momentum, $\boldsymbol{p}$, must be calculated taking into account the quantum-potential and the spin energy, and not just the classical potential as was done in the Bohr-Sommerfeld theory. Finally, the action variable, J, for the particle motion alone depends in the location of the orbit. Only the sum of this action variable and the one corresponding to the spin angular momentum is quantized in a way that is independent of the location of the orbit.

We are now in a position to see how our model accounts for the quantization of the angular momentum. To do this, we first note that the action variable is proportional to $\oint \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{x}$ which is an integral representing the total *vorticity* of the fluid inside the circuit in question. Thus, we have obtained a simple physical interpretation of the action variable. Then, because of the relationship $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\nabla\varphi)$, the connection (36) is established between this vorticity, and the changes of body angle, $\Delta\psi_0$ and $\Delta\varphi$, which occur as we go around a circuit. Finally, because we have a continuous field of spinning bodies, $\Delta\psi_0 + \Delta\varphi$ and $\Delta\psi - \Delta\varphi$ must be even multiples of 2π. Thus, the quantization of vorticity and therefore of angular momentum reflects basically the requirement that we have a continuous and single-valued field of spinning bodies, whose orbital motion is coordinated to their spin motion in the way implied by the relationship, $\boldsymbol{v} = (\hbar/2m)(\nabla\psi + \cos\theta\,\nabla\varphi)$.

– On the Process of Measurement of the Spin.

We shall now indicate in general terms how the process of measurement of the spin is to be treated in the model given in this paper. To do this, we shall first review briefly a similar treatment of the analogous problem of a non-spinning particle (described by the *Schrödinger equation* and not by the Pauli equation) going around an atomic nucleus with orbital angular momentum, $\hbar$.

Now a common method of measuring the spin of an atom is to place the atom in a non-homogeneous magnetic field. This field then separates atoms according to their component of the angular momentum along the direction of the field. To show how this process of measurement is treated in terms of the causal interpretation of the quantum theory, we shall use a method developed in a previous paper [9]. Now the general initial wave function of the electron with orbital angular momentum of $\hbar$ is

$$\Psi = a_{-1}\Psi_{-1}(\boldsymbol{x}) + a_0\Psi_0(\boldsymbol{x}) + a_1\Psi_1(\boldsymbol{x})\,, \tag{42}$$

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[9] D. Bohm: *Phys. Rev.*, **85**, 180 (1952).

where Ψ_{-1}, Ψ_0, Ψ_1 represent the wave functions of electrons with Z components of the angular momentum of respectively, $-\hbar$, 0 and $\hbar$, while a_{-1}, a_0, a_1, are the corresponding coefficients (in general complex) for the expansion of the wave function, $\Psi(\boldsymbol{x})$. The complete wave function for the combined system consisting of electron plus the atomic nucleus (whose coordinates we denote by $\boldsymbol{y}$) is then

$$f_0(\boldsymbol{y})\big(a_{-1}\Psi_{-1}(\boldsymbol{x}) + a_0\Psi_0(\boldsymbol{x}) + a_1\Psi_1(\boldsymbol{x})\big)\,, \tag{43}$$

where $f_0(\boldsymbol{y})$ represents a wave packet describing the fact that the atomic nucleus is fairly well-localized in space.

Now when we apply a magnetic field, $\mathfrak{H}$, (the direction of which we take to be that of the Z axis) then each of the three parts of the wave function described above will begin to oscillate at a different frequency. Moreover, if the magnetic field is inhomogeneous, the three parts will begin to separate in space. Indeed, after some time, the wave function will be transformed into

$$\Phi = f_0(\boldsymbol{y} - \boldsymbol{\Delta})a_{-1}\exp[i\alpha_{-1}]\Psi_{-1}(\boldsymbol{x}) + f_0(\boldsymbol{y})a_0\exp[i\alpha_0]\Psi_0(\boldsymbol{x}) + \\ + f_0(\boldsymbol{y} + \boldsymbol{\Delta})a_1\exp[i\alpha_1]\Psi_1(\boldsymbol{x})\,, \tag{44}$$

where α_{-1}, α_0, α_1 are changes of the phase angles which have resulted from the effects of the magnetic field, and $\boldsymbol{\Delta}$ is the motion of the nucleus under the action of this field. Now eventually $\boldsymbol{\Delta}$ becomes much bigger than the width of the wave packet $f_0(\boldsymbol{y})$, so that the three parts of the wave function cease to interfere with each other, and obtain a classically distinct separation. When this happens, then as has been shown (9), the particle-like inhomogeneity associated with the electron must have entered one of the separate packets. The probability that it is any one of them is given by

$$P_{-1} = |a_{-1}|^2, \qquad P_0 = |a_0|^2, \qquad P_1 = |a_1|^2. \tag{45}$$

Thereafter, the other packets play no role, so that they can be neglected.

We see then that what really happens in a measurement is that the measuring apparatus provides a general environment (in this case, the inhomogeneous magnetic field) in which the wave function is transformed from whatever it may have been initially into an eigen-function of the « observable » that is being measured (in this case, the Z component of the angular momentum).

Thus, in the case under discussion, the actual orbital angular momentum of the electron can vary continuously but only certain stable quantized values are possible, which can exist indefinitely without change, in the environment supplied by the inhomogeneous magnetic field. Which of the three possible stable values will actually be obtained is not determined in an individual case, if we merely specify the initial wave function, $\Psi(\boldsymbol{x})$. To determine, the actual result, we should have to specify also the initial location, $\xi(t_0)$, of the particle, and the general irregular motions in the Ψ field which lead to a statistical behavior that is described by the probability distribution, $P=|\Psi|^2$ for the particles. But as we have seen, the probability of obtaining any particular result in a statistical aggregate of cases is determined in terms of the coefficients of the initial wave function with the aid of eq. (45).

In the causal interpretation of the quantum theory, a typical measurement process of the kind that can be carried out in connection with the atomic level is therefore not really a measurement of the detailed properties of the underlying system (which is assumed to consist of the Ψ field plus the particle that moves in it). Rather, it represents a kind of statistical response to certain over-all properties of the Ψ field and of the particle. In this sense, the « observables » are rather analogous to the pressure and temperature of macroscopic physics or to the macroscopic variables used in hydrodynamics, since these likewise do not describe any detailed properties of the underlying molecular motions, but rather general statistical properties of the system as a whole.

Let us now consider how this theory of measurements could be extended to the Pauli spin theory. To do this, we should need a theory of the many-body problem, corresponding to our theory of the many-body-Schrödinger equation [9], because as we have seen, we have had to discuss the interaction of the electron with the atomic nucleus in order to develop a theory of measurements. Such a theory is now being developed in terms of an extension of our model to include second quantization; and preliminary results suggest that such an extension will be possible, although many of the details remain to be worked out. For the present, however, we shall merely state that it appears that the theory of measurements can be carried out for the spin theory in a way that is essentially the same as what has been done for the Schrödinger equation without spin. For example, in the case of a spinning electron in an atom in an « s » state, there will be two possible basic solutions, one corresponding to spin up, and the other to spin down, with the spin measured in any desired direction. Hence, the initial wave function can be expressed as

$$\Psi = a_+\Psi_+(\boldsymbol{x}) + a_-\Psi_-(\boldsymbol{x}) .$$

Then, as in the case that we have already discussed of a particle of angular momentum, $\hbar$, obeying Schrödinger's equation, the effect of the inhomogeneous magnetic field will be to transform this wave function either into Ψ_+ or into Ψ_-, with respective probabilities, $P_+=|a_+|^2$, and $P_-=|a_-|^2$, of obtaining either result. Hence, as in the Schrödinger theory, the appearance of quantized possible values for the spin in a measurement will be the result of a reaction to the magnetic field of the system as a whole, (consisting of fluid with spinning bodies, and the particle-like inhomogeneity, described in Sec. **1**, which is the counter part of the particle appearing in Schrödinger's equation). Thus, what is obtained in what is now called a measurement of the spin will have no direct and simple relationship to the spins of the bodies constituting the fluid. Indeed, as we have seen, the components of the latter can vary continuously, but nevertheless, in a magnetic field, the overall motion of the whole system is such that the total angular momentum eventually settles down either to $\hbar/2$ or to $-\hbar/2$. (We may make here an analogy to certain kinds of classical non-linear oscillators, which after being disturbed eventually settle down to one of a number of possible stable modes of oscillation). It is clear then that the spin as it is now measured should be considered as a higher-level property, having a relationship to the assumed spinning bodies that is somewhat analogous to the relationship between macroscopic variables, such as pressure and temperature, and the underlying atomic variables.

6. – Summary and Conclusions.

In this paper and in the previous paper (A) we have developed a model for the Pauli equation in terms of a fluid composed of spinning bodies, which contribute an « intrinsic angular momentum » to the total angular momentum of the system. This model has the property that if the bodies are at any time all spinning with their angular momenta parallel to their principal axis of symmetry, then they will continue to satisfy this condition for all time. On the other hand, it is possible for the angular momentum $\boldsymbol{S}$ to have a general orientation; and in this case, the component of $\boldsymbol{S}$ normal to the principal axis will turn with an angular velocity, ω, that depends on how fast the body happens to be spinning and in the torques acting on the body. For processes with characteristic times, $\tau \gg 1/\omega$, the component of $\boldsymbol{S}$ normal to the principal axis will average out to zero, and the Pauli theory will provide a good approximation. But for processes in which τ is of the order of $1/\omega$ or less, the Pauli equation will no longer apply, and the full general set of non-linear equations will be needed. This means that our model already implies the possibility of a break down in connection with sufficiently high frequencies, and therefore with sufficiently high energies, of the whole general scheme connected with the usual interpretation of the quantum theory, which is based in an essential way on the assumption that the fundamental equations of the theory will *always* be linear.

In connection with our discussion of the theory of measurements in Sec. **5**, it was seen that the spinning bodies of which our fluid is assumed to be constituted are not identical with the spin « observables » of the usual quantum theory, but that rather, they constitute a lower level, in terms of which the spin « observables » are determined as overall and in general statistical properties of the fluid. For example, the characteristic quantized way in which the spin angular momentum manifests itself at the atomic level was seen in Sec. **4** to follow from conditions of single-valuedness applying to the motion of the fluid as a whole, which are such that even though the spin motions of the bodies are continuous, the overall motion has certain discrete possible stationary values for the angular momentum.

In sum, then, it may be said that we have, for the case of the Pauli equation, *explained* the quantum theory in terms of the motions of new entities existing at a sub quantum-mechanical level. We call the new level « sub quantum-mechanical » because the laws of quantum-mechanics do not apply there. Rather, the laws of quantum mechanics emerge as overall and statistical relationships arising on the basis of the lower level laws, as for example, the laws of ordinary hydrodynamics arise on the basis of lower level laws governing the atomic motions.

Naturally, to make an explanation of the quantum theory possible, we have had to postulate something, viz, the fluid composed of spinning bodies (since without assuming something we can never explain anything). It may then be asked what we have gained by making such a postulate. First of all, we have gained the possibility of seeing in a rational way how all of the phenomena of atomic physics could be connected by means of a set of general causally determined motions, so that we do not have to regard atomic phenomena as mysterious processes which take place in a way that could never even be conceived of. (In this connection, let us recall that there has existed a widespread general impression that to obtain such a rationally understandable explanation of quantum phenomena in general and of the electron spin in particular would be impossible.) Secondly, whenever one obtains a rational explanation of a wide range of phenomena past experience in science has shown that this explanation generally suggests fruitful new avenues of approach to problems, which would not even have been suspected if the phenomena had not been thus explained, but rather had simply been accepted as things that « just happen » for no particular reason whatever. For example, the atomic theory, suggested originally by the effort to explain the laws of chemical combination and the gas laws in terms of the properties of atoms, was eventually able to explain many new kinds of phenomena (e.g., Brownian motion, viscosity, gaseous discharges, etc.) and suggested important new directions of research (e.g., Rutherford scattering, electron theory of metals, etc.). As for the question of wether the assumption of a sub quantum-mechanical level will eventually prove to be fruitful in a similar way, this can of course be answered definitively only in the future. Nevertheless, one can already see good reasons why this approach may be on the right track, even if, perhaps, not correct in all of its details. Thus, the characteristic new phenomena of modern high energy physics is the appearance of a whole host of « elementary » particles, which can be « created », « destroyed » and transformed into each other. The very fact that these processes of creation, destruction and transformation are possible suggests strongly that the so-called elementary particles are not really elementary, but rather, that they arise on the basis of motions of new kinds of entities that are still more fundamental. Thus, what is suggested is a *new level*, below that of the « elementary » particles, out of which these particles arise as some kind of moving structures.

Now, we have already seen that to explain the quantum theory causally, we have already had to postulate a sub-quantum mechanical level, out of the motions of which the usual quantum mechanical properties of things arose as overall characteristics (e.g., quantization). Now, as long as the basic equations governing the system are *linear*, nothing new can arise in these overall characteristics, not already treated in the well-known solutions of the Pauli equation. But as we have seen, it is just in connection with sufficiently high energy processes that the equations of our model can become *non-linear*. Now, it is well known that non-linear equations have, in general, many modes of stable motion. Each of these modes would manifest itself at the atomic level in connection with new rules for quantization and for the determination of other overall properties of the system, which we would interpret in terms of the appearance of a new kind of « particle ». Thus, the way is opened up for a treatment of the processes of creation, destruction, and transformation of elementary « particles », as well as for a calculation of which kinds of « particles » can exist, and of what some of their properties are, since the new « particles » could correspond to new modes of overall motion of the underlying fluid.

Of course, we do not believe that a model based on an explanation of the Pauli equation will really be adequate for the purposes described in the previous paragraph because it is not relativistic. A model based on an explanation of the Dirac equation (and better still with second quantization) should however give a much more accurate treatment than would be possible with the model given in this paper. Present work indicates that models can already be found which reproduce most of the features of the Dirac equation and many of those of second quantization. The completion of this work would then lay the foundation for an attack on the properties of the new level, including those connected with the creation, destruction and transformation of elementary particles. In any case, it is clear that new directions of investigation could thus be opened up, going outside the framework of theories that fit into the current general scheme of the quantum theory.