Systems of Self-Gravitating Bosons with a Cut-off in Their Distribution Function. Newtonian Treatment (*) (**).

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Summary. — We examine the Newtonian equilibrium configurations of a system of bosons undergoing quantum condensation, with a distribution function with a cut-off in the momentum space. Bounded configurations with a core of condensed particles surrounded by an uncondensed phase are obtained. The results are compared and contrasted with the ones in which the spatial divergences are removed by a cut-off in density. The well-known solution corresponding to fully condensed configurations is obtained for suitable values of the central density.

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1. – Introduction.

A quantum system of self-gravitating bosons admits a configuration of equilibrium for a number of particles less than the critical value $(m_{\text{Pl}}/m)^2$, the pressure caused by the quantum-mechanical uncertainty principle keeping the system from collapsing under its gravity [1].

At the opposite limit, the classical systems of self-gravitating bosons or fermions admit configurations of equilibrium for any value of the number of particles [2], but in order to obtain finite values of the masses and radii, a direct cut-off in the density must be introduced. This method leads to the usual definition of an *isothermal core*.

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The spatial divergences of the classical systems can be also removed by introducing a cut-off in the velocity distribution function in phase space [3, 4]. This corresponds to the request that the particles with kinetic energy exceeding an escape energy are lost to the system.

More recently a distribution function with energy cut-off has been proposed for particles fulfilling Fermi-Dirac and Bose-Einstein statistics [5]. In the fermionic case, the existence and stability of self-gravitating configurations have been analysed in a series of papers, from the classical regime [6] to the semi-degenerate and fully degenerate regime [7], both in Newtonian and relativistic regime.

In this work we present a Newtonian treatment of self-gravitating systems of bosons with an energy cut-off in the distribution function: we consider isothermal and spherically symmetric systems, from the classical to the fully condensed regime. As a result we obtain configurations where a core of condensed bosons is surrounded by an envelope of uncondensed bosons. This treatment is compared and contrasted with the one in which the spatial divergence is removed by a direct cut-off in the density [8].

2. – Distribution functions with a cut-off in momentum space.

We consider a system of bosons, with the same mass m and an isotropic velocity distribution function. We assume in complete generality that the distribution function of such particles in momentum space is given by [5]

$$(1)\qquad \begin{cases} \mathrm{d}n = \dfrac{g}{h^3}\dfrac{1-\exp[(\varepsilon-\varepsilon_c)/kT]}{\exp[(\varepsilon-\mu)/kT]-1}\,\mathrm{d}^3p(\varepsilon) & \text{for } \varepsilon \leqslant \varepsilon_c\,, \\ \mathrm{d}n = 0 & \text{for } \varepsilon > \varepsilon_c\,, \end{cases}$$

where $g = 2s+1$ is the spin multiplicity of quantum states, h the Planck constant, μ the chemical potential and T the temperature of the Bose gas. In the limit $\varepsilon_c \to \infty$, which corresponds to the absence of a cut-off, we have

$$(2)\qquad \mathrm{d}n = \frac{g}{h^3}\frac{1}{\exp[(\varepsilon-\mu)/kT]-1}\,\mathrm{d}^3p(\varepsilon)\,,$$

which is the well-known distribution function in the Bose-Einstein statistics.

We are interested in the nonrelativistic regime of the gas, where $kT \ll mc^2$. Therefore starting from the general distribution of eq. (1), the pressure p and the mass density ρ can be expressed by

$$(3)\qquad p = \frac{8\sqrt{2}\pi}{3}\frac{gm^4c^5}{h^3}\beta^{5/2}\int_0^W \frac{1-\exp[x-W]}{\exp[x-\theta]-1}\,x^{3/2}\,\mathrm{d}x$$

and

$$(4)\qquad \rho = 4\sqrt{2}\pi\,\frac{gm^4c^3}{h^3}\beta^{3/2}\int_0^W \frac{1-\exp[x-W]}{\exp[x-\theta]-1}\,x^{1/2}\,\mathrm{d}x\,,$$

where

$$(5)\qquad \theta = \mu/kT\,, \qquad W = \frac{\varepsilon_c}{kT} \quad \text{and} \quad \beta = \frac{kT}{mc^2}\,.$$

The thermodynamic functions defined in eqs. (3), (4) depend on four different parameters: the mass of the particle m, the temperature parameter β, the degeneracy parameter θ and the dimensionless energy cut-off parameter W. In the limit $W \to \infty$, which corresponds to the absence of a cut-off, eq. (4) with $\theta = 0$ defines, as a function of the temperature β, the mass density ρ_{cond} at which the condensation sets in for a gas of bosons [9]. Thus, we obtain

$$\rho_{\text{cond}} = 4\sqrt{2}\pi \frac{gm^4 c^3}{h^3} \beta^{3/2} \Gamma(3/2)\, \zeta(3/2)\,, \tag{6}$$

where Γ and ζ are, respectively, the gamma-function and the Riemann zeta-function. Equation (6) also defines, for a given density ρ, the condensation temperature β_{cond} of a Bose gas.

If we consider a system of bosons not interacting and in an infinite volume, for $\beta < \beta_{\text{cond}}$ ($\rho > \rho_{\text{cond}}$) a finite number of particles undergoes *condensation* into a level of zero momentum $\boldsymbol{p} = 0$. This phenomenon is known as the Bose-Einstein condensation in momentum space [10].

We assume that the Bose-Einstein condensation exists in systems of bosons with cut-off in the distribution function in momentum space. Therefore eq. (4) with $\theta = 0$ defines, for any value of the energy cut-off parameter W, a condensation density $\rho_{\text{cond}}(W)$ and a condensation temperature $\beta_{\text{cond}}(W)$. For a given value of W, the condition $\beta \gg \beta_{\text{cond}}(W)$ [$\rho \ll \rho_{\text{cond}}(W)$] corresponds to the classical regime of a Bose system. This regime is reached in the limit $\theta \to -\infty$. The condition $\beta \geqslant \beta_{\text{cond}}(W)$ [$\rho \leqslant \rho_{\text{cond}}(W)$] defines an intermediate regime with $\theta \leqslant 0$. For $\beta < \beta_{\text{cond}}(W)$ [$\rho > \rho_{\text{cond}}(W)$] the Bose gas with energy cut-off in the distribution function presents a mixture of two thermodynamic phases: an uncondensed phase, composed of particles with momentum $\boldsymbol{p} \neq 0$, and a condensed phase, composed of particles with $\boldsymbol{p} = 0$. In the limit $\beta \to 0$, all bosons are condensed in the ground state with $\boldsymbol{p} = 0$ and the pressure of the gas vanishes.

All the above considerations clearly apply only to a gas of bosons occupying an infinite volume. If we now turn to a system of self-gravitating bosons, the situation is different: the gravitation induces a confinement on the system and the concept of equation of state fails when applied to the condensed phase.

A new method of studing the fully condensed system of self-gravitating bosons was introduced by Bonazzola and Ruffini [1]. This method is considered here in order to obtain a more general case of a system of partially condensed bosons.

3. – Equilibrium equations for uncondensed systems.

We consider a static, isothermal and spherically symmetric system. In the equilibrium configurations of such a system the mass density is a radial function decreasing from the centre to the border of the configuration. As a consequence, for fixed values of the central density ρ_0 and the temperature parameter β, we have uncondensed configurations with $\rho_0 \leqslant \rho_{\text{cond}}$ or partially condensed systems with $\rho_0 > \rho_{\text{cond}}$. This classification describes, respectively, systems in which a inner region of uncondensed bosons is surrounded by particles in the classical regime and systems in which a core of partially condensed bosons is surrounded by uncondensed particles. In the limit $\theta \to -\infty$ we recover the well-known classical isothermal spheres and, for $\beta \to 0$, systems of fully condensed particles are approached.

In the Newtonian limit the equations for gravitational equilibrium are given by

$$\frac{dV}{dr} = \frac{GM_r}{r^2} \tag{7}$$

and

$$\frac{dM_r}{dr} = 4\pi r^2 \rho \,, \tag{8}$$

where V is the gravitational potential, M_r the mass within a given radius r and ρ the mass density. For uncondensed systems eq. (7) can be rewritten in the form

$$\frac{dp}{dr} = -\frac{GM_r \rho}{r^2} \,, \tag{9}$$

with the pressure p and the density ρ given by eqs. (3), (4) with $\theta \leq 0$.

Due to the presence of the gravitational potential V, we can impose constraints on the energy cut-off parameter W and the degeneracy parameter θ. For the energy cut-off parameter W we have

$$W = -\frac{V}{\beta c^2} \,, \tag{10}$$

while for the degeneracy parameter θ we have [6]

$$\theta = W + \theta_R \,, \tag{11}$$

where θ_R is the degeneracy parameter at the surface of the configuration. Clearly for any given equilibrium configuration, the temperature parameter β is a constant, while θ and W are functions of the radial coordinate r. On the other hand, since θ and W are related by eq. (11), we can express the pressure p and the mass density ρ in terms of W. In this way we obtain quantities which depend on the radial coordinate r only through the parameter W. Equation (9) then becomes

$$\frac{dW}{dr} = -\frac{G}{\beta c^2} \frac{M_r}{r^2} \tag{12}$$

and eqs. (8)-(12) must be integrated with the boundary conditions

$$M_0 = 0 \quad \text{and} \quad W(0) = W_0 \tag{13}$$

up to the radius R at which $W(R) = 0$ or, equivalently, $\rho(R) = 0$.

4. – Equilibrium equations for partially condensed systems.

In this section we consider systems of bosons with a central density $\rho_0 > \rho_{\text{cond}}$. We indicate by R_c the radial coordinate at which the density becomes equal to the condensation density: for $r \leq R_c$ the configuration is composed of a condensed phase of N_c bosons in their ground state in gravitational equilibrium with a phase of uncondensed bosons; for $r > R_c$ the system is only composed of an uncondensed phase.

For $N_c \ll (m_{Pl}/m)^2$ the condensed phase can be treated following the Newtonian treatment of [1]. The uncondensed phase, instead, can be treated with the Newtonian formalism of sect. 3. In this approximation the equations for the gravitational equilibrium are given by eqs. (7), (8), with the mass density ρ here rewritten as

$$\begin{cases} \rho = \rho_{nc} + \rho_c & \text{for } 0 < r \leqslant R_c , \\ \rho = \rho_{nc} & \text{for } R_c < r \leqslant R , \end{cases} \tag{14}$$

where ρ_{nc} is given by eq. (4) with the condition (10), and the degeneracy parameter θ varying as

$$\begin{cases} \theta = 0 & \text{for } 0 < r \leqslant R_c , \\ \theta = W + \theta_R & \text{for } R_c < r \leqslant R . \end{cases} \tag{15}$$

The Schrödinger equation for a particle of mass m in the presence of the gravitational potential V is given by

$$\Delta\psi + (2m/\hbar^2)(E - mV)\psi = 0 . \tag{16}$$

The condensed phase of N_c bosons can be considered like a quantum system in its ground state with the density ρ_c expressed by

$$\rho_c = mN_c \psi^* \psi , \tag{17}$$

where ψ is the ground-state eigenfunction of the Schrödinger equation, depending on the radial coordinate r. The wave function is normalized to unity by the condition

$$\int_0^{R_c} 4\pi\psi^* \psi r^2 \, dr = 1 . \tag{18}$$

We integrate the system of equations (7), (8), (16) and (18) from the centre up to the radius R_c with the boundary conditions

$$M_0 = 0 \quad \text{and} \quad \rho(0) = \rho_0 . \tag{19}$$

As usual the eigenfunction and the eigenvalue of the ground state are determined by requiring that the wave function ψ has no nodes.

For $r > R_c$ the gravitational equilibrium is governed by eqs. (8) and (12) with the boundary conditions

$$M(R_c) = mN_c + \int_0^{R_c} 4\pi\rho_{nc} r^2 \, dr \quad \text{and} \quad W(R_c) = -\theta_R . \tag{20}$$

5. – Results of numerical integrations.

The integration of the equilibrium equations has been performed assuming selected values of the temperature parameter β, of the central energy cut-off parameter W_0 and of the degeneracy parameter at the surface of the configuration θ_R. The mass of the particles has been fixed at the value $m = 1$ GeV/c^2, the spin at the value $s = 0$.

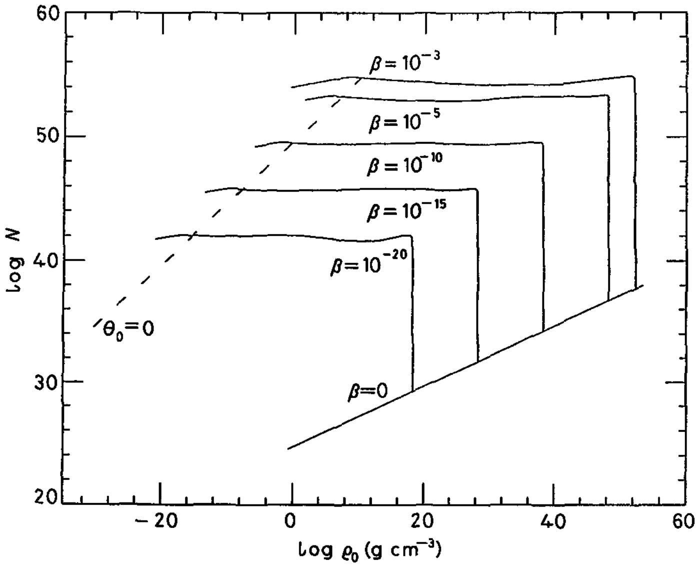

Fig. 1. – The number N of self-gravitating bosons of the Newtonian equilibrium configurations is given as a function of the central density ρ_0 for values of the temperature parameter β in the range $10^{-20} \leq \beta \leq 10^{-3}$. The degeneracy parameter at the surface of the configuration has been fixed at the value $\theta_R = -1$, the mass of the particles at the value $m = 1$ GeV/c^2 and the spin at the value $s = 0$. The dash-dotted line corresponds to configurations with $\rho = \rho_{cond}$, while the continuous line labelled with $\beta = 0$ corresponds to the fully condensed solution of Ruffini and Bonazzola.

In fig. 1 the number N of self-gravitating boson is given as a function of the central density ρ_0 for selected values of β, in the case of configurations with $\theta_R = -1$. The line along which $\theta_0 = 0$ gives N as a function of ρ_0 for configurations with $\rho_0 = \rho_{cond}$ and corresponds to the transition between the uncondensed configurations and the configurations with a core of partially condensed bosons. The line with $\beta = 0$ corresponds to the fully condensed configurations.

Along each curve with fixed values of β, there is a sequence of four different families of equilibrium configurations. We have a first family of classical configurations where N increases for increasing values of the central density until a maximum value of N is reached. At this point a second family of uncondensed configurations occurs where N decreases until the condensation value of the density is reached. The behaviour of these families corresponding to configurations of uncondensed bosons differs from the ones obtained by Ingrosso and Ruffini [8] due to the choice of the cut-off. In that case, where a direct cut-off in density was considered, the number of particles N of the equilibrium configurations is always decreasing with increasing values of the central density.

For ever-increasing values of the central density we have a third family of partially condensed configurations along which N is approximately a constant, while the number of condensed bosons N_c increases up to a value $N_c \sim \beta^{1/2}(m_{Pl}/m)^2$. At this point a sharp decrease of N occurs, followed by the last family corresponding to the fully condensed configurations, increasing in N for increasing values of the central density.

For a number of condensed bosons $N_c \sim (m_{Pl}/m)^2$ corresponding to values of the temperature parameter $\beta \sim 1$, the gravitational energy and the thermal energy of the particles become of the order of the rest mass energy. In this case correction coming from special and general relativistic effects must be taken into account. This has been carried out by Bonazzola and Ruffini [1] for a system of fully condensed bosons. The results is that no equilibrium configurations exist for fully condensed systems of $N_c \gtrsim (m_{Pl}/m)^2$ bosons, while, for a number of particles $N_c \ll (m_{Pl}/m)^2$, the relativistic treatment approaches the Newtonian approximation developed in this paper.

6. – Conclusion.

We have seen in the previous sections the Newtonian treatment of self-gravitating systems of bosons with a cut-off in their distribution function. The families of equilibrium configurations, for selected values of the temperature parameter β, are finite both in masses and radii. They largely differ from the ones obtained by including a direct cut-off in density, for low values of the central density corresponding to the uncondensed regime. For large values or the central density the well-known results of the fully condensed configurations are recovered.

The results obtained in the present paper have been generalized to the general relativistic regime and the stability analysis of the equilibrium configurations has been developed as well. These issues will be published in a forthcoming paper [11].

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