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New Solutions for Charged Anisotropic Fluid Spheres in General Relativity.

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Summary. — The general analytical solutions for charged fluid distribution with anisotropic pressure are obtained. These solutions depend on an arbitrary generating function and the choice of an anisotropic function which measures the degree of anisotropy. As an illustration of the procedure some physically important examples are considered.

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1. – Introduction.

In recent years the solutions of Einstein's field equations corresponding to fluid distribution with anisotropic pressure have generated great interest among physicists [1-10]. These solutions are relevant in the study of relativistic astrophysics as model of compact object which has anisotropic pressure [11]. Recently, Rago [12] has presented an anisotropic solution which is a generalization of the static solution of isotropic fluid spheres [13]. Singh *et al.* [14, 15] have studied static anisotropic fluid spheres with non-uniform density and in higher-dimensional space-time. The charged-matter distribution problems in general relativity also have received considerable attention. Patino and Rago [16] have found some new solutions for charged fluid spheres. Singh *et al.* [17, 18] have extended Bayin's work [19] to the case of charged fluid spheres and Gaete and Hojman's work [20] to the case of magnetofluids.

The object of this paper is to extend the work of Rago [12] in the presence of an electromagnetic field.

2. – Field equations and conventions.

We will consider the standard coordinate and the line element as given by

$$\mathrm{d}s^2 = \exp[\nu]\,\mathrm{d}t^2 - \exp[\lambda]\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{1}$$

Here ν and λ are functions of the radial coordinate r alone.

Einstein's field equations have the following well-known form:

$$R_{ji} - \frac{1}{2}Rg_{ij} = -8\pi(T_{ij} + E_{ij}), \tag{2}$$

where the energy-momentum tensor T_{ij} for anisotropic fluid distribution is defined by

$$T_{ij} = (\rho_\mathrm{m} + p_\mathrm{r})u_i u_j - p_\perp g_{ij}(p_\mathrm{r} - p_\perp)x_i x_j, \tag{3}$$

where u^i is the fluid four-velocity vector $u^i = \delta^i_4 \exp[-\nu/2]$, x^i is unit space-like vector in the radial direction $x^i = \delta^i_1 \exp[-\lambda/2]$, ρ_m is the energy density of matter, p_r is the pressure in the direction of x_i and $p_\perp$ is the pressure on the two-space orthogonal to x_i.

The energy-momentum tensor of electromagnetic field is given by

$$E_{ij} = \frac{1}{4\pi}\left[g^{kl}F_{ik}F_{jl} - \frac{1}{4}g_{ij}F_{kl}F^{kl}\right], \tag{4}$$

where F_{ij} is the electromagnetic-field tensor defined in terms of the four-potential A_i as

$$F_{ij} = A_{j;\,i} - A_{i;\,j}. \tag{5}$$

The electromagnetic-field equations are given by

$$F_{ij;\,k} + F_{jk;\,i} + F_{ki;\,j} = 0, \tag{6}$$

$$F^{ij}{}_{;\,j} = -4\pi J^i. \tag{7}$$

Here J^i is the four-current density. The combined Einstein-Maxwell equations for line element (1) can be expressed as

$$8\pi\rho_\mathrm{m} + \frac{\theta^2}{r^4} = \exp[-\lambda]\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \tag{8}$$

$$8\pi\rho_\mathrm{r} - \frac{\theta^2}{r^4} = \exp[-\lambda]\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2}, \tag{9}$$

$$8\pi\rho_1 + \frac{\theta^2}{r^4} = \frac{\exp[-\lambda]}{2}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} + \frac{\nu' - \lambda'}{r}\right), \tag{10}$$

$$p_\mathrm{r}' + (p_\mathrm{r} + \rho_\mathrm{m})\frac{\nu''}{2} = \frac{2}{r}(p_\perp - p_\mathrm{r}) + \frac{1}{8\pi r^4}\frac{\mathrm{d}Q^2}{\mathrm{d}r}, \tag{11}$$

where

$$Q(r) = 4\pi \int_0^r r^2 \rho_e \, dr, \tag{12}$$

is the charge within a sphere of radius r and the charge density ρ_e is related to the proper charge density $\bar{\rho}_e$ by

$$\rho_e = \bar{\rho}_e \exp[\lambda/2]. \tag{13}$$

Equation (8) can be integrated to give

$$\exp[-\lambda] = 1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}, \tag{14}$$

where we have introduced the mass function $m(r)$ of the fluid distribution defined as

$$m(r) = \int_0^r \left(4\pi\rho_m r^2 + \frac{QQ'}{r}\right) dr. \tag{15}$$

By use of eqs. (11) and (14) from eq. (9), we have

$$8\pi p_r - \frac{Q^2}{r^4} + \frac{1}{r^2} = \tag{16}$$

$$= \left[1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right]\left[\frac{1}{r^2} - 2\left\{p_r' - \frac{2}{r}(p_\perp - p_r) - \frac{QQ'}{4\pi r^4}\right\}\right] \Big/ r(p_r + \rho_m).$$

Now, we define a generating function

$$G(r) = \frac{[1 - 2m/r + Q^2/r^2]}{[1 + 8\pi p_r r^2 - Q^2/r^2]} \tag{17}$$

and an anisotropic function

$$W(r) = \frac{4(p_r - p_\perp)}{(\rho_m + p_r)} G(r). \tag{18}$$

With the help of eqs. (15), (17) and (18), eq. (16) can be written as

$$p_r' + \frac{(1 - G + W)(1 - 3G - G'r)}{Gr(1 + G - W)} p_r = \frac{QQ'}{2\pi r^4(1 + G - W)} - \tag{19}$$

$$- \frac{(1 - G + W)}{2\pi r^3(1 + G - W)}\left[1 - G - G'r - (1 + G - G')\frac{Q^2}{r^2} + \frac{2QQ'}{r} G\right].$$

It is clear that for given $G(r)$, $W(r)$ and $Q(r)$ as known functions of r the linear differential equation (19) can be integrated to give the general solution

$$p_r = \exp\left[-\int B\,\mathrm{d}r\right]\left[p_0 + \int C\exp\left[\int B\,\mathrm{d}r\right]\mathrm{d}r\right], \tag{20}$$

where p_0 is an arbitrary integration constant and functions $B(r)$ and $C(r)$ are given by

$$B(r) = \frac{(1-3G-G'r)(1-G+W)}{rG(1+G-W)}, \tag{21}$$

$$C(r) = \frac{QQ'}{2r^4(1+G-W)} - \tag{22}$$

$$-\frac{(1-G+W)}{8r^3G(1+G-W)}\left[1-\frac{Q^2}{r^2}-\left(1+\frac{Q^2}{r^2}-\frac{2QQ'}{r}\right)G-\left(1-\frac{Q^2}{r^2}\right)G'r\right].$$

Once p_r is known, the matter density ρ_m can be easily calculated from eqs. (15) and (17) thus obtaining

$$8\pi\rho_m = \frac{1}{r^2}\cdot \tag{23}$$

$$\cdot\left[1-G\left(1+24\pi p_r' r^2+8\pi p_r' r^3-\frac{2QQ'}{r}+\frac{Q^2}{r^2}\right)-G'\left(r+8\pi p_r r^3-\frac{Q^2}{r^2}\right)-\frac{Q^2}{r^2}\right].$$

After obtaining $p_r(r)$ and $\rho_m(r)$, the tangential pressure $p_\perp$ can be found from eq. (18),

$$p_\perp = p_r - \frac{W(\rho_m + p_r)}{4G}. \tag{24}$$

Finally, taking into account eqs. (11), (14)-(17), the metric coefficients can be expressed as

$$\exp[-\lambda] = G\left(1+8\pi p_r r^2 - \frac{Q^2}{r^2}\right), \tag{25}$$

$$\exp[-\nu] = \frac{A^2}{r}\exp\left[\int\frac{\mathrm{d}r}{rG}\right]. \tag{26}$$

Here A^2 is also an integration constant.

3. – Illustration of the method.

We should like to point out that any given functions $G(r)$, $W(r)$ and charge distribution $Q(r)$ generate static anisotropic spherically symmetric solutions of Einstein-Maxwell equations. For a physically meaningful solution the generating function G must satisfy some general requirements. Assuming a non-divergent pressure at the origin, the regularity conditions at the origin $r = 0$ $(m(r)/r \to 0,$

$Q^2/r^2 \to 0$, $\exp[\lambda] \to 1$ as $r \to 0$) imply that $\lim_{r \to 0} G(r) = 1$. If $G = 1$, $W = 0$ ($p_r = p_\perp$) and $Q = 0$, one obtain Minkowski flat space-time.

If we consider

$$G(r) = \frac{(1 - 2M/r + e^2/r^2)\, r^3}{(1 - e^2/r^2)} \tag{27}$$

and

$$Q(r) = e\,, \tag{28}$$

then

$$\exp[\lambda] = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}, \tag{29}$$

$$\exp[\nu] = \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right). \tag{30}$$

By this way one can get, with $p_0 = 0$ in eq. (20), a corresponding Reissner-Nördstrom solution, irrespective of the choice of the anisotropic function. Any interior solution must join smoothly to Reissner-Nördstrom metric at the surface $r = r_0$ of the fluid distribution. For this requirement we must demand continuity of generating function at $r = r_0$,

$$G(r_0) = G^{\mathrm{RN}}(r_0) = \frac{(1 - 2M/r_0 + e^2/r_0^2)}{(1 - e^2/r_0^2)}\,, \tag{31}$$

$$Q(r_0) = e\,. \tag{32}$$

Equation (32) indicates the continuity of the radial electric field assuming no charge concentration at the boundary surface. One can easily see that there is no junction condition imposed on the anisotropic function w.

If we consider that charge density is constant, then eq. (12) implies that $Q(r) \sim r^3$. The appropriate junction condition at r_0 yields

$$Q(r) = e(r/r_0)^3\,. \tag{33}$$

Further, we assume

$$G(r) = 1 - ar^2 \tag{34}$$

and

$$W(r) = -\,ar^2\,, \tag{35}$$

where a is a constant. This choice is also physically reasonable, because function $G(r) \sim 1$ as $r \sim 0$. The value of the constant is to be calculated in order to satisfy the boundary conditions (31) and (32). Then

$$a = \frac{2(M/r_0 - e^2/r_0^2)}{(r_0 - e^2)} . \tag{36}$$

With the help of eqs. (33)-(35), the expression of the solutions, from eqs. (20)-(26), can be written as

$$8\pi p_r = 8\pi p_0 + 6K^2 r^2 , \tag{37}$$

$$8\pi\rho_m = 3a + 8\pi p_0 (5ar^2 - 3) + (36ar^2 - 26) K^2 r^2 , \tag{38}$$

$$8\pi p_\perp = 8\pi p_0 + 6K^2 r^2 + \frac{ar^2}{4(1 - ar^2)} [3a + 8\pi p_0 (5ar^2 - 2) + (35ar^2 - 20) K^2 r^2], \tag{39}$$

$$\exp[-\lambda] = (1 - ar^2)(1 + 8\pi p_0 r^2 + 5K^2 r^4), \tag{40}$$

$$\exp[\nu] = \frac{A^2}{(1 - ar^2)^{1/2}} , \tag{41}$$

where

$$K = e/r_0^3 .$$

We will consider again the choice

$$G(r) = b \quad (\text{const}), \tag{42}$$

$$W(r) = c \quad (\text{const}), \tag{43}$$

$$Q = Kr^3 . \tag{44}$$

Substituting these values into eqs. (20)-(26), we get the expressions for physical variables as

$$8\pi p_r = 8\pi p_0 r^{-D} + V r^{-2} - NK^2 r^2 , \tag{45}$$

$$8\pi\rho_m = 8\pi b p_0 (D - 3) r^{-D} + (1 - b - bV) r^{-2} - (1 - b - 5bN) K^2 r^2 , \tag{46}$$

$$\begin{aligned} 8\pi p_\perp = 8\pi p_0 &\left[1 - \frac{c}{4b}(1 + bD - 3b)\right] r^{-D} + \\ &+ \left[V - \frac{c}{4b}(1 - b)(1 + V)\right] r^{-2} - \left[N - \frac{c}{4b}(1 + N - b - 5bN)\right] K^2 r^2 , \end{aligned} \tag{47}$$

$$\exp[-\lambda] = 8\pi b p_0 r^{2-D} - (N + b) K^2 r^4 + b + V, \tag{48}$$

$$\exp[\nu] = A^2 r^{(1-b)/b} , \tag{49}$$

where

$$D = \frac{(1-3b)(1-b+c)}{b(1+b-c)}\,, \tag{50}$$

$$N = \frac{5b^2 - 18b + 5bc + c + 1}{5b^2 - 2b - 5bc + c + 1}\,, \tag{51}$$

$$V = \frac{(b-1)(1-b+c)}{b^2 - 6b - bc + c + 1}\,. \tag{52}$$

This solution represents the uniformly anisotropic charged fluid distribution which is an anisotropic charged analogue of Tolman-V solution [21] with a slight change in notation (his n corresponding to $(1-b)/2b$). For a neutral isotropic sphere (*i.e.* $K=0$, $c=0$) Tolman's results are recovered.

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