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The Classification of Spherically Symmetric Space-Times.

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Summary. — A complete classification of spherically symmetric space-times according to their isometries and metrics (or classes of metrics) is obtained by solving the Killing equations. It is demonstrated that a symmetry given by Turkowski does not correspond to any metric.

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1. - Introduction.

Spherically symmetric space-times are not only the simplest to deal with mathematically, but have great physical relevance. However, there is no easily accessible literature available which provides a complete, concise classification of such space-times. Such a classification was achieved [1] and has already been useful in proving a singularity theorem without reference to a positive-energy condition [2] and for discussion of Ricci collineations [3]. More importantly still, the methods developed for these space-times have been extended to plane symmetric static space-times [4] and to cylindrically symmetric static space-times [5]. There will surely be many other applications and developments to arise from this work.

Ever since Einstein's equations were written down in 1915, there has been much interest in investigating the properties of their solutions [6]. Special interest has been focused on their symmetry properties. These properties are given locally by the solutions, k^a, of the Killing equations

$$g_{ab,c}k^c + g_{ca}k^c{}_{,b} + g_{bc}k^c{}_{,a} = 0 \qquad (a, b, c, \ldots = 0, 1, 2, 3), \tag{1.1}$$

where g_{ab} is the metric tensor. The solutions, called Killing vectors, can be regarded as the generators of a Lie algebra. Normally, the Einstein equations are solved for a specific stress-energy tensor, T^{ab}, to obtain g_{ab}. Now, since g_{ab} are known functions of x^a, eqs. (1.1) form a linear system of first-order partial differential equations for four functions, k^a, of the four variables, x^a. However, if we do not restrict T^{ab}, and hence g_{ab} are not known functions of x^a, the system by eqs. (1.1) is a non-linear system of ten first-order partial differential equations for the fourteen functions ($10g_{ab}$'s and $4k^a$'s of the four variables x^a).

The group-theoretic approach to solving eqs. (1.1) was pioneered by Eisenhart [7] who studied 2- and 3-dimensional spaces. In the former case there are 3 equations for 5 functions of 2 variables and in the latter 6 equations for 9 functions of 3 variables. He classified *all* the metrics along with their symmetry structures for these cases. This work was extended by Petrov [8] to 4-dimensional spaces. He claimed to have obtained a complete (and exhaustive) classification of gravitational fields admitting groups of motion of all possible orders. However, the inverse problem, as to which groups of motions correspond to a given type of gravitational field, remains unsolved (p. 132 of ref. [8]). By restricting attention to spherical symmetry and staticity, Bokhari and Qadir [9] were able to achieve a complete classification and exhaustive list of that class of *space-times*. In this paper we remove the condition of staticity and obtain an exhaustive list and complete classification of *all* spherically symmetric space-times.

It is pertinent to point out at this stage that this classification is based upon an elimination procedure and gives an exhaustive list of *all* possible metrics (or classes of metrics) classified according to their isometries. However, there is a redundancy, as many of the metrics appear repeatedly. The curvature invariants are calculated and metrics which appear different but are in fact the same, expressed in some other coordinates, are eliminated.

It is found that there are symmetries which have not been given explicitly by Petrov, *i.e.* all G_7's and some G_6's. These are given in sect. 2 to sect. 3. Further, we have unique metrics (depending only on some parameters) for the higher symmetries and classes of metrics for the lower symmetries. The properties of some of these space-times are given in table II at the end.

The plan of the paper is as follows. In sect. 2 the Killing equations will be solved analytically to give k^a in terms of explicit functions of two of the spatial coordinates, the metric coefficients and five arbitrary functions of the other two coordinates. Then the resulting equations are completely solved in sect. 2 and sect. 3. The results are summarised and discussed in the concluding section.

2. – Explicit solution of the Killing equations in ϑ and ϕ.

We will use the following notation:

$$x^a = (t, r, \vartheta, \phi) \qquad (a = 0, 1, 2, 3);$$

the partial derivative will be denoted by a comma «,» and partial derivatives relative to t and r by a dot «$\cdot$» and prime «$'$», respectively;

spherically symmetric metrics will be written as [6]

$$ds^2 = \exp[\nu]\, dt^2 - \exp[\lambda]\, dr^2 - r^2 \exp[\mu]\, d\Omega^2 , \tag{2.1}$$

where ν, λ, μ are functions of the variables t and r and $d\Omega^2 (= d\vartheta^2 + \sin^2\vartheta\, d\phi^2)$ is the usual solid-angle element squared;

$\boldsymbol{K} = k^a(\partial/\partial x^a)$ will represent the Killing vector field with components k^a;

G_r will represent [6, 8] an r-dimensional group and $\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots$ its generators, *i.e.* $G_r = \langle \boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_r \rangle$;

$[X_a, X_b]$ is the Lie-bracket. B_j $(j = 1, 2, 3)$, K, L, will be used to represent arbitrary functions of t and r arising from integration;

$\otimes$ stands for the direct product whereas $\circledS$ stands for the semi-direct product.

It is worth stressing that the spherically symmetric metric given by eq. (2.1) cannot be further simplified in general. For this metric eqs. (1.1) reduce to

(2.2) $$\dot{\nu}k^0 + \nu' k^1 + 2k^0{}_{,0} = 0\,,$$

(2.3) $$\exp[\nu]k^0{}_{,1} - \exp[\lambda]k^1{}_{,0} = 0\,,$$

(2.4) $$\exp[\nu]k^0{}_{,2} - r^2 \exp[\mu]k^2{}_{,0} = 0\,,$$

(2.5) $$\exp[\nu]k^0{}_{,3} - r^2 \sin^2\vartheta \exp[\mu]k^3{}_{,0} = 0\,,$$

(2.6) $$\dot{\lambda}k^0 + \lambda' k^1 + 2k^1{}_{,1} = 0\,,$$

(2.7) $$\exp[\lambda]k^1{}_{,2} + r^2 \exp[\mu]k^2{}_{,1} = 0\,,$$

(2.8) $$\exp[\lambda]k^1{}_{,3} + r^2 \sin^2\vartheta \exp[\mu]k^3{}_{,1} = 0\,,$$

(2.9) $$\dot{\mu}k^0 + \left(\mu' + \frac{2}{r}\right)k^1 + 2k^2{}_{,2} = 0\,,$$

(2.10) $$k^2{}_{,3} + \sin^2\vartheta k^3{}_{,2} = 0\,,$$

(2.11) $$\dot{\mu}k^0 + \left(\mu' + \frac{2}{r}\right)k^1 + 2\,\mathrm{ctg}\,\vartheta k^2 + 2k^3{}_{,3} = 0\,.$$

We start by evaluating k^a in terms of explicit functions of φ and ϑ. Taking the partial derivatives of eqs. (2.7) and (2.8) with respect to ϕ and ϑ, respectively and using eq. (2.10) and also the same procedure for eqs. (2.4) and (2.5), we get

(2.12) $$k^i{}_{,32} - \mathrm{ctg}\,\vartheta k^i{}_{,3} = 0 \qquad (i = 0, 1)\,.$$

Adding $\dot{\mu}$ times eq. (2.12) with $(i = 0)$ and $(\mu' + 2/r)$ times eq. (2.12) with $(i = 1)$ and using eqs. (2.9) and (2.10) yields $k^2{}_{,3}$ and k^3 as explicit functions of ϑ. Using these values in the equation obtained by comparing eqs. (2.9) and (2.11) we get k^3 as an explicit function of ϑ and ϕ, while k^2 still depends on an arbitrary function of ϑ. Again using the value of k^2 and k^3 so obtained in eqs. (2.4) and (2.8), and using eqs. (2.9) and (2.11) we obtain k^0 and k^1 as explicit functions of ϑ and k^2 as an explicit function of ϑ and ϕ.

Using the values of k^a obtained so far in the original Killing equations we finally obtain

(2.13) $$k^0 = r^2 \exp[\mu - \nu][-\sin\vartheta(\dot{B}_1 \sin\phi - \dot{B}_2 \cos\phi) + \dot{B}_3 \cos\vartheta] + K\,,$$

(2.14) $$k^1 = -r^2 \exp[\mu - \lambda][-\sin\vartheta(B_1' \sin\phi - B_2' \cos\phi) + B_3' \cos\vartheta] + L\,,$$

(2.15) $$k^2 = -[B_1 \sin\phi - B_2 \cos\phi]\cos\vartheta + B_3 \sin\vartheta + (c_1 \sin\phi - c_2 \cos\phi)\,,$$

(2.16) $$k^3 = -[B_1 \cos\phi + B_2 \sin\phi]\,\mathrm{cosec}\,\vartheta + (c_1 \cos\phi + c_2 \sin\phi)\,\mathrm{ctg}\,\vartheta + c_3\,,$$

subject to the conditions

$$2\ddot{B}_j + (2\dot{\mu} - \dot{\nu})\dot{B}_j - \nu' \exp[\nu - \lambda] B_j' = 0, \tag{2.17}$$

$$2\dot{B}_i' + \left(\mu' + \frac{2}{r} - \nu'\right)\dot{B}_j + (\dot{\mu} - \dot{\lambda}) B_i' = 0, \tag{2.18}$$

$$2B_j'' + \left(2\mu' + \frac{4}{r} - \lambda'\right) B_j' - \dot{\lambda} \exp[\lambda - \nu]\dot{B}_j = 0, \tag{2.19}$$

$$-\dot{\mu} \exp[\mu - \nu]\dot{B}_j + \left(\mu' + \frac{2}{r}\right) \exp[\mu - \lambda] B_j' + \frac{2}{r^2} B_j = 0, \tag{2.20}$$

$$2\dot{K} + \dot{\nu} K + \nu' L = 0, \tag{2.21}$$

$$\exp[\nu] K' - \exp[\lambda]\dot{L} = 0, \tag{2.22}$$

$$2L' + \lambda' L + \dot{\lambda} K = 0, \tag{2.23}$$

$$\dot{\mu} K + \left(\mu' + \frac{2}{r}\right) L = 0, \tag{2.24}$$

where c_j are the coefficients for the three generators of the minimal group for spherical symmetry, $SO(3)$.

The problem is now reduced to finding eight functions of two variables (t and r), namely, B_j, K, L, ν, λ, and μ from a system of sixteen coupled non-linear partial differential equations. We consider all possibilities of the coefficients of the solid angle $d\Omega^2$, *i.e.* $\exp[\mu + 2 \ln r]$ in eq. (2.24):

case I $\qquad \dot{\mu} = 0, \quad \mu' + \frac{2}{r} = 0;$

case II $\qquad \dot{\mu} = 0, \quad \mu' + \frac{2}{r} \neq 0;$

case III $\qquad \dot{\mu} \neq 0, \quad \mu' + \frac{2}{r} = 0;$

case IV $\qquad \dot{\mu} \neq 0, \quad \mu' + \frac{2}{r} \neq 0.$

An alternate, but equivalent approach could be to consider the invariant

$$J = (\underline{\nabla} r^2 \exp[\mu])^2 = [(r^2 \exp[\mu])^{\cdot}]^2 \exp[-\nu] - [(r^2 \exp[\mu])']^2 \exp[-\lambda], \tag{2.25}$$

to obtain a classification. Again there are four cases:

Case I'. If $J \equiv 0$, then either $r^2 \exp[\mu] = a^2 = \text{const}$, (which corresponds to case I) or $\underline{\nabla} r^2 \exp[\mu]$ is null (which gives metric (3.9) of case IV);

Case II′. If $J < 0$, then $r^2 \exp[\mu] = r^2$ can be achieved and it corresponds to case II;

Case III′. If $J > 0$, then $r^2 \exp[\mu] = t^2$ can be achieved and it corresponds to case III;

Case IV′. If $J = 0$ (but $\not\equiv 0$), then this corresponds to case IV.

Though this classification has certain advantages (in that it would avoid redundancy) we use the other classification (and exclude redundant metrics by checking each metric separately) on account of calculational convenience.

Cases I-III are comparatively simple to solve. The solution is given in the appendix for completeness. In *case* I, where $\mu = \ln(a^2/r^2)$, it is proved that either: A) $\nu = \nu(r)$, $\lambda = 0$; or B) $\nu = 0$, $\lambda = \lambda(t)$ (metric (2.1)). For both of these subcases the minimal isometry group is $SO(3) \otimes \mathbf{R}$, where the only difference is that for A) $\mathbf{R}$ is time-like and for B) it is space-like. Now for the space-times admitting higher symmetries it is straightforward to see from eq. (2.20) that $B_j = 0$ and solution of eqs. (2.21)-(2.23) gives in subcase A):

$$\nu(r) = \ln \cosh^2(A + \sqrt{-\alpha}\, r) \qquad (\alpha < 0), \tag{2.26}$$

$$\nu(r) = \ln(B + r)^2 \qquad (\alpha = 1), \tag{2.27}$$

$$\nu(r) = \ln \cos^2(C + \sqrt{\alpha}\, r) \qquad (\alpha > 0), \tag{2.28}$$

$$\nu(r) = Dr \qquad (\alpha = 0,\ D \neq 0), \tag{2.29}$$

$$\nu(r) = 0 \qquad (\alpha = 0,\ D = 0). \tag{2.30}$$

The corresponding Killing vectors in this case involve six arbitrary constants. Thus the metrics given by $\lambda = 0$ and ν given by eqs. (2.26)-(2.30) admit six isometries. The metrics with ν given by eqs. (2.26)-(2.28) admit the symmetry groups $SO(3) \otimes SO(1, 2)$, $SO(3) \otimes SO(1, 1) \ltimes \mathbf{R}^2$ and $SO(3) \otimes SO(2, 1)$, respectively. Equations (2.29) and (2.30) give metrics corresponding to those given by eqs. (2.26) and (2.27). This is because, since there is a unique two-dimensional metric of signature zero corresponding to any given constant scalar curvature, only three of the metrics corresponding to eqs. (2.26)-(2.30) can be distinct. The metrics with $\lambda = 0$, ν given by eq. (2.26) include the Bertotti-Robinson metric which represents the non-null homogeneous Einstein-Maxwell fields [6, 10, 11]. The classification of case B) is now simple because $\lambda(t)$ can be obtained by replacing r by t in functions $\nu(r)$ of case A) and corresponding Killing vectors can be obtained by the transformations $k^0(t, r) \leftrightarrow k^1(r, t)$. Hence the metric with $\nu = 0$, $\lambda = \ln[\cos^2(c + \sqrt{\alpha}\, t)]$ includes the Bertotti-Robinson–type metric, representing a homogeneous non-null electromagnetic field discussed by Cahen, Lorey and Stephani [6, 12, 13].

Case II gives all the spherically symmetric static space-times given by Bokhari and Qadir [9]. The three spaces of constant curvature, namely de Sitter (negative curvature), Minkowski (zero curvature) and anti-de Sitter (positive curvature) are included. These space-times admit a G_{10}, thus the spherically symmetric space-times admitting G_{10} are all known.

Case III gives only one extra metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}t^2}{(t^2/T^2) - 1} - \mathrm{d}r^2 - t^2\,\mathrm{d}\Omega^2\,, \tag{2.31}$$

where T is a constant. This metric also admits an isometry group $SO(1, 3) \otimes \boldsymbol{R}$ similar to the anti-Einstein universe with a difference that here $\boldsymbol{R}$ is space-like and the energy density is positive.

Case IV, which is complicated and lengthy, is discussed in the next section.

3. – Solution of case IV.

The procedure depends upon the analysis of the arbitrary constants involved in the solutions of the systems of equations (2) and (3), where system (2) consists of eqs. (2.21)-(2.24) and system (3) of eqs. (2.17)-(2.20). For this purpose we apply a theorem (theorem 1 of ref. [7]) to two of the special cases: one when $m = 1$; and the other when $m = 2$, (where m represents the number of dependent variables involved in a system of partial differential equations). According to this theorem, if $m = 1$, the solution of the system contains at most one arbitrary constant and when $m = 2$, the solution has at most two arbitrary constants.

For system (2), eq. (2.24) gives

$$L = -\frac{\dot{\mu}}{\mu' + 2/r}\,K \equiv -\sigma(t, r)K\,. \tag{3.1}$$

Using eqs. (2.21)-(2.23) two independent expressions for K' can be obtained. Comparing these expressions yields

$$\frac{1}{2\sigma}\left[\exp[\lambda - \nu]\{\sigma^2(\dot{\nu} - \sigma\nu') - 2\sigma\dot{\sigma}\} + (2\sigma' - \dot{\lambda} + \sigma\lambda')\right]K \equiv P(t, r)K = 0\,. \tag{3.2}$$

Equation (3.2) implies that either $P(t, r) = 0$ or $L = K = 0$. If $P(t, r) = 0$, $K \neq 0$, the system (2) is reduced to a system with $m = 1$. Thus the solution of this system has at most one arbitrary constant. Therefore, for system (2), either $K = L = 0$ or if $P(t, r) = 0$, $K \neq 0$, we obtain one Killing vector corresponding to the functions K and L.

For the investigation of system (3), we put $\exp[\mu + 2\ln r - (\nu/2)]\dot{B}_j = f_j$, $\exp[\mu + 2\ln r - (\lambda/2)]B_j' = g_j$ and correspondingly eqs. (2.17)-(2.20) reduce to three subsystems, each having $m = 2$. Therefore, system (3) has a solution with either six arbitrary constants or three arbitrary constants, or $f_j = g_j = 0$ (*i.e.* $B_j = 0$).

Now combining the possible arbitrary constants involved in the solutions of these systems with the three arbitrary constants c_j, we have the possibilities listed in table I. Thus there are only three cases to be discussed here: *A*) the space-times admitting G_4; *B*) the space-times admitting G_6; and *C*) the space-times admitting G_7.

In case *A*) where $G_4 = SO(3) \otimes \boldsymbol{R}$, $\boldsymbol{R}$ being $\langle X_0 \rangle$ such that $X_0 \perp X_j$, that is, $[X_0, X_j] = 0$, $\forall j$; $\langle X_1, X_2, X_3 \rangle = SO(3)$. Since the cases when X_0 is time-like or space-like have been given in sect. 2, we only need to discuss the possibility when X_0

TABLE I. – *The possible combinations of the solutions according to the arbitrary constants involved in the systems* (2) *and* (3) *for the case* $\dot{\mu} \neq 0$, $\mu' + 2/r \neq 0$.

Serial no.	No. of arbitrary constants involved in the solutions of		Total no. of arbitrary constants involved including three for spherical symmetry	Remarks
	system (2)	system (3)		
1	one	six	ten	space times admitting G_{10}. All discussed in sect. 2
2	one	three	seven	to be discussed
3	one	$B_j = 0$	four	to be discussed
4	$K = L = 0$	six	nine	not possible Fubini [14]
5	$K = L = 0$	three	six	to be discussed
6	$K = L = 0$	$B_j = 0$	three	metric (2.1)

is null. Now

(3.3) $$X_0 = K(t, r)\,\partial/\partial t + L(t, r)\,\partial/\partial r\,.$$

The existence of X_0 will be ensured later by using the condition $P(t, r) = 0$. For X_0 to be null $L = \pm \exp[(\nu - \lambda)/2]K$. Hence comparison of $(\ln K)^{\cdot\,\prime}$ and $(\ln K)^{\prime\,\cdot}$, using system (2) gives

(3.4) $$(\nu' \exp[(\nu - \lambda)/2])' = (\dot{\lambda} \exp[(\lambda - \nu)/2])^{\cdot}\,,$$

an integrability condition for $\dot{K}$ and K'. Using this condition we define the transformations

(3.5) $$\mathrm{d}u = \exp[\nu/2] \cosh\psi\,\mathrm{d}t + \exp[\lambda/2] \sinh\psi\,\mathrm{d}r\,,$$

(3.6) $$\mathrm{d}v = \exp[\nu/2] \sinh\psi\,\mathrm{d}t + \exp[\lambda/2] \cosh\psi\,\mathrm{d}r\,,$$

from the variables t, r to u, v, where $\psi = \int \dot{\lambda} \exp[(\lambda - \nu)/2]\,\mathrm{d}r$. It is easy to verify that these transformations with condition (3.4) satisfy the integrability conditions (theorem I, ref. [7]). These transformations reduce the metric (2.1) into the form

(3.7) $$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - \exp[\mu + 2\ \ln r]\,\mathrm{d}\Omega^2\,.$$

Hence $\dot{K} = K' = 0$, *i.e.* $K = c_0$ (c_0 is an arbitrary constant) and $L = \pm c_0$. Putting this value of L in eq. (3.1) yields

$$\dot{\mu}\left(\mu' + \frac{2}{r}\right)^{-1} = \mp 1 = \sigma, \tag{3.8}$$

which identically satisfies eq. (3.2). Now eq. (3.8) gives

$$\mu + 2 \ln r = f(t \pm r).$$

Thus the metric given by eq. (3.7) reduces to

$$ds^2 = dt^2 - dr^2 - \exp[f(z)]\, d\Omega^2 \qquad (z = t \pm r) \tag{3.9}$$

and the corresponding Killing vector is

$$\begin{aligned} \boldsymbol{K} = {} & c_0 \partial/\partial t \pm c_0 \partial/\partial r + \\ & + (c_1 \sin\phi - c_2 \cos\phi)\partial/\partial\vartheta + [\operatorname{ctg}\vartheta(c_1 \cos\phi + c_2 \sin\phi) + c_3]\partial/\partial\phi . \end{aligned} \tag{3.10}$$

This $\boldsymbol{K}$ involves four arbitrary constants and the corresponding symmetry group is $G_4 = SO(3) \otimes \boldsymbol{R}$, where $\boldsymbol{R} = \langle \partial/\partial t \pm \partial/\partial r \rangle$ is null. To verify that $B_j = 0$ put the metric components from eq. (3.9) into eq. (2.20), to obtain

$$- df/dz \exp[f(z)] \dot{B}_j \mp df/dz \exp[f(z)] B_j' + 2B_j = 0, \tag{3.11}$$

where $z = t \mp r$. Therefore, G_4 is the group of maximum mobility for the metric given by eq. (3.9). This completes the classification of case A). The metric given by eq. (3.9) represents a class of non-static metrics depending on one arbitrary function of the variable z.

In case B) $K = L = 0$ and the solutions of system (3) have three arbitrary constants (table I) corresponding to each B_j. Thus the space-times admit a $G_6 \supseteq SO(3)$. Our task is, now, to determine these G_6 and the corresponding metrics or classes of metrics. For this purpose we consider k^a with $K = L = 0$ and subject to the conditions of system (3).

We write generators of G_6 as $\boldsymbol{X}_{j+3}$ corresponding to each B_j and $\boldsymbol{X}_j$ corresponding to each c_j, present in the expressions of k^a. Then we have

$$\boldsymbol{X}_1 = \sin\phi\, \partial/\partial\vartheta + \operatorname{ctg}\vartheta \cos\phi\, \partial/\partial\phi, \tag{3.12}$$

$$\boldsymbol{X}_2 = -\cos\phi\, \partial/\partial\vartheta + \operatorname{ctg}\vartheta \sin\phi\, \partial/\partial\phi, \tag{3.13}$$

$$\boldsymbol{X}_3 = \partial/\partial\phi, \tag{3.14}$$

$$\begin{aligned} \boldsymbol{X}_4 = {} & -r^2 \exp[\mu - \nu] \dot{B}_1 \sin\vartheta \sin\phi\, \partial/\partial t + r^2 \exp[\mu - \lambda] B_1' \sin\vartheta \sin\phi\, \partial/\partial r - \\ & - B_1 (\cos\vartheta \sin\phi\, \partial/\partial\vartheta + \operatorname{cosec}\vartheta \cos\phi\, \partial/\partial\phi), \end{aligned} \tag{3.15}$$

$$\boldsymbol{X}_5 = -\partial \boldsymbol{X}_4/\partial\phi, \tag{3.16}$$

with B_1 replaced by B_2 in this equation,

$$\begin{aligned} \boldsymbol{X}_6 = {} & -r^2 \exp[\mu - \nu] \dot{B}_3 \cos\vartheta\, \partial/\partial t + \\ & + r^2 \exp[\mu - \lambda] B_3' \cos\vartheta\, \partial/\partial r + B_3 \sin\vartheta\, \partial/\partial\vartheta . \end{aligned} \tag{3.17}$$

Now using the closure property of G_6 and evaluating the commutators $[X_p, X_q]$ ($\forall p, q = 1, \ldots, 6$), we have

$$[X_1, X_4] = -r^2 \exp[\mu - \nu]\dot{B}_1 \cos\vartheta\,\partial/\partial t + r^2 \exp[\mu - \lambda]B_1' \cos\vartheta\,\partial/\partial r + B_1 \sin\vartheta\,\partial/\partial\vartheta\,.$$

Clearly, for the algebra to be closed we must have $B_1 = bB_3$ (where $b \neq 0$ is an arbitrary constant). Thus

$$[X_1, X_4] = bX_6\,, \qquad [X_2, X_4] = 0\,,$$

$$[X_3, X_4] = -r^2 \exp[\mu - \nu]\dot{B}_1 \sin\vartheta \cos\phi\,\partial/\partial t + r^2 \exp[\mu - \lambda]B_1' \sin\vartheta \cos\phi\,\partial/\partial r - - B_1(\cos\vartheta \cos\phi\,\partial/\partial\vartheta + \operatorname{cosec}\vartheta \sin\phi\,\partial/\partial\phi)\,,$$

and again for the algebra to be closed we must have $B_1 = (b/e)B_2$ (where $e \neq 0$ is an arbitrary constant). Hence the closure of G_6 implies that $B_1 = bf(t, r)$, $B_2 = ef(t, r)$, $B_3 = f(t, r)$, where $f(t, r)$ is an arbitrary function. Thus after rescaling X_4 and X_5, we have

$$X_4 = -r^2 \exp[\mu - \nu]\dot{f} \sin\vartheta \sin\phi\,\partial/\partial t + r^2 \exp[\mu - \lambda]f' \sin\vartheta \sin\phi\,\partial/\partial r - -f \cos\vartheta \sin\phi\,\partial/\partial\vartheta - f \operatorname{cosec}\vartheta \cos\phi\,\partial/\partial\phi\,, \tag{3.18}$$

$$X_5 = -\partial X_4/\partial\phi\,, \tag{3.19}$$

$$X_6 = -r^2 \exp[\mu - \nu]\dot{f} \cos\vartheta\,\partial/\partial t + r^2 \exp[\mu - \lambda]f' \cos\vartheta\,\partial/\partial r + f \sin\vartheta\,\partial/\partial\vartheta\,. \tag{3.20}$$

The other commutation relations are

$$\begin{cases} [X_1, X_5] = 0\,, \quad [X_2, X_5] = X_6\,, \quad [X_3, X_5] = X_4\,, \\ [X_1, X_6] = -X_4\,, \quad [X_2, X_6] = -X_5\,, \quad [X_3, X_6] = 0\,, \\ [X_4, X_5] = H(t, r)X_3\,, \quad [X_4, X_6] = -H(t, r)X_1\,, \\ [X_5, X_6] = -H(t, r)X_2\,, \end{cases} \tag{3.21}$$

where $H(t, r) = r^2 \exp[\mu - \nu]\dot{f}^2 - r^2 \exp[\mu - \lambda]f'^2 + f^2$, and using system (3) we have $\dot{H}(t, r) = H'(t, r) = 0$. Thus $H(t, r) = \alpha$ an arbitrary constant. Correspondingly: $G_6 \equiv SO(1, 3)$ for $\alpha > 0$; $G_6 \equiv SO(3) \otimes \boldsymbol{R}^3$ ($\alpha = 0$); $G_6 \equiv SO(4)$ ($\alpha < 0$). The metric given by eq. (2.1) is now subject to the constraints

$$2\ddot{f} + (2\dot{x} - \dot{\nu})\dot{f} - \nu' \exp[\nu - \lambda]f' = 0\,, \tag{3.22}$$

$$2\dot{f}' + (x' - \nu')\dot{f} + (\dot{x} - \dot{\lambda})f' = 0\,, \tag{3.23}$$

$$2f'' + (2x' - \lambda')f' - \dot{\lambda} \exp[\lambda - \nu]\dot{f} = 0\,, \tag{3.24}$$

$$-\dot{x} \exp[x - \nu]\dot{f} + x' \exp[x - \lambda]f' + 2f = 0\,, \tag{3.25}$$

$$\exp[x - \nu]\dot{f}^2 - \exp[x - \lambda]f'^2 + f^2 = \alpha \qquad (\alpha \gtreqless 0)\,, \tag{3.26}$$

where $x = \mu(t, r) + 2 \ln r$. Different choices of $f(t, r)$ and «α» would lead us to different classes. We discuss all possibilities of $f(t, r)$: *a*) $\dot{f} = 0, f' \neq 0$; *b*) $\dot{f} \neq 0, f' = 0$; *c*) $\dot{f} \neq 0$, $f' \neq 0$. First we consider case *a*). Equations (3.22)-(3.26) imply that

$x = x_0(t) + x_1(r)$ and $\lambda = \lambda_0(t) + \lambda_1(r)$ with $x_0(t) = \lambda_0(t)$ and

$$\exp\left[\frac{1}{2}x_1\right] = A\sinh(\sqrt{\alpha}\,r + C) \qquad (\alpha > 0),$$

$$\exp\left[\frac{1}{2}x_1\right] = D \pm r \qquad (\alpha = 0),$$

$$\exp\left[\frac{1}{2}x_1\right] = E\sin(\sqrt{-\alpha}\,r + \varepsilon) \qquad (\alpha < 0),$$

where A, C, D, E, ε are constants of integration. Putting $\alpha = 1/r_0^2$ for $\alpha > 0$ and $\alpha = -1/r_0^2$ for $\alpha < 0$ and redefining the variables, the possible metrics are, therefore,

$$\text{(3.27)} \qquad ds^2 = dt^2 - \exp[x_0(t)]\left[dr^2 + r_0^2\sinh^2\left(\frac{r}{r_0}\right)d\Omega^2\right],$$

$$\text{(3.28)} \qquad ds^2 = dt^2 - \exp[x_0(t)][dr^2 + r^2 d\Omega^2],$$

$$\text{(3.29)} \qquad ds^2 = dt^2 - \exp[x_0(t)]\left[dr^2 + r_0^2\sin^2\left(\frac{r}{r_0}\right)d\Omega^2\right],$$

for $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$, respectively. Now for the maximum group of mobility for these metrics, we appeal to systems (2)-(3) and require G_6 to be the maximal group of motions we must have $[\exp[-x_0] - (1/2)r_0^2\ddot{x}_0] \neq 0$, $\ddot{x}_0 \neq 0$ and $[\exp[-x_0] + (1/2)r_0^2\ddot{x}_0] \neq 0$, respectively. The metrics given by eqs. (3.27)-(3.29) are conformally related to the anti-de Sitter, Minkowski and de Sitter space-times with symmetry groups $SO(1,3)$, $SO(3) \ltimes \boldsymbol{R}^3$ and $SO(4)$, respectively. These metrics represent Robertson-Walker (Friedmann) [15] cosmologies.

For case *b*), where $\dot{f} \neq 0$, $f' = 0$,

$$\text{(3.30)} \qquad ds^2 = \exp[x_1(r)]\left[dt^2 - t_0^2\cosh^2\left(\frac{t}{t_0}\right)d\Omega^2\right] - dr^2,$$

where $x_1(r)$ is an arbitrary function of r subject to the condition

$$\text{(3.31)} \qquad \left[\exp[-x_1] + \frac{1}{2}t_0^2 x_1''\right] \neq 0,$$

and t_0 is an arbitrary constant. In this case there exists no solution for $\alpha \leqslant 0$. The metric given by eq. (3.30) is conformally related to the metric given by eq. (2.31) and admits $SO(1,3)$ as the maximal symmetry group. It represents a class of space-times depending on one arbitrary function of one variable. If condition (3.31) is not satisfied, the metric reduces to the Minkowski metric.

Next we come to case *c*). Here $\dot{x} \neq 0$, $x' \neq 0$, $\dot{f} \neq 0$, $f' \neq 0$. Using eq. (3.25) in eqs. (3.22)-(3.24), one can write two equations in $\dot{f}$ and f'. A non-trivial solution of these two equations establishes a condition on the metric coefficients. Using this condition and the condition obtained by comparing the expressions for $(\dot{f})'$ and $(f')^{\cdot}$

we obtain

$$(3.32)\quad (x'\dot{f}-\dot{x}f')[\{(\dot{x}-\dot{\lambda})\exp[(\lambda-\nu)/2]\}^{\cdot}-\{(x'-\nu')\exp[(\nu-\lambda)/2]\}']=0\,.$$

Now putting $\exp[x-\nu]\dot{f}=F$ and $\exp[x-\lambda]f'=G$ into eqs. (3.22)-(3.25) and comparing $(\dot{F})'$ with $(F')^{\cdot}$ or $(\dot{G})'$ with $(G')^{\cdot}$ yields

$$(3.33)\quad [(\dot{x}-\dot{\lambda})\exp[(\lambda-\nu)/2]]^{\cdot}-$$

$$-[(x'-\nu')\exp[(\nu-\lambda)/2]]'-2\exp[-x+[(\nu+\lambda)/2]]=0\,.$$

Condition (3.32) implies that either

$$x'\dot{f}-\dot{x}f'=0 \text{ or } [(\dot{x}-\dot{\lambda})\exp[(\lambda-\nu)/2]]^{\cdot}-[(x'-\nu')\exp[(\nu-\lambda)/2]]'=0\,.$$

If $[(\dot{x}-\dot{\lambda})\exp[(\lambda-\nu)/2]]^{\cdot}-[(x'-\nu')\exp[(\nu-\lambda)/2]]'=0$, then eq. (3.33) implies that

$$(3.34)\quad \exp[-x+[(\nu+\lambda)/2]]=0\,,$$

which is not possible. Thus

$$(3.35)\quad x'\dot{f}-\dot{x}f'=0\,.$$

Again, writing $\exp[\nu/2]F=S$ and $\exp[\lambda/2]G=T$, in eqs. (3.22)-(3.25) and using condition (3.35) yields $S\dot{S}-T\dot{T}=SS'-TT'=0$. Therefore, $[S^2-T^2]^{\cdot}=[S^2-T^2]'=0$. Using eq. (3.26) we have $[\exp[x](\alpha-f^2)]^{\cdot}=[\exp[x](\alpha-f^2)]'=0$. Therefore, $\exp[x](\alpha-f^2)=\beta$, here β is a constant of integration. Using this value of f, with $\beta=0$, in eqs. (3.22)-(3.25) and comparing the resulting equations gives the condition (3.4), for which the metric attains the form given by eq. (3.7). Here $P(t,r)=0$. For this type of metric there are four Killing vectors. For $\beta\neq 0$ the same procedure shows that there are ten Killing vectors.

Now we discuss case *C*), where the solution of system (2) admits one arbitrary constant whereas that of system (3) admits three arbitrary constants (table I). Since system (2) admits one arbitrary constant, the corresponding Killing vector $\boldsymbol{X}_0$ is of the form given by eq. (3.3). The other six Killing vectors are $\boldsymbol{X}_p$ represented by eqs. (3.12)-(3.14) and (3.18)-(3.20). Using the closure property of G_7 we have

$$(3.36)\quad [\boldsymbol{X}_0,\boldsymbol{X}_j]=0\,,$$

$$(3.37)\quad [\boldsymbol{X}_0,\boldsymbol{X}_4]=$$

$$=-H_1(t,r)\exp[x-\nu]\dot{f}\sin\vartheta\sin\phi\,\partial/\partial t+H_2(t,r)\exp[x-\lambda]f'\sin\vartheta\sin\phi\,\partial/\partial r-$$

$$-H_3(t,r)[\cos\vartheta\sin\phi f\partial/\partial\vartheta+\operatorname{cosec}\vartheta\cos\phi f\partial/\partial\phi]\,,$$

$$(3.38)\quad [\boldsymbol{X}_0,\boldsymbol{X}_5]=-\partial[\boldsymbol{X}_0,\boldsymbol{X}_4]/\partial\phi\,,$$

$$(3.39)\quad [\boldsymbol{X}_0,\boldsymbol{X}_6]=$$

$$=-H_1(t,r)\exp[x-\nu]\dot{f}\cos\vartheta\,\partial/\partial t+H_2(t,r)\exp[x-\lambda]f'\cos\vartheta\,\partial/\partial r+$$

$$+H_3(t,r)f\sin\vartheta\,\partial/\partial\vartheta\,,$$

where

(3.40) $H_1(t, r) =$

$$= \frac{1}{\dot{f}}\left[\exp[\nu-\lambda]f'K' + \frac{1}{2}\nu'\exp[\nu-\lambda]f'K + \frac{1}{2}(x'\dot{f}-\dot{x}f')L + \frac{1}{2}\dot{\lambda}f'L\right],$$

(3.41) $$H_2(t, r) = \frac{1}{f'}\left[\exp[\lambda-\nu]\dot{f}\dot{L} + \frac{1}{2}\dot{\lambda}\exp[\lambda-\nu]\dot{f}L - \frac{1}{2}(x'\dot{f}-\dot{x}f')K + \frac{1}{2}\nu'\dot{f}L\right],$$

(3.42) $$H_3(t, r) = \frac{1}{f}(K\dot{f} + Lf').$$

Thus the necessary and sufficient conditions for the algebra to be closed are

(3.43) $$\dot{f} \neq 0, \quad f' \neq 0, \quad H_j(t, r) = \beta,$$

β being a constant greater than, less than or equal to zero.

If $\beta = 0$, then $G_7 = \langle X_0, \ldots, X_6\rangle$ such that $[X_0, X_p] = 0$, $\forall p$ and $G_7 \supseteq G_6 \supseteq SO(3)$, where G_6 is $SO(1, 3)$, $SO(3) \otimes_s \mathbf{R}^3$ and $SO(4)$ according to whether $\alpha > 0$, $\alpha = 0$ and $\alpha < 0$, respectively, and the Lie algebra for these groups is given by eq. (3.21). Thus in this case the possible symmetry groups may be $SO(1, 3) \otimes \mathbf{R}$, $[SO(3) \otimes_s \mathbf{R}^3] \otimes \mathbf{R}$ and $SO(4) \otimes \mathbf{R}$. To determine the metrics corresponding to these algebras, consider the conditions (3.43) with $j = 3$, $\beta = 0$. In this case a non-trivial solution of eqs. (2.24) and (3.43) will exist if eq. (3.35) holds. But for $\dot{x} \neq 0$, $x' \neq 0$, $\dot{f} \neq 0$, $f' \neq 0$ and $x'\dot{f} - \dot{x}f' = 0$ solutions of eqs. (3.22)-(3.26) admit only G_4 or G_{10}.

Now we consider the case when $\beta \neq 0$. Rescaling X_0 so that β is absorbed in it (or, equivalently, setting $\beta = 1$), we have

(3.44) $$[X_0, X_{j+3}] = X_{j+3}$$

and other commutation relations are given by eqs. (3.21). These Lie algebras are not included in Petrov's discussion, but are discussed by Turkowski [16] for $\alpha = 0$.

For a non-trivial solution of K and L, from eqs. (2.24) and (3.43) with $j = 3$, $\beta = 1$, we have the requirement that eq. (3.35) does not hold. Solving eqs. (3.43) for all j gives

(3.45) $$\exp[\nu-\lambda]f'[2K' + \nu'K + \dot{\lambda}\exp[\lambda-\nu]L] = 2\dot{f} + \dot{x}f$$

and

(3.46) $$\exp[\lambda-\nu]\dot{f}[2\dot{L} + \dot{\lambda}L + \nu'\exp[\nu-\lambda]K] = 2f' + \dot{x}f.$$

These equations may be combined to give

(3.47) $$4\dot{f}f'(\exp[\nu]K' - \exp[\lambda]\dot{L}) + 4\exp[\nu+\lambda][\exp[-\lambda]f'^2 - \exp[-\nu]\dot{f}^2] + {}$$
$$+ f[x'\exp[-\lambda]f' - \dot{x}\exp[-\nu]\dot{f}] = 0.$$

Using eqs. (2.22), (3.25) and (3.26), we have $\alpha = 0$. Thus if $\dot{x} \neq 0 \neq x'$ and the space-times admit a G_7, the only G_6 included in the G_7 has the Lie algebra given by eq. (3.21) with $\alpha = 0$. This Lie algebra is similar to that of Turkowski [16], *i.e.*

TABLE II. – *Properties of the spherically symmetric space-times of the type $V_2 \oplus S^2$ admitting G_6 as the maximal symmetry group.*

Metric	R_{0101}	R_{00}	R_{11}	R
(2.26)	$\alpha \cosh^2(A + \sqrt{-\alpha}\, r)$	$-\alpha \cosh^2(A + \sqrt{-\alpha}\, r)$	α	$-2(\alpha + a^{-2})$
(2.27)	—	—	—	$-2a^{-2}$
(2.28)	$\alpha \cos^2(c + \sqrt{\alpha}\, r)$	$-\alpha \cos^2(c + \sqrt{\alpha}\, r)$	α	$-2(\alpha + a^{-2})$
(2.29)	$-\frac{1}{4} D^2 \exp[Dr]$	$\frac{1}{4} D^2 \exp[Dr]$	$-\frac{D^2}{4}$	$2\left(\frac{D^2}{4} - a^{-2}\right)$
(2.26) $r \leftrightarrow t$	$-\alpha \cosh^2(A + \sqrt{-\alpha}\, t)$	α	$-\alpha \cosh^2(A + \sqrt{-\alpha}\, t)$	$2(\alpha - a^{-2})$
(2.27) $r \leftrightarrow t$	—	—	—	$-2a^{-2}$
(2.28) $r \leftrightarrow t$	$-\alpha \cos^2(C + \sqrt{\alpha}\, t)$	α	$-\alpha \cos^2(C + \sqrt{-\alpha}\, t)$	$2(\alpha - a^{-2})$
(2.29) $r \leftrightarrow t$	$\frac{1}{4} D^2 \exp[Dt]$	$-\frac{D^2}{4}$	$\frac{D^2}{4} \exp[Dt]$	$-\left(\frac{D^2}{4} + 2a^{-2}\right)$
(2.30)	—	—	—	$-2a^{-2}$

Metric	κT_{00}	κT_{11}	κT_{22}	Petrov type	Identification
(2.26)	$\frac{\cosh^2(A + \sqrt{-\alpha}\, r)}{a^2}$	$-a^{-2}$	$-\alpha a^2 \begin{cases} \alpha = -a^{-2} \\ \alpha \neq -a^2 \end{cases}$	O D	static Bertotti-Robinson anisotropic perfect fluid
(2.27)	$a^{-2}(B + r)^2$	$-a^{-2}$	—	D	anisotropic perfect fluid
(2.28)	a^{-2}	$-a^{-2}$	$-\alpha a^2$	D	anisotropic perfect fluid
(2.29)	$a^{-2} \exp[Dr]$	$-a^{-2}$	$\frac{1}{4} a^2 D^2 \begin{cases} D^2 = 4a^{-2} \\ D^2 \neq 4a^{-2} \end{cases}$	O D	non-null homo-isotropic Einstein-Maxwell field anisotropic perfect fluid
(2.26) $r \leftrightarrow t$	a^{-2}	$\frac{\cosh^2(A + \sqrt{-\alpha}\, t)}{a^2}$	αa^2	D	(non-static) anisotropic perfect fluid
(2.27) $r \leftrightarrow t$	a^{-2}	$a^{-2}(B + t)^2$	—	D	anisisotropic perfect fluid (non-static)
(2.28) $r \leftrightarrow t$	$2(\alpha - a^{-2})$	$-\frac{\cos^2(C + \sqrt{\alpha}\, t)}{a^2}$	$\alpha a^2 \begin{cases} \alpha = a^{-2} \\ \alpha \neq a^{-2} \end{cases}$	O D	non-static, Cahen, Lorey, Stephani anisotropic perfect fluid
(2.29) $r \leftrightarrow t$	a^{-2}	$-\frac{\exp[Dt]}{a^2}$	$-\frac{1}{4} a^2 D^2 \begin{cases} D^2 = 4a^{-2} \\ D^2 \neq 4a^{-2} \end{cases}$	D D	homo-isotropic perfect fluid anisotropic perfect fluid
(2.30)	a^{-2}	$-a^{-2}$	—	D	anisotropic perfect fluid

$[SO(3) \circledS \mathbf{R}^3] \circledS \mathbf{R}$. However, in this case eqs. (2.24) and (3.43) with $j = 3$, $\beta = 1$ have no non-trivial solutions by virtue of eq. (3.35). Thus there is no space-time corresponding to Turkowski's algebra!

4. – Conclusion.

The main point of the paper is that we have been able to obtain a complete classification of spherically symmetric space-times according to their isometries and metrics. Thus we have a complete list of *all* metrics, or classes of metrics, corresponding to each possible isometry group along with the group's infinitesimal generators. In the process we also found some metrics not previously known and demonstrated that Turkowski's symmetry does not apply to any 4-dimensional spherically symmetric space-time. This complete classification has already been used to prove a singularity theorem [2] without reference to a positive-energy condition, for classifying Ricci collineations [3] and to extend the classification to other symmetries [4, 5].

There are five different classes of metrics admitting G_4 as the maximal group of motions. These classes depend on one or two arbitrary functions of one variable. It was proved that there does not exist any spherically symmetric space-time admitting G_5 as the maximal group of motions. The space-times admitting G_6 as the maximal group of motions have symmetry structures $SO(3) \otimes SO(1, 2)$; $SO(3) \otimes SO(1, 1) \circledS \circledS \mathbf{R}^2$; $SO(3) \otimes SO(2, 1)$; $SO(4)$; $SO(1, 3)$ and $SO(3) \circledS \mathbf{R}^3$. These space-times include: metrics with $\lambda = 0$, ν given by eq. (2.26), representing the Bertotti-Robinson metric; a metric with $\nu = 0$ and λ given by eq. (2.26) with r replaced by t, satisfying the conditions for a non-null electromagnetic field (discussed by Cahen, Lorey and Stephani) and the Robertson-Walker space-times given by metrics (3.27)-(3.29) (see table II).

The spherically symmetric space-times admitting G_7 as the maximal group of motions which previously included the Einstein and anti-Einstein space-times only have proved to also include a non-static space-time admitting the symmetry group $SO(1, 3) \otimes \mathbf{R}$. This admits a symmetry group similar to the anti-Einstein universe.

* * *

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APPENDIX

Case I. The metric given by eq. (2.1) reduces to

$$\mathrm{d}s^2 = \exp[\nu(t, r)]\,\mathrm{d}t^2 - \exp[\lambda(t, r)]\,\mathrm{d}r^2 - a^2\,\mathrm{d}\Omega^2\,, \tag{A.1}$$

where $(\exp[\nu]\,\mathrm{d}t^2 - \exp[\lambda]\,\mathrm{d}r^2)$ is the metric for a V_2 space and «a» is a constant. We first consider the case of the group of motions G_1 of a V_2, then either [7]: A) ν and λ are functions of r only; or B) ν and λ are functions of t only.

In case A), redefine r so that $\lambda = 0$. The curves of the parameter t are the trajectories of motion. Thus G_1 is generated by the operator $X_0 = \partial/\partial t$, a time-like Killing vector ($g_{ab}X_0^a X_0^b > 0$). Thus the space-times in this case are all static. Now in order to determine whether this V_2 can admit more than one motion, we consider the conditions (2.21)-(2.23) with $\dot{\nu} = 0$ and $\lambda = 0$, we get $L' = 0$ and

$$2\dot{K} + \nu' L = 0 \,, \tag{A.2}$$

$$\exp[\nu] K' - \dot{L} = 0 \,. \tag{A.3}$$

These equations easily yield

$$\ddot{L} - \alpha L = 0 \,, \tag{A.4}$$

$$\nu'' = -2\alpha \exp[-\nu] \,, \tag{A.5}$$

where α is a separation constant. Equations (A.2)-(A.5) then yield the metric with ν given by eqs. (2.26)-(2.30). The corresponding Killing vectors for these cases are, respectively,

$$\begin{aligned} K = {} & [c_0 - \operatorname{tgh}(A + \sqrt{-\alpha}\, r)\{c_4 \sin(\sqrt{-\alpha}\, t) + c_5 \cos(\sqrt{-\alpha}\, t)\}]\, \partial/\partial t + \\ & + \{c_4 \cos(\sqrt{-\alpha}\, t) + c_5 \sin(\sqrt{-\alpha}\, t)\}]\, \partial/\partial r + (c_1 \sin\phi - c_2 \cos\phi)\, \partial/\partial\vartheta + \\ & + [\operatorname{ctg}\vartheta (c_1 \cos\phi + c_2 \sin\phi) + c_3]\, \partial/\partial\phi \,; \end{aligned} \tag{A.6}$$

$$\begin{aligned} K = {} & \left[c_0 - \frac{1}{B + r}(c_4 \sinh t + c_5 \cosh t\right] \partial/\partial t + (c_4 \cosh t + c_5 \sinh t)\, \partial/\partial r + \\ & + (c_1 \sin\phi - c_2 \cos\phi)\, \partial/\partial\vartheta + [\operatorname{ctg}\vartheta(c_1 \cos\phi + c_2 \sin\phi) + c_3]\, \partial/\partial\phi \,; \end{aligned} \tag{A.7}$$

$$\begin{aligned} K = {} & [c_0 + \operatorname{tg}(C + \sqrt{\alpha}\, r)\{c_4 \sinh(\sqrt{\alpha}\, t) + c_5 \cosh(\sqrt{\alpha}\, t)\}]\, \partial/\partial t + \\ & + \{c_4 \cosh(\sqrt{\alpha}\, t) + c_5 \sinh(\sqrt{\alpha}\, t)\}\, \partial/\partial r + (c_1 \sin\phi - c_2 \cos\phi)\, \partial/\partial\vartheta + \\ & + [\operatorname{ctg}(c_1 \cos\phi + c_2 \sin\phi) + c_3]\, \partial/\partial\phi \,; \end{aligned} \tag{A.8}$$

$$\begin{aligned} K = {} & \left[c_0 - \frac{1}{2} D\left(\frac{1}{2} c_4 t^2 + c_5 t\right) - \frac{c_4}{D} \exp[-Dr - E]\right] \partial/\partial t + (c_4 t + c_5)\, \partial/\partial r + \\ & + (c_1 \sin\phi - c_2 \cos\phi)\, \partial/\partial\vartheta + [\operatorname{ctg}\vartheta(c_1 \cos\phi + c_2 \sin\phi) + c_3]\, \partial/\partial\phi \,; \end{aligned} \tag{A.9}$$

$$\begin{aligned} K = {} & (c_0 + c_4 r)\, \partial/\partial t + (c_4 t + c_5)\, \partial/\partial r + (c_1 \sin\phi - c_2 \cos\phi)\, \partial/\partial\vartheta + \\ & + [\operatorname{ctg}\vartheta(c_1 \cos\phi + c_2 \sin\phi) + c_3]\, \partial/\partial\phi \,. \end{aligned} \tag{A.10}$$

All other metrics given by eq. (A.1) with $\nu(t, r) \equiv \nu(r)$ and $\lambda(t, r) = 0$, admit a $G_4 = SO(3) \otimes \boldsymbol{R}$ as the maximal group of motions, where $\boldsymbol{R}$ stands for time translation.

The classification of the case B) is now obvious. The functions $\lambda(t)$ can be obtained by replacing r by t in the functions $\nu(r)$ of the case A) and the corresponding Killing vectors can be obtained by the transformations

$$k^0(t, r) \leftrightarrow k^1(r, t) \,.$$

Case II. Putting $\dot{\mu} = 0$ in eq. (2.24) yields $L = 0$. Therefore eq. (2.23) implies that either A) $\dot{\lambda} = 0$, *i.e.* $\lambda = \lambda(r)$ or B) $K = 0$. The latter case will be discussed at the end.

Using eqs. (2.21) and (2.22) we get $\dot{\nu}' = 0$. Thus for the case A), absorbing the time part of ν in the definition of time, $\nu(t, r) \equiv \nu(r)$, $\lambda(t, r) \equiv \lambda(r)$ and $\mu(t, r) = 0$ in metric (2.1).

For this metric $K = c_0$, an arbitrary constant, and using eq. (2.20), eqs. (2.17)-(2.19) reduce to

(A.11) $$2r\ddot{B}_j = \nu' \exp[\nu] B_j = 0 ,$$

(A.12) $$\{2(-\exp[\lambda] + 1) - \nu' r\} \dot{B}_j = 0 ,$$

(A.13) $$[2(\exp[\lambda] - 1) - \lambda' r] B_j = 0 .$$

There are two possibilities consistent with eq. (A.13). The first is a) $B_j = 0$, and hence the metric admits at least the isometry group G_4 of case A). Therefore, these space-times are again static. The second is b), where $B_j \neq 0$ and $\exp[\lambda] = 1/(\alpha r^2 + 1)$ ($\alpha \gtreqless 0$). Equation (A.12) further allows two possibilities. Either: b)* $\dot{B}_j = 0$; or b)** $\dot{B}_j \neq 0$ and $\exp[\nu] = \alpha r^2 + 1$. Now using eq. (A.11), these possibilities give:

(A.14) $$\nu = 0 , \qquad \lambda = -\ln(\alpha r^2 + 1) \qquad (\alpha \gtrless 0);$$

(A.15) $$\nu = \ln(\alpha r^2 + 1) , \qquad \lambda = -\ln(\alpha r^2 + 1) \qquad (\alpha \gtrless 0);$$

(A.16) $$\nu = \lambda = 0 \qquad (\alpha = 0).$$

The metrics given by eq. (A.14) represent the Einstein and «anti-Einstein» universes admitting seven isometries ($SO(4) \otimes \boldsymbol{R}$ and $SO(1, 3) \otimes \boldsymbol{R}$, respectively). For completeness we give the extra isometries

(A.17) $$\begin{cases} X_4 = -(\alpha r^2 + 1)^{1/2} \sin\vartheta \sin\phi\, \partial/\partial r - \\ \qquad - \dfrac{(\alpha r^2 + 1)^{1/2}}{r} [\cos\vartheta \sin\phi\, \partial/\partial\vartheta + \operatorname{cosec}\vartheta \cos\phi\, \partial/\partial\phi] , \\ X_5 = -\partial X_4/\partial\phi , \\ X_6 = -(\alpha r^2 + 1)^{1/2} \cos\vartheta\, \partial/\partial r + \dfrac{(\alpha r^2 + 1)^{1/2}}{r} \sin\vartheta\, \partial/\partial\vartheta . \end{cases}$$

The metrics given by eq. (A.15) represent the de Sitter ($\alpha < 0$) and anti-de Sitter ($\alpha > 0$) universes admitting ten isometries ($SO(1, 4)$ and $SO(2, 3)$, respectively). The extra isometries are

$$\text{(A.18)}\quad \begin{cases} X_4 = -\dfrac{r}{Jr_0}\cos\sqrt{\alpha}\,t\,\sin\vartheta\,\sin\phi\,\dfrac{\partial}{\partial t} - J\cos\sqrt{\alpha}\,t\,\sin\vartheta\,\sin\phi\,\dfrac{\partial}{\partial r} - \\ \qquad\qquad - \dfrac{J}{r}\cos\sqrt{\alpha}\,t\,\cos\vartheta\,\sin\phi\,\dfrac{\partial}{\partial\vartheta} - \dfrac{J}{r\sin\vartheta}\cos\sqrt{\alpha}\,t\,\cos\phi\,\dfrac{\partial}{\partial\phi}\,, \\ X_5 = -\partial X_4/\partial\phi\,, \\ X_6 = \dfrac{r}{Jr_0}\cos\vartheta\,\sin\sqrt{\alpha}\,t\,\dfrac{\partial}{\partial t} - J\cos\vartheta\,\cos\sqrt{\alpha}\,t\,\dfrac{\partial}{\partial r} - \dfrac{J}{r}\sin\vartheta\,\cos\sqrt{\alpha}\,t\,\dfrac{\partial}{\partial\vartheta}\,, \\ X_7 = r_0\,\partial X_4/\partial t\,, \qquad X_8 = r_0\,\partial X_5/\partial t\,, \qquad X_9 = r_0\,\partial X_6/\partial t\,, \\ J(r) \equiv \sqrt{1+\alpha r^2}\,. \end{cases}$$

Here the trigonometric functions get replaced by hyperbolic functions when the argument becomes imaginary. The metric given by eq. (A.16) is the well-known Minkowski space-time admitting the Poincaré group, $SO(1,3)\circledS \boldsymbol{R}^4$. The extra generators are

$$\text{(A.19)}\quad \begin{cases} X_4 = -r\sin\vartheta\,\sin\phi\,\dfrac{\partial}{\partial t} - t\sin\vartheta\,\sin\phi\,\dfrac{\partial}{\partial r} - \\ \qquad\qquad - \dfrac{t}{r}\cos\vartheta\,\sin\phi\,\dfrac{\partial}{\partial\vartheta} - \dfrac{t}{r}\,\mathrm{cosec}\,\vartheta\,\cos\phi\,\dfrac{\partial}{\partial\phi}\,, \\ X_5 = -\partial X_4/\partial\phi\,, \\ X_6 = -r\cos\vartheta\,\dfrac{\partial}{\partial t} - t\cos\vartheta\,\dfrac{\partial}{\partial r} + \dfrac{t}{r}\sin\vartheta\,\dfrac{\partial}{\partial\vartheta}\,, \\ X_7 = -\sin\vartheta\,\sin\phi\,\dfrac{\partial}{\partial r} - \dfrac{1}{r}\cos\vartheta\,\sin\phi\,\dfrac{\partial}{\partial\vartheta} - \dfrac{1}{r}\,\mathrm{cosec}\,\vartheta\,\cos\phi\,\dfrac{\partial}{\partial\phi}\,, \\ X_8 = -\partial X_7/\partial\phi\,, \qquad X_9 = -\cos\vartheta\,\dfrac{\partial}{\partial r} + \dfrac{1}{r}\sin\vartheta\,\dfrac{\partial}{\partial\vartheta}\,. \end{cases}$$

Again all other metrics admit a G_4 as the maximal group of motions as in case A). This completes the classification of case A).

Case III. Putting $\mu' + 2/r = 0$ in eq. (2.24) yields $K = 0$. Therefore, eq. (2.21) implies that either A) $\nu' = 0$, or, equivalently, $\nu = \nu(t)$; or B) $L = 0$. Using eqs. (2.22) and (2.23) we get $\dot\lambda' = 0$. Proceeding as before (with $\dot\nu' = 0$) we now find that $\nu(r,t) \equiv \nu(t)$, $\lambda(r,t) \equiv \lambda(t)$ and $\mu(r,t) \equiv \ln(t^2/r^2)$. Here $L = c_0$ and we obtain equations analogous to eqs. (A.11)-(A.13) with $r \leftrightarrow t$, $\nu \leftrightarrow \lambda$ and $\exp[\nu] \leftrightarrow -\exp[\nu]$. Hence the minimal G_4 obtained now has a *space-like* instead of a *time-like* translation isometry. Only one new space-time with higher symmetry is obtained by applying the above transformation to the Einstein metric and the corresponding isometry group becomes $SO(1,3)\otimes \boldsymbol{R}$ ($\boldsymbol{R}$ being space-like). For the generators of this algebra we also need to transform $k^0 \leftrightarrow k^1$.

In case II B) (or III B)) $K = L = 0$ and $\dot{\mu} = 0$ (or $\mu' + 2/r = 0$). We can now redefine variables so that the coefficient of the solid angle is r^2 (or t^2). Using eqs. (3.22)-(3.25) and following the previous procedure we find that there are no additional space-times contained in these cases.

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