Existence and Stability Criteria for Circular Geodesics in the Vicinity of a Reissner-Nordström Black Hole.

A. ARMENTI jr.

Department of Physics, Villanova University - Villanova, Pa.

(ricevuto il 13 Novembre 1973)

Summary. — Circular timelike geodesics in the vicinity of a Reissner-Nordström black hole of mass m and charge q are considered. These geodesics exist for all $r > (3m/2)[1 + (1 - 8q^2/9m^2)^{\frac{1}{2}}]$, and are stable for all $r > r_s$, where r_s is the largest real root of $r^3 - 6mr^2 + 9q^2 r - 4q^4/m = 0$. For $q^2 = 0$ these expressions reduce to the familiar Schwarzschild results $r > 3m$ and $r > 6m$, respectively; for $q^2 = m^2$ they reduce to $r > 2m$ and $r > 4m$.

In Schwarzschild co-ordinates $x_0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$ with $c = G = 1$, the Reissner-Nordström metric may be written (1)

$$ds^2 = -\Phi\,dt^2 + \Phi^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{1}$$

with

$$\Phi = 1 - \frac{2m}{r} + \frac{q^2}{r^2}. \tag{2}$$

The metrical form (1) also encompasses the Schwarzschild (2) metric ($\Phi = 1 - 2m/r$) and the de Sitter (3) universe ($\Phi = 1 - r^2/R^2$, $0 < r < R$).

---

(1) H. REISSNER: *Ann. der Phys.*, **50**, 106 (1916); G. NORDSTRÖM: *Proc. K. Akad. Wet. Amsterdam*, **20**, 1238 (1918).

(2) K. SCHWARZSCHILD: *Sitzber preuss. Akad. Wiss., Physik.-math. Kl.*, **189**, 189 (1916); J. DROSTE: *Proc. W. Akad. Wet. Amsterdam.*, **19**, 197 (1916).

(3) W. DE SITTER: *Mon. Not. Roy. Astr. Soc.*, **76**, 699 (1916); **77**, 155 (1916). Circular geodesics do not exist in the empty de Sitter universe as is readily confirmed with the aid of eq. (9b).

The motion of test particles in the field of the static metric (1) may be determined from the Lagrangian (4)

$$\mathscr{L} \equiv \frac{\mathrm{d}s^2}{\mathrm{d}\tau^2} = -\Phi\dot{t}^2 + \Phi^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)\,, \tag{3}$$

where dots denote differentiation with respect to proper time. The absence of explicit t- or φ-dependence in (3) leads to two conserved quantities

$$E \equiv \Phi\dot{t}\,, \qquad J \equiv r^2\sin^2\theta\dot{\varphi}\,, \tag{4}$$

where E and J correspond to the energy per unit mass and angular momentum per unit mass of the test particle. The motions are planar ($\theta = \pi/2$, $\dot{\theta} = 0$) and are governed by the orbit equation

$$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = \frac{E^2 - \Phi}{J^2} - \Phi\varrho^2\,, \tag{5}$$

or equivalently

$$\frac{\mathrm{d}^2\varrho}{\mathrm{d}\varphi^2} = -\frac{1}{2}\left[\left(\varrho^2 + \frac{1}{J^2}\right)\Phi' + 2\varrho\Phi\right], \tag{6}$$

where $\varrho \equiv 1/r$ and $\Phi' = \mathrm{d}\Phi(\varrho)/\mathrm{d}\varrho$. For circular motion $\mathrm{d}^2\varrho/\mathrm{d}\varphi^2 = \mathrm{d}\varrho/\mathrm{d}\varphi = 0$ and consequently, for such motions

$$J^2 = \frac{-r^2\Phi'}{\Phi' + 2r\Phi} \tag{7}$$

and

$$E^2 = \frac{2r\Phi^2}{\Phi' + 2r\Phi}\,. \tag{8}$$

Circular motions require and energy and angular momentum E and J which are both real and finite. Hence circular motions *exist* only for those values of r for which both

$$\Phi' + 2r\Phi > 0 \tag{9a}$$

and

$$\Phi' < 0\,, \tag{9b}$$

are satisfied. Equations (9) define an *existence* domain on the r-axis whose

(4) See, *e.g.*, C. MØLLER: *The Theory of Relativity* (Oxford, 1952), p. 228.

threshold value corresponds to the radius at which the orbital velocity of the test particle reaches the local velocity of light.

The stability of circular geodesics may be investigated by means of the standard stability analysis which is based on a Lagrangian formalism (5) and which leads to the condition

$$2\varrho\Phi'^2 + \Phi(\Phi' - \varrho\Phi'') < 0 . \tag{10}$$

It is interesting to note that the radius of the smallest stable circle (defined by equality in (10)) is also the radius at which E^2 and J^2 achieve their minimum values

$$E^2_{\min} = \frac{4\Phi\Phi'}{3\Phi' + \varrho\Phi''}, \qquad J^2_{\min} = \frac{-2r\Phi'^2}{3\Phi\Phi' + \varrho\Phi\Phi''} . \tag{11}$$

Hence for spherically symmetric metrics of the form (1), a *stability threshold* for circular geodesics occurs at that value of r for which the slopes of the energy and angular-momentum curves vanish (Fig. 1 and 2). *The Schwarz-*

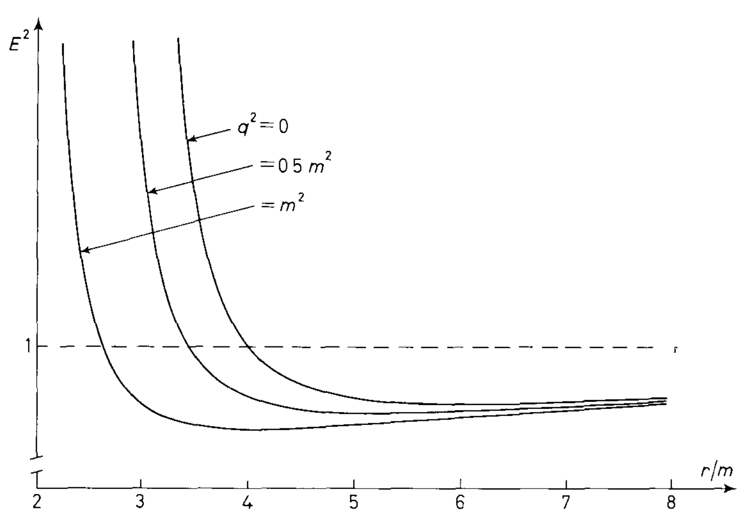

Fig. 1. – E^2 *vs.* r/m for circular timelike geodesics about a Reissner-Nordström black hole. For $q^2 = 0$, E^2 reaches its minimum value $E^2_{\min} = 8/9$ at $r = 6m$. For $q^2 = m^2$, the minimum value is $E^2_{\min} = 27/32$ at $r = 4m$. For every value of q^2, $E^2 \to 1$ as $r/m \to \infty$.

(5) E. T. WHITTAKER: *Analytical Mechanics*, 4th ed., Chap. VII (Cambridge, 1937); see also A. ARMENTI jr. and P. HAVAS: *Relativity and Gravitation*, edited by C. G. KUPER and A. PERES (London, 1971), p. 1.

schild geometry:

$$\Phi = 1 - 2m\varrho\,, \qquad \Phi' = -2m\,, \qquad \Phi'' = 0\,. \tag{12}$$

From (5), (7) and (8) we have

$$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = \frac{E^2-1}{J^2} + \frac{2m}{J^2}\varrho - \varrho^2 + 2m\varrho^3\,, \tag{13}$$

$$J^2 = \frac{mr}{1-3m/r}\,, \qquad E^2 = \frac{(1-2m/r)^2}{1-3m/r}\,. \tag{14}$$

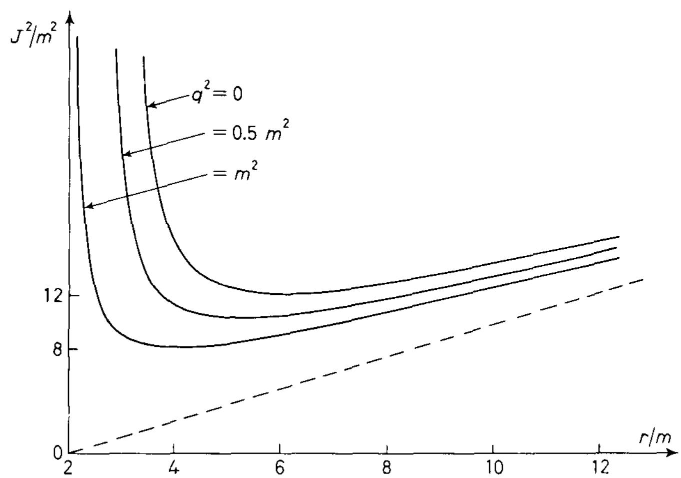

Fig. 2. – J^2/m^2 *vs.* r/m for circular timelike geodesics about a Reissner-Nordström black hole. For $q^2 = 0$, J^2 reaches its minimum value $J^2_{\min} = 12m^2$ at $r = 6m$. For $q^2 = m^2$, the minimum value is $J^2_{\min} = 8m^2$ at $r = 4m$. For every value of q^2, $J^2 \to mr$ (dashed line) as $r/m \to \infty$.

From eqs. (9), (10) and (11) we find

$$\begin{cases} r > 3m & \text{(existence threshold)}\,, \\ r > 6m & \text{(stability threshold)} \end{cases} \tag{15}$$

and

$$J^2_{\min} = 12m^2\,, \qquad E^2_{\min} = 8/9\,. \tag{16}$$

The Reissner-Nordström geometry:

$$\Phi = 1 - 2m\varrho + q^2\varrho^2\,, \qquad \Phi' = 2q^2\varrho - 2m\,, \qquad \Phi'' = 2q^2\,. \tag{17}$$

From eqs. (5), (7) and (8) we obtain

$$\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\varphi}\right)^2 = \frac{E^2-1}{J^2} + \frac{2m}{J^2}\varrho - \left(1+\frac{q^2}{J^2}\right)\varrho^2 + 2m\varrho^3 - q^2\varrho^4\,, \tag{18}$$

$$J^2 = \frac{mr-q^2}{1-3m/r+2q^2/r^2}\,, \qquad E^2 = \frac{(1-2m/r+q^2/r^2)^2}{1-3m/r+2q^2/r^2}\,. \tag{19}$$

J^2 and E^2 are plotted in Fig. 1 and 2 for several values of q^2. From eqs. (9) the existence conditions for the Reissner-Nordström case are

$$r^2 - 3mr + 2q^2 > 0\,, \qquad r - q^2/m > 0\,. \tag{20}$$

The existence threshold value defined by these two equations is

$$r > \frac{3m}{2}\left[1 + (1-8q^2/9m^2)^{\frac{1}{2}}\right], \tag{21}$$

which for $0 < q^2 \leqslant m^2$ is always *smaller* than $3m$. Thus circular motions in the Reissner-Nordström field exist not only for $r > 3m$ as in the Schwarz-

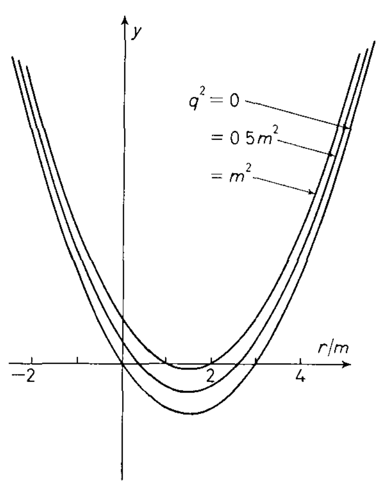

Fig. 3. – Existence criteria for circular timelike geodesics about a Reissner-Nordström black hole. The existence threshold is defined by the larger of the two roots of the quadratic $y \equiv r^2 - 3mr + 2q^2 = 0$. For $q^2 = 0$ this root has the value $e = 3m$. For $q^2 = m^2$ it has the value $r = 2m$.

schild case but also in the region $2m < r \leqslant 3m$ (Fig. 3). The lower limit $r = 2m$ is achieved at $q^2 = m^2$, that is at the transition between a Reissner-Nordström black hole and a naked singularity.

From eq. (10) the stability condition for the Reissner-Nordström case becomes

$$r^3 - 6mr^2 + 9q^2r - 4q^4/m > 0 \,. \tag{22}$$

For $q^2 \ll m^2$ this condition is equivalent to

$$r > 6m[1 - q^2/4m^2] \,. \tag{23}$$

Hence stable circles exists in the Reissner-Nordström field down to radii which are in all cases smaller than the Schwarzschild stability threshold $r = 6m$. In fact, in the limit $q^2 \to m^2$, $r_s \to 4m$ (Fig. 4). Finally, for $q^2 = m^2$

$$J^2_{\min} = 8m^2 \,, \qquad E^2_{\min} = 27/32 \,. \tag{24}$$

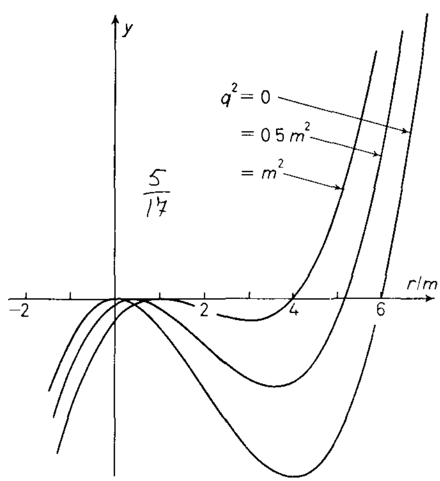

Fig. 4. – Stability criteria for circular timelike geodesics about a Reissner-Nordström black hole. The stability threshold is defined by the largest real root of the cubic $y \equiv r^3 - 6mr^2 + 9q^2 r - 4q^4/m = 0$. For $q^2 = 0$ this root has the value $r = 6m$. For $q^2 = m^2$ it has the value $r = 4m$.

In summary, circular timelike geodesics about a Reissner-Nordström black hole exist and are stable down to radii which are smaller than the corresponding thesholds for the Schwarzschild case. This results from the fact that a Reissner-Nordström source of mass m and charge q is formally equivalent (for geodesics) to a Schwarzschild source with variable mass [6]

$$m^* \equiv m - \frac{q^2}{2r} \,. \tag{25}$$

---

[6] Equation (25) transforms the Schwarzschild metric and orbit equation (13) into the Reissner-Nordström metric and orbit equation (18).

Interestingly, the presence of electric charge of either sign on the source *reduces* the effective mass of the source, again only as far as the motion of uncharged test particles (geodesics) are concerned (7). It is this lowered effective mass which accounts for the lowered thresholds for circular geodesics in the Reissner-Nordström case. The negative sign in (25) reflects a fundamental difference between mass and charge, *viz.* that like masses attract, while like charges repel. Specifically, the sign difference in (25) arises from the fact that in assembling a Reissner-Nordström source having mass and charge, the respective energies of formation and hence the equivalent masses must invariably be of opposite sign. For *charged* test particles, the magnitude of the effective mass depends also on whether the source and test particle charges are alike or unlike, and hence on whether the electric interaction energy (and its equivalent mass) is negative or positive.

---

(7) This is true for noncircular geodesics as well. In the case of quasi-elliptic motions, for example, the exact perihelion advance associated with the Reissner-Nordström solution is *less* than that associated with the Schwarzschild solution, being smaller by a factor of m^*/m (A. Armenti jr.: to be published).

● RIASSUNTO (*)

Si prendono in considerazione geodetiche temporali circolari in prossimità di una buca nera di Reissner e Nordström di massa m e carica q. Tali geodetiche esistono per ogni $r > (3m/2)[1 + (1 - 8q^2/9m^2)^{\frac{1}{2}}]$ e sono stabili per ogni $r > r_s$, in cui r_s è la massima radice reale di $r^3 - 6mr^2 + 9q^2 r - 4q^4/m = 0$. Per $q^2 = 0$ queste espressioni si riducono ai soliti risultati di Schwarzschild $r > 3m$ e $r > 6m$, rispettivamente; per $q^2 = m^2$ si riducono a $r > 2m$ e $r > 4m$.

(*) *Traduzione a cura della Redazione.*

Критерии существования и устойчивости для круговых геодезических линий в окрестности черной дыры Рейснера-Нордстрема.

Резюме (*). — Рассматриваются круговые времениподобные геодезические линии в окрестности черной дыры Рейснера-Нордстрема с массой m и зарядом q. Эти геодезические линии существуют для всех $r > (3m/2)[1 + (1 - 8q^2/9m^2)^{\frac{1}{2}}]$, и являются стабильными для всех $r > r_s$, где r_s есть наибольший вещественный корень уравнения $r^3 - 6mr^2 + 9q^2 r - 4q^4/m = 0$. Для $q^2 = 0$ эти выражения сводятся к известным результатам Шварцшильда $r > 3m$ и $r > 6m$, соответственно. Для $q^2 = m^2$ они сводятся к $r > 2m$ и $r > 4m$.

(*) *Переведено редакцией.*