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An Inequality Stronger than Bell's Inequality.

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Summary. — It is shown that, for a system consisting of two spin-$\frac{1}{2}$ particles, every local hidden-variable theory must satisfy a certain inequality D. It is shown that the inequality D implies Bell's inequality. An example is given in which Bell's inequality is satisfied while inequality D is violated. The same example shows that a nonempty set of (logically) possible results exists which are consistent both with quantum mechanics and with Bell's inequality, but which are not consistent with inequality D and hence not consistent with a local hidden-variable theory. A necessary and sufficient condition is given for Bell's inequality and inequality D to be equivalent; this condition is not generally satisfied for the system we consider.

1. – Introduction.

To date, in order to find experiments whose predictions are consistent either with quantum mechanics (QM) or with a local hidden-variable theory (l.h.v.), only Bell's inequality has been available [1].

According to FREEDMAN and CLAUSER the data from their experiment « in agreement with quantum mechanics, violate these predictions (*viz.* those imposed by the presence of local hidden variables) to high statistical accuracy, thus providing strong evidence against local hidden-variable theories » [2].

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(1) J. S. BELL: *Physics*, **1**, 195 (1965).
(2) S. J. FREEDMAN and J. F. CLAUSER: *Phys. Rev. Lett.*, **28**, 938 (1972).

Nevertheless according to some authors (3), with whom we agree, it seems advisable that, for matters so critical and important other experiments be performed.

Unfortunately, for some experiments that can be performed today and for which we believe that QM and any l.h.v. theory should lead to incompatible predictions, Bell's inequality does not allow one to demonstrate such a conjecture. Let us consider, for instance, the physical system S consisting of two photons γ obtained by positronium decay. If this system is described, in agreement with QM, by a mixture of 2nd type, then, as was shown (4), information about polarization correlations of the two γ obtained by Compton scattering does not violate Bell's inequality.

Moreover there are stronger reasons to think that Bell's inequality is too much of a « weak » condition: the theory proposed by JAUCH (5) can never violate Bell's inequality, as shown in (3). We recall that according to Jauch theory mixtures of 2nd type do not exist in Nature, while a physical system consisting of two subsystems on which separate measures can be performed is described by a mixture of 1st type. Nevertheless for the system S the predictions on polarization correlations obtained by Compton scattering are different, as shown by JAUCH (5), depending on whether QM or Jauch theory is valid. Bell's inequality in this case does not seem suitable to discriminate between cases we know are different (6).

Actually for the system consisting of two spin-$\frac{1}{2}$ particles in a singlet state we find an inequality D which implies Bell's inequality but which, in general, is not implied by the latter. We shall show also that there are results consistent with QM and with Bell's inequality, but incompatible with a l.h.v. theory, because they violate inequality D.

The example on which we shall give the proof of the above-mentioned results shows that inequality D is stronger then an inequality found by SELLERI (8).

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(3) V. CAPASSO, D. FORTUNATO and F. SELLERI: *Intern. Journ. Theor. Phys.*, **7**, 319 (1973).
(4) V. CAPASSO, D. FORTUNATO and F. SELLERI: *Riv. Nuovo Cimento*, **2**, 149 (1970).
(5) J. M. JAUCH: *Rendiconti S.I.F.*, Course IL (New York, N. Y., 1970), p. 20.
(6) One could ask: « Then, is it not sufficient to perform that experiment for which QM and Jauch's theory predict different results without using Bell's inequality? »; we can give the following answer: « the results obtained by JAUCH are still unsuitable for an immediate comparison with experimental data, because they do not take into account the finite volume of the source of photons and the finite volume of the targets where Compton effect takes place; these volumes should not be taken too small, if, for reasonable source intensities and in reasonable time intervals, we want to observe a number of events large enough to be statistically meaningful ». People are working in Catania on this problem in the experiment they are carrying out (7).
(7) G. FARACI, S. NOTARRIGO, A. R. PENNISI and D. GUTKOWSKI: *Boll. S.I.F.*, No. 93, 39 (1972).
(8) F. SELLERI: *Lett. Nuovo Cimento*, **3**, 581 (1972).

Moreover it seems to us that the method we have followed is suitable to be generalized to further conditions, different from l.h.v. The Pool axiomatics (9), for instance, makes essentially use of the conditional probability on which is based our treatment.

The next questions to be investigated are:

i) Can inequality D make the experiment on Compton scattering of photons of the system an « experimentum crucis »?

ii) Is inequality D necessarily satisfied by Jauch's theory?

iii) Does inequality D increase the likelihood in the statistical sense of the hypothesis that, in the Freedman and Clauser experiment, a l.h.v. theory is not valid?

We do not know yet the anwers to these questions, but even if inequality D does not allow one, as in the above-mentioned questions, to operate a discrimination sharper than that one operated by Bell's inequality, it seems to us that, in the scheme of thought that guided us in finding inequality D, it is possible to find even stronger conditions.

2. – We do not start, as BELL does, from correlation functions $P(a, b)$ but from a set Λ (the set of the values of hidden variable λ). On $\mathbf{P}(\Lambda)$ (the set of all subsets of Λ) a positive measure P is defined simply additive and normalized (probability). Hence our hypotheses are, from a mathematical point of view, more general then those of BELL, because we do not require the existence of a probability density $p(\lambda)$, $\lambda \in \Lambda$. On the contrary, ours are the most general hypotheses under which it is permissible to speak of probability (10): let us recall that complete additivity of the measure P is not required.

We shall suppose that there exist two functions:

$$f_1 : \Omega \times \Lambda \to \{-1, +1\}, \qquad f_2 : \Omega \times \Lambda \to \{-1, +1\}, \tag{2.1}$$

where

$$\Omega = \Theta \times \Phi, \qquad \Theta = \{\theta | 0 \leqslant \theta \leqslant \pi\}, \qquad \Phi = \{\varphi | 0 \leqslant \varphi \leqslant 2\pi\}. \tag{2.2}$$

The physical meaning of such functions is the following: for fixed $a \in \Omega$, $b \in \Omega$, $\lambda \in \Lambda$, $f_1(a, \lambda)$ takes the value ± 1 if and only if the spin component of

(9) J. C. T. POOL: *Events, observables and operations and the mathematical approach to quantum theory*, lectures presented at the NATO *Advanced Study Institute on Quantum Mechanics and Ordered Linear Spaces* (1973), to be published in *Lecture Notes in Physics* (Berlin).

(10) B. DE FINETTI: *Teoria della probabilità* (Torino, 1970).

a particle relative to the direction a has the value $\pm\frac{1}{2}$, $f_2(b,\lambda)$ takes the value ± 1 if and only if the spin component of the other particle relative to the direction b has the value $\pm\frac{1}{2}$.

Locality, in the same sense as Bell's, is expressed by the hypotheses that f_1 is independent of b and f_2 is independent of a [11].

So, in order that, as in quantum mechanics, for an arbitrary direction the total spin component (given by the sum of the spin components of the two particles relative to that direction) be zero, let us impose the condition

$$\forall a\in\Omega\,,\qquad \forall\lambda\in\Lambda\,,\qquad f_1(a,\lambda)=-f_2(a,\lambda)\,. \tag{2.3}$$

Let us introduce the function

$$f_{ra}(\lambda):\Lambda\to\{-1,+1\} \tag{2.4}$$

defined by

$$\forall\lambda\in\Lambda\,,\qquad f_{ra}(\lambda)=f_r(a,\lambda)\,,\qquad r=1,2\,. \tag{2.5}$$

In order that the mean value of the spin component of a particle relative to an arbitrary direction be zero, like in QM, let us impose the condition

$$\forall a\in\Omega\,,\qquad P\{f_{ra}^{-1}(+1)\}=P\{f_{ra}^{-1}(-1)\}=\tfrac{1}{2}\,,\qquad r=1,2\,. \tag{2.6}$$

Let us define

$$k_{ij}=P\{f_{1a_i}^{-1}(+1)\cap f_{1a_j}^{-1}(+1)\}\,,\qquad a_i,\ a_j\in\Omega; \tag{2.7}$$

according to (2.3) we have

$$k_{ij}=P\{f_{1a_i}^{-1}(+1)\cap f_{2a_j}^{-1}(-1)\}\,. \tag{2.8}$$

(2.9) *Theorem.* $\forall i,j,k\in\mathscr{I}$, where $\mathscr{I}$ is a set of indexes and each index refers to a unitary vector, the inequalities

1) $k_{ik}\leqslant\min(k_{ij},k_{ik})+\min(\frac{1}{2}-k_{ij},\frac{1}{2}-k_{ik})$,

2) $k_{jk}\geqslant\max(k_{ij}+k_{ik}-\frac{1}{2},\frac{1}{2}-k_{ij}-k_{ik})=|\frac{1}{2}-k_{ij}-k_{ik}|$

must be valid. From now on inequality 2) will be called inequality D. Before giving the proof let us introduce, in order to make the formalism simpler, the following notation:

$$\forall i\in\mathscr{I}\,,\qquad f_i^+=f_{1a_i}^{-1}(+1)\,,\qquad f_1^-=f_{1a_i}^{-1}(-1)\,. \tag{2.9.1}$$

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[11] It is easily shown that the subsequent condition (2.3) implies the independence of f_1, f_2 from hidden variables of the instrument.

Let us go on to give the proof of 1):

(2.9.2) $k_{jk} = P\{f_j^+ \cap f_k^+\} = P\{(f_j^+ \cap f_k^+) \cap (f_i^+ \cup f_i^-)\} = P\{f_j^+ \cap f_k^+ \cap f_i^+\} + P\{f_j^+ \cap f_k^+ \cap f_i^-\}$,

$$P\{f_j^+ \cap f_k^+ \cap f_i^+\} \leqslant \begin{cases} P\{f_i^+ \cap f_j^+\} = k_{ij}\,, \\ P\{f_i^+ \cap f_k^+\} = k_{ik}\,, \end{cases}$$

$$P\{f_j^+ \cap f_k^+ \cap f_i^-\} \leqslant \begin{cases} P\{f_i^- \cap f_j^+\} = \frac{1}{2} - k_{ij}\,, \\ P\{f_i^- \cap f_k^+\} = \frac{1}{2} - k_{ik}\,; \end{cases}$$

inequality 1) is therefore proved.

We shall give the proof of inequality D by means of lemmata from (2.9.3) to (2.9.6).

(2.9.3) *Proposition.* The following alternative holds: either

$$P\{f_i^+ \cap f_j^+\} + P\{f_i^+ \cap f_k^+\} = \tfrac{1}{2} \quad \text{and} \quad P\{f_i^- \cap f_j^+\} + P\{f_i^- \cap f_k^+\} = \tfrac{1}{2}$$

or f_i', f_i'', $L > 0$ can be found, such that [12]

$$\{f_i', f_i''\} = \{f_i^+, f_i^-\}\,,$$

$$P\{f_i' \cap f_j^+\} + P\{f_i' \cap f_k^+\} = \tfrac{1}{2} + L$$

and

$$P\{f_i'' \cap f_j^+\} + P\{f_i'' \cap f_k^+\} = \tfrac{1}{2} - L\,.$$

Proof. Proposition (2.9.3) follows from the equality

$$1 = P\{f_j^+\} + P\{f_k^+\} = P\{(f_i^+ \cup f_i^-) \cap f_j^+\} + P\{(f_i^+ \cup f_i^-) \cap f_k^+\} =$$
$$= P\{f_i^+ \cap f_j^+\} + P\{f_i^+ \cap f_k^+\} + P\{f_i^- \cap f_j^+\} + P\{f_i^- \cap f_k^+\}\,.$$

(2.9.4) If the first proposition of alternative (2.9.3) is valid, then inequality D is satisfied.

This proposition is obvious.

(2.9.5) If the second proposition of alternative (2.9.3) is valid, then

$$P\{f_i' \cap f_j^+ \cap f_k^+\} \geqslant L\,.$$

Proof.

$$\tfrac{1}{2} + L = P\{f_i' \cap f_j^+\} + P\{f_i' \cap f_k^+\} = P\{(f_i' \cap f_j^+) \cup (f_i' \cap f_k^+)\} + P\{f_i' \cap f_j^+ \cap f_k^+\}\,;$$

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(12) $\{f_i^+, f_i^-\}$ denotes the set whose elements are only f_i^+ and f_i^-.

from the previous relation it follows immediately that

$$P\{f_i' \cap f_j^+ \cap f_k^+\} - L = \tfrac{1}{2} - P\{(f_i' \cap f_j^+) \cup (f_i' \cap f_k^+)\} \geqslant 0 .$$

(2.9.6) If the second proposition of alternative (2.9.3) is valid, then inequality D is satisfied.

Proof.

$$P\{f_j^+ \cap f_k^+\} = k_{jk} \geqslant P\{f_i' \cap f_j^+ \cap f_k^+\} \geqslant L$$

according to (2.9.5); furthermore from the definition of L in (2.9.3) it follows

$$L = |\tfrac{1}{2} - k_{ij} - k_{ik}| .$$

Proposition (2.9.6) has thus been proved.

(2.9.7) The proof of inequality D can be obtained from (2.9.3), (2.9.4) and (2.9.6).

Proposition (2.9) has thus been proved.

3. – Inequality 1) is equivalent to Bell's inequality.

3'1. – $P(a_i, a_j)$ is defined as in BELL (1), *viz.* it is the mean value of the product of the results of the measurements performed on a particle along the direction a_i and on the other particle along the direction a_j. An easy calculation gives

(3.1.1) $$P(a_i, a_j) = 1 - 4k_{ij} , \qquad \forall i, j \in \mathscr{I} .$$

3'2. – We remind the reader that Bell's inequality can be written

(3.2.1) $$|P(a_i, a_j) - P(a_i, a_k)| \leqslant 1 + P(a_j, a_k) ,$$

which, because of (3.1.1), becomes

(3.2.2) $$k_{jk} \leqslant \tfrac{1}{2} - |k_{ik} - k_{ij}| ,$$

an inequality which is equivalent to inequality 1).

4. – Inequality D implies Bell's inequality, *i.e.* the set A_D of the triplets R_1, R_2, R_3 such that

$$0 \leqslant R_1 \leqslant \tfrac{1}{2} , \qquad 0 \leqslant R_2 \leqslant \tfrac{1}{2} , \qquad 0 \leqslant R_3 \leqslant \tfrac{1}{2}$$

and

$$\forall i, j, k:\{i, j, k\} = \{1, 2, 3\},$$
$$R_i \geqslant |\tfrac{1}{2} - R_j - R_k|$$

is a proper subset of the set A_B of the triplets R_1, R_2, R_3 such that

$$0 \leqslant R_1 \leqslant \tfrac{1}{2}, \qquad 0 \leqslant R_2 \leqslant \tfrac{1}{2}, \qquad 0 \leqslant R_3 \leqslant \tfrac{1}{2}$$

and

$$\forall i, j, k:\{i, j, k\} = \{1, 2, 3\},$$
$$R_i \leqslant \tfrac{1}{2} - |R_j - R_k|.$$

The proof is obtained by the following **4·1** and **4·2**.

4·1. – Let Q be the square whose side is $[0, \frac{1}{2}]$. $\forall (R_2, R_3) \in Q$, let $A''_D(R_2, R_3)$ be the set of the triplets R_1, R_2, R_3 such that

$$\forall i, j, k:\{i, j, k\} = \{1, 2, 3\}, \qquad R_i \geqslant |\tfrac{1}{2} - R_j - R_k|.$$

Let A'_D be the set union of the sets $A''_D(R_2, R_3)$ when the pair (R_2, R_3) describes the square Q.

By taking into account that $\forall (R_2, R_3) \in Q$ the set of values for R_1 allowed by the inequalities D is not empty, it is easily proved that

$$A'_D(R_2, R_3) = A_D(R_2, R_3).$$

4·2. – However fixed $(R_2, R_3) \in Q$, it is proved by direct calculation that the set of values of R_1 allowed by inequality D is a closed interval, which we denote by $[d_1(R_2, R_3), d_2(R_2, R_3)]$, and it is a subset (proper or not according to the pair (R_2, R_3)) of the set of values of R_1 allowed by inequality B, which is also a closed interval, and we denote it by $[b_1(R_2, R_3), b_2(R_2, R_3)]$.

We have

$$\forall (R_2, R_3) \in Q, \qquad b_2(R_2, R_3) = d_2(R_2, R_3) = \tfrac{1}{2} - |R_2 - R_3|.$$

(4.2.1) For $R_2 + R_3 \geqslant \frac{1}{2}$

$$b_1(R_2, R_3) = d_1(R_2, R_3) = |\tfrac{1}{2} - R_2 - R_3|.$$

(4.2.2) For $R_2 + R_3 \leqslant \frac{1}{2}$

$$b_1(R_2, R_3) = R_2 + R_3 - \tfrac{1}{2} \leqslant 0 \leqslant d_1(R_2, R_3) = |\tfrac{1}{2} - R_2 - R_3|.$$

4'3. – An example for case (4.2.2).

(4.3.1) The numerical values chosen, satisfying the conditions required by (4.2.2), are the following:

$$k_{ij}=\frac{85}{200}, \qquad k_{ik}=\frac{10}{200}.$$

(4.3.2) From inequality D it follows that

$$k_{jk}-k_{ik}\leqslant \tfrac{1}{2}-k_{ij}\leqslant k_{jk}+k_{ik}.$$

(4.3.3) Permuting the indexes we have

$$k_{ij}-k_{ik}\leqslant \tfrac{1}{2}-k_{jk}\leqslant k_{ij}+k_{ik},$$

(4.3.4) from which, substituting the values chosen, we obtain

$$\frac{5}{200}\leqslant k_{jk}\leqslant \frac{25}{200}.$$

(4.3.5) Let us choose $\tilde{k}_{jk}=0$, *i.e.* a value which violates inequality D; we will show that it does not violate Bell's inequality.

(4.3.6) We have, owing to (3.1.1),

$$P(a_i, a_j)=-\frac{35}{50}, \qquad P(a_i, a_k)=\frac{40}{50}, \qquad P(a_j, a_k)=1,$$

$$|P(a_i, a_j)-P(a_i, a_k)|=\frac{75}{50}, \qquad 1+P(a_j, a_k)=\frac{100}{50};$$

with analogous calculations it can be verified that Bell's inequality is not violated for all permutations of indexes i, j, k.

5. – Comparison between Bell's inequality, inequality D and QM for example (4.3) previously considered.

5'1. — Since equivalence between Bell's inequality and inequality 1) has been proved, the limitations on k_{jk} given by Bell's inequality can be obtained as intersection between the sets of values allowed by inequality 1) (written in the equivalent form (3.2.2)), for all the permutations of indexes i, j, k.

(5.1.1) An easy calculation will show that Bell's inequality gives the single nontrivial limitation

$$k_{jk} \leqslant \frac{25}{200},$$

while inequality D gives the two limitations (4.3.4).

(5.1.2) The conditional probability that the measure of the spin component of a particle along direction a_i gives the result $+\frac{1}{2}$ under condition that the measure of the spin component of the other particle along direction a_j has given the result $-\frac{1}{2}$ is expressed by

$$\frac{P\{f_i^+ \cap f_j^+\}}{P\{f_j^+\}}, \tag{5.2.1}$$

and it is equal to $2k_{ij}$ because of (2.6) and (2.8).

(5.2.2) This probability, according to QM is expressed by

$$\cos^2 \frac{\widehat{a_i a_j}}{2}.$$

If we assign to k_{ij} and k_{ik} the values of the previous example, then QM gives for k_{jk} the following limitations:

$$\frac{0.565}{200} \leqslant k_{jk} \leqslant \frac{43.36}{200}.$$

Notice that values of $k_{jk} \in (0.565/200, 5/200)$ are consistent with QM and incompatible with a l.h.v. theory and that this can be asserted on the grounds of inequality D and not on the grounds of Bell's inequality.

Note added in proofs.

In a subsequent paper we shall answer to some remarks by S. J. FREEDMAN and S. NOTARRIGO.

● RIASSUNTO

Si dimostra che, per un sistema costituito da due particelle di spin $\frac{1}{2}$, ogni teoria a variabili nascoste locali deve soddisfare ad una certa diseguaglianza D. Si dimostra che la diseguaglianza D implica la diseguaglianza di Bell. Si dà un esempio in cui la diseguaglianza di Bell è soddisfatta, mentre la diseguaglianza D è violata. Il medesimo esempio mostra che esiste un insieme non vuoto di risultati (logicamente) possibili che sono compatibili tanto con la meccanica quantistica quanto con la diseguaglianza di Bell, ma che non sono compatibili con la diseguaglianza D e quindi non sono compatibili con una teoria a variabili nascoste locali. Si dà una condizione, in generale non soddisfatta per il sistema considerato, necessaria e sufficiente affinché la diseguaglianza di Bell e la diseguaglianza D siano equivalenti.

Неравенство более строгое, чем неравенство Белла.

Резюме (*). — Показывается, что для системы, состоящей из двух частиц со спином $\frac{1}{2}$, любая локальная теория со скрытыми переменными должна удовлетворять определенному неравенству D. Показывается, что неравенство D заключает в себе неравенство Белла. Приводится пример, в котором неравенство Белла удовлетворяется, тогда как неравенство D нарушается. Этот пример показывает, что существует непустая система (логически) возможных результатов, которая согласуется и с квантовой механикой и с неравенством Белла, но которая не согласуется с неравенством D и, следовательно, не согласуется с локальной теорией со скрытыми переменными. Приводятся необходимые и достаточные условия, чтобы неравенство Белла и неравенство D были бы эквивалентны. Это условие в общем случае не удовлетворяется для рассматриваемой нами системы.

(*) *Переведено редакцией.*