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Classical and Recent Formulations for Linear Elasticity

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Summary

The paper presents a review of the classical formulations for linear elasticity, known as principles in solid mechanics. The analogy between the first variation of the energy functionals and the weak forms is pointed out. Several recent developments in first order system least squares, applied to elasticity are discussed. The paper concludes with a mathematical formulation of a new mixed least squares method and a discussion about its future development.

1 INTRODUCTION

> *"The essential fact is that all the pictures which science now draws of nature, and which alone seem capable of according with the observational facts, are mathematical pictures."*
>
> **James Jeans**

The underlying concept of the work presented in this paper came as a result of discussions related to the mixed least squares numerical analysis technique, which was recently developed for solving problems in linear elasticity. In recent years, several new formulations have emerged and a natural question is how they compare with the classical principles, and what are the similarities and the difference between them. A detailed study of the literature on the existing formulations has showed that there are contradictory interpretations of the classical weak forms or principles, which creates a problem in their understanding and numerical implementation. There is a difference in the terminology used by mathematical and engineering oriented researchers, which creates problems in the successful communication of ideas between them. In many discussions on one or another formulation very little attention is apparently paid to the way the different types of boundary conditions are imposed. In order to determine the possibility of applying the different formulations, it is important to state which boundary conditions are essential and which are natural. To clarify some ambiguities existing in the literature, a review is presented of the classical variational formulations for linear elasticity: virtual work, potential energy, virtual forces, complementary energy, Herrmann, Hellinger-Reissner and Hu-Washizu principles. The equivalence between the first variation of the classical energy functionals and the corresponding weak forms is pointed out. The manner by which the boundary conditions are imposed (natural or strong form) is stated. It is well-known that the classical Galerkin

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mixed formulations require compatibility between the approximation for the different variables, expressed by the Ladyzhenskaya - Babuška - Brezzi condition. The paper describes the general concept of the formulations used to circumvent this condition such as Galerkin least squares and first order least squares systems. During the course of the last several years different versions of first order systems least squares (FOSLS) methods have been developed. So far, these methods have been most popular in the mathematical literature. The general idea involves minimization of least squares functionals formed by representation of the higher order partial differential equations as a system of first order equations. Usually, the least squares method is associated with squaring the residual of the higher order partial differential equation. It is well-known that this leads to squaring of the condition number of the resulting linear system and makes the standard version of the least squares method difficult to use. This problem does not exist in the FOSLS methods, where the squaring involves low (first) order partial differential equations. A specific form of a first order systems least squares method is the mixed least squares method for solving problems in elasticity, which the authors developed in recent years [60]. The paper continues with a description of the mathematical formulation, which involves separate approximations for stresses and displacements, allows continuous or discontinuous displacement approximation and results in a positive definite coefficient matrix, which is suitable to be solved by using multilevel iterative solvers. A unique feature of the method is that the displacements can be approximated by different functions over different parts of the problem domain. The initial computational results illustrated in previous publications show that the method gives excellent results for both displacements and stresses for compressible and incompressible materials (plane strain, Poisson ratio exactly equal to 0.5). For the first time, it seems, a numerical formulation involving independent approximations for displacements and stresses, exhibits stability for low order interpolations in the incompressible limit, without additional stabilization techniques (which are necessary for the Hellinger-Reissner formulation). The method gives excellent results for stresses at the inter-element contacts. Its capability to work in the incompressible limit makes it attractive for future extension to nonlinear problems involving plasticity and contact interfaces.

2 PRELIMINARIES

2.1 Strong Form of the Linear Elasticity Equations

Consider a body of volume Ω which is bounded by a surface $\partial\Omega$, and subjected to the action of body forces $\mathbf{f} = (f_x, f_y, f_z)^T$, surface tractions $\mathbf{t} = (\tilde{t}_x, \tilde{t}_y, \tilde{t}_z)^T$, and surface displacements $\mathbf{g} = (g_x, g_y, g_z)^T$. The physical behavior of the linear elastic body is governed by the following equations.

2.1.1 Equations of equilibrium

The stress state at a point of the body is determined by the second order symmetric stress tensor $\boldsymbol{\sigma}$ which satisfies the equations of equilibrium

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} = \mathbf{f}. \tag{1}$$

2.1.2 Relationships between strains and displacement

The deformed state at a point of the body is described by the symmetric second order strain tensor $\boldsymbol{\epsilon}$. In the geometrically linear theory of elasticity, the strain tensor $\boldsymbol{\epsilon}$ is related to the displacement vector $\mathbf{u}$ in terms of the relationships

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right] \tag{2}$$

2.1.3 Stress-strain relationships

For a linear elastic material, stresses are expressed in terms of strains by the relationship $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon}$, where $\mathbf{C}$ is the fourth order elastic constitutive tensor. Strains are expressed in terms of stresses by $\boldsymbol{\epsilon} = \mathbf{C}^{-1}\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\sigma}$, where $\mathbf{D}$ is the fourth order elastic compliance tensor. In this paper we use ν as Poisson ratio, E as modulus of elasticity, and G as shear modulus.

2.1.4 Boundary conditions

The body surface $\partial\Omega$ can be split into two non-overlapping parts depending on the boundary conditions. Denote the part where tractions are prescribed by $\partial\Omega_t$ and the part where displacements are specified by $\partial\Omega_u$. The boundary conditions are given by the following expressions

$$\boldsymbol{t} = \tilde{\boldsymbol{t}} \tag{3}$$

$$\boldsymbol{u} = \tilde{\boldsymbol{u}} \tag{4}$$

2.2 Definitions

In this paper we adopt the following definitions for *variation of a function* and *variation of a functional.*

Variation of a function. Consider the functions u and $\eta = u + \xi v$ where ξ is a positive number. The function η can be made arbitrarily close to u by choosing ξ small enough. The term ξv is called a variation in u and also often is written as δu.

Variation of a functional of one variable. Consider the functional $J(u)$. The quantity $\delta J(u, v)$ is called the first variation in the functional J at u and is given by

$$\delta J(u, v) = lim_{\xi \to 0} \frac{1}{\xi}[J(u + \xi v) - J(u)] = \frac{\partial}{\partial \xi} J(u + \xi v)|_{\xi=0.} \tag{5}$$

Note that the definition of variation of a functional involves a second function v. It will be shown in the following sections that it is extremely important to define what kind of restrictions are imposed on this function.

3 CLASSICAL WORK FORMULATIONS FOR LINEAR ELASTICITY

In the following sections we describe the classical weak formulations in linear elasticity. We show that they are equivalent to the first variation of the corresponding energy functionals.

3.1 Virtual Displacement (Virtual Work) Formulation

The virtual displacement formulation is derived based on the equations of equilibrium (1). Let $\mathbf{v} = (v_x, v_y, v_z)^T$ be a vector of test functions satisfying homogeneous boundary conditions $\mathbf{v} = \mathbf{0}$ on the displacement part of the boundary $\partial\Omega_u$. Let us form a dot product of the vector, representing the residual of the equations of equilibrium with the vector $\mathbf{v}$ and integrate over the domain Ω

$$\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \boldsymbol{f}) \cdot \boldsymbol{v} \, d\,\Omega = 0 \tag{6}$$

Integration by parts of the left-hand side of equation (6) leads to the weak form

$$-\int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\nabla v}) \; d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\,\Omega + \int_{\partial\Omega_u} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v} \, d\,\partial\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v} \, d\,\partial\Omega = 0 \tag{7}$$

The integral over $\partial\Omega_u$ in equation (7) is equal to zero, because the test functions $\mathbf{v}$ satisfy homogeneous boundary conditions on this part of the boundary. On the other hand, using the symmetry of the stress tensor, in the case of small strain theory we can show that $\boldsymbol{\sigma} : (\nabla v) = (\boldsymbol{\sigma} : \boldsymbol{\epsilon})$. In variational calculus the test functions $\boldsymbol{v}$ have the meaning of variations of displacements $\boldsymbol{v} = \boldsymbol{\delta u}$. The equation (7) becomes

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta\epsilon}\, d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\delta u}\, d\,\Omega - \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{\delta u}\, d\,\partial\Omega_t = 0 \tag{8}$$

Equation (8) expresses the principle of virtual displacements. It is the first variation of the virtual work functional. The physical meaning of $\left(\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\delta\epsilon}\, d\Omega\right)$ is the strain energy, or the internal work, the term $\left(\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\delta u}\, d\Omega\right)$ denotes the work done by body forces, and the term $\int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{\delta u}\, d\,\partial\Omega_t$ is the work done by the surface tractions.

3.2 Potential Energy Formulation

It is readily seen that equation (8) is the first variation of the functional

$$F_{vd} = \frac{1}{2}\int_\Omega \boldsymbol{C}\,\boldsymbol{\epsilon} : \boldsymbol{\epsilon}\, d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}\, d\,\Omega - \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}\, d\,\partial\Omega_t \tag{9}$$

which is known as the potential energy functional. The term $\frac{1}{2}(\boldsymbol{C}\,\boldsymbol{\epsilon} : \boldsymbol{\epsilon})$ is a quadratic form, having the physical meaning of strain energy density.

The principle of minimum potential energy is a special case of the principle of virtual work. Washizu [68] pointed out that both principles are equivalent under the following assumptions: First, there exist a positive definite function, which can be derived from the relationships between strains and stresses; second, the relationships between strains and displacements are given by the linear equations (2); and third, the body forces and the surface tractions are conservative and can be derived by differentiation of potential functions.

The solution to the linear elasticity problem consists of those admissible displacements (u_x, u_y, u_z), which satisfy the displacement boundary conditions and lead to a stationary point of the total potential energy. In the case where the body forces and the surface tractions do not depend on displacements, the principle of stationarity of the total potential energy reduces to the principle of minimum potential energy.

3.3 Virtual Forces (Complementary Virtual Work) Formulation

The complementary virtual work formulation is conjugate to the virtual displacements formulation. It is derived from the relationships between strains and displacements. Let $\boldsymbol{\tau}$ be a symmetric tensor satisfying zero traction boundary conditions

$$\boldsymbol{\tau} \cdot \boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \partial\Omega_t \tag{10}$$

and the homogeneous equations of equilibrium

$$\nabla \cdot \boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad \Omega \tag{11}$$

Let us contract the residual of the geometric relationships (2) with $\boldsymbol{\tau}$ and integrate over the domain Ω:

$$\int_{\Omega}\left[\boldsymbol{\epsilon}-\frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}+(\boldsymbol{\nabla}\boldsymbol{u})^{T}\right)\right]:\boldsymbol{\tau}\,d\Omega=\mathbf{0} \tag{12}$$

After integration by parts of the left-hand side of (12), the following expression is obtained

$$\int_{\Omega}\boldsymbol{\epsilon}:\boldsymbol{\tau}\,d\Omega+\int_{\Omega}(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})\cdot\boldsymbol{u}\,d\Omega-\int_{\partial\Omega_u}(\boldsymbol{\tau}\cdot\boldsymbol{n})\cdot\boldsymbol{u}\,d\,\partial\Omega-\int_{\partial\Omega_t}(\boldsymbol{\tau}\cdot\boldsymbol{n})\cdot\boldsymbol{u}\,d\,\partial\Omega=0 \tag{13}$$

The substitution $\boldsymbol{\tau}\cdot\boldsymbol{n}=\mathbf{0}$ on $\partial\Omega_t$, $\boldsymbol{u}=\boldsymbol{g}$ on $\partial\Omega_u$, and $\nabla\cdot\boldsymbol{\tau}=\mathbf{0}$ into (13) leads to the following weak form

$$\int_{\Omega}\boldsymbol{\epsilon}:\boldsymbol{\tau}d\,\Omega-\int_{\partial\Omega_u}\boldsymbol{g}\cdot(\boldsymbol{\tau}\cdot\boldsymbol{n})d\,\partial\,\Omega=\mathbf{0} \tag{14}$$

If the test function $\boldsymbol{\tau}$ is selected to be equal to a variation of stresses $\boldsymbol{\delta\sigma}$, then equation (14) becomes

$$\int_{\Omega}\boldsymbol{\epsilon}:\boldsymbol{\delta\sigma}d\,\Omega-\int_{\partial\Omega_u}\boldsymbol{g}\cdot(\boldsymbol{\delta\sigma}\cdot\boldsymbol{n})d\,\partial\,\Omega=\mathbf{0} \tag{15}$$

The term $\left(\int_{\Omega}\boldsymbol{\epsilon}:\boldsymbol{\delta\sigma}\right)$ denotes the complementary energy. The weak form expressed by equation (15) represents the principle of complementary virtual work in mechanics of solids. It is valid for any constitutive relationship between stresses and strains.

3.4 Complementary Energy Formulation

Equation (15) represents a first variation of the functional

$$F_{cw}=\frac{1}{2}\int_{\Omega}\boldsymbol{D\sigma}:\boldsymbol{\sigma}d\,\Omega-\int_{\partial\Omega_u}\boldsymbol{g}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{n})d\,\partial\,\Omega \tag{16}$$

which is known as complementary energy functional. The term $\frac{1}{2}\left(\boldsymbol{D\sigma}:\boldsymbol{\sigma}\right)$ denotes the complementary energy density.

The complementary energy formulation is a special case of the complementary virtual work expression. Both formulations are equivalent under the same assumptions for equivalence between virtual work and potential energy formulations.

According to the complementary energy principle the solution of the linear elasticity problem is a symmetric stress tensor satisfying the equations of equilibrium and the prescribed traction boundary conditions, and minimizing the complementary energy functional F_{cw}. The stress tensor $\boldsymbol{\sigma}$ must have components which are square integrable functions, and its divergence $(\nabla\cdot\boldsymbol{\sigma})$ must also be square integrable $(\boldsymbol{\sigma}\in H(div,\Omega))$. The complementary energy formulation is difficult to use because the stress tensor functions have to satisfy the boundary conditions and the equations of equilibrium a priori. It is a formidable task to choose such a tensor, which satisfies these equations a priori in a domain of arbitrary shape.

4 MIXED FORMULATIONS

In the following section we describe classical two and three-field mixed formulations.

4.1 Hellinger-Reissner Formulation

The Hellinger-Reissner formulation involves two independent types of unknowns: displacements and stresses. Hellinger [28] formulated a functional of strains and stresses and postulated that the *"canonic"* equations (the equations of equilibrium and the constitutive equations) can be obtained from this functional. Hellinger discussed that when strains are expressed in terms of stresses, another form of this functional can be obtained, (reminding us of the complementary energy statement) from which the equations of equilibrium can be recovered. In order to obtain displacements corresponding to the stresses, the conditions of compatibility should be fulfilled.

Reissner [53] formulated a variational theorem stating that *"Among all states of stress and displacement which satisfy the boundary conditions of prescribed surface displacement, the actually occurring state of stress and displacement is determined by the variational equation*

$$\delta\left\{\int_\Omega F\,d\Omega-\int_{\partial\Omega}(\tilde{t}_x u_x+\tilde{t}_y u_y+\tilde{t}_z u_z)\right\}d\,\partial\Omega=0.\text{''} \tag{17}$$

Reissner [53] assumed that the integral over $\partial\Omega_t$ vanishes because the variations of displacements $\boldsymbol{\delta u}$ are equal to zero on the traction part of the boundary. The original form of the Reissner principle expressed by equation (62) in the Appendix, implies that displacements are essential boundary conditions and the tractions are natural boundary conditions.

Washizu [68] discussed two forms of the Hellinger-Reissner principle which are given in detail in the Appendix.

The two forms of the Hellinger-Reissner formulation can be derived from the complementary energy functional by introducing the equilibrium equations as additional terms. One alternative is to impose them as penalty terms. A disadvantage of penalty formulations in general is that the physics of the problems is sometimes affected in order to satisfy purely mathematical conditions. The rate of convergence of the iterative methods used to solve the resulting linear system is slowed down. A better alternative, having more physical meaning, is to introduce the equilibrium conditions in terms of Lagrange multipliers. Let us augment the complementary energy functional with the term $\int_\Omega \boldsymbol{\lambda}\cdot(\nabla\cdot\boldsymbol{\sigma}-\boldsymbol{f})d\Omega$ which also reflects an energy state. It is clear that the Lagrange multipliers $\boldsymbol{\lambda}$ must have the physical meaning of displacements ($\boldsymbol{\lambda}=\boldsymbol{u}$). The augmented complementary energy functional takes the form:

$$F_{ace_1}=\frac{1}{2}\int_\Omega \boldsymbol{D\sigma}:\boldsymbol{\sigma}d\,\Omega-\int_{\partial\Omega_u}\boldsymbol{g}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{n})d\,\partial\,\Omega+\int_\Omega \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\sigma}-\boldsymbol{f})d\Omega \tag{18}$$

It can be readily recognized that (18) is the form of the Hellinger-Reissner functional given by equation (63) in the Appendix. If the term $\int_\Omega \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\sigma})d\,\Omega$ is integrated by parts and substituted in (18), then the result is

$$F_{ace_2}=\frac{1}{2}\int_\Omega \boldsymbol{D\sigma}:\boldsymbol{\sigma}d\,\Omega-\int_{\partial\Omega_t}\boldsymbol{u}\cdot(\boldsymbol{\sigma}\cdot\boldsymbol{n})d\,\partial\,\Omega-\int_\Omega \boldsymbol{u}\cdot\boldsymbol{f}d\Omega+\int_\Omega \nabla\boldsymbol{u}:\boldsymbol{\sigma}d\Omega \tag{19}$$

It is quite clear that expression (19) is similar to the second form of the Hellinger-Reissner functional expressed by equation (64). It should be noted that the addition of Lagrange multiplier terms changes the mathematical nature of the problem. The complementary energy formulation is a pure minimization problem, where the equilibrium and boundary conditions have to be imposed a priori on the selected trial functions. The Hellinger-Reissner formulation represents a saddle point, or constraint minimization problem, where the equilibrium equations are introduced in terms of Lagrange multipliers having the physical meaning of displacements.

Next it is shown that both forms of the Hellinger-Reissner functional are equivalent to corresponding weak forms obtained from the elasticity equations. The condition for stationarity of a two-field functional $F(\boldsymbol{u}, \boldsymbol{\sigma})$ is given by the variational equations:

$$\begin{aligned} \delta F(\boldsymbol{u}, \boldsymbol{\sigma}) \mid_{\boldsymbol{u}=const} &= 0 \\ \delta F(\boldsymbol{u}, \boldsymbol{\sigma}) \mid_{\boldsymbol{\sigma}=const} &= 0 \end{aligned} \tag{20}$$

The application of the stationarity conditions (20) to the Hellinger-Reissner functional in (18) yields the following set of equations:

$$\begin{aligned} \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{\delta\sigma}) \cdot \boldsymbol{u}\, d\Omega - \int_{\partial\Omega_u} (\boldsymbol{\delta\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}\, d\,\partial\Omega - \int_\Omega \boldsymbol{D\sigma} : \boldsymbol{\delta\sigma}\, d\Omega &= 0 \\ \int_\Omega (\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}) \cdot \boldsymbol{\delta u}\, d\Omega &= 0 \end{aligned} \tag{21}$$

Next, let us consider the equations of equilibrium and the constitutive equations expressed in terms of displacements and stresses. Let $\boldsymbol{\tau}$ be vector test functions and $\boldsymbol{\tau}$ be tensor test functions. The displacement-stress weak mixed form is given by the equations:

$$\begin{aligned} \int_\Omega \left[\frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right) - \boldsymbol{D\sigma}\right] : \boldsymbol{\tau}\, d\Omega &= 0 \\ \int_\Omega (\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}) \cdot \boldsymbol{v}\, d\Omega &= 0 \end{aligned} \tag{22}$$

If the first of the equations (22) is integrated by parts, and if $\boldsymbol{\tau} \cdot \boldsymbol{n} = 0$ on the traction part of the boundary $\partial\Omega_t$, then the weak mixed form becomes

$$\begin{aligned} \int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \cdot \boldsymbol{u}\, d\Omega - \int_{\partial\Omega_u} (\boldsymbol{\tau} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}\, d\,\partial\Omega - \int_\Omega \boldsymbol{D\sigma} : \boldsymbol{\tau}\, d\Omega &= 0 \\ \int_\Omega (\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}) \cdot \boldsymbol{v}\, d\Omega &= 0 \end{aligned} \tag{23}$$

It can be readily seen that the substitution of $\boldsymbol{v} = \boldsymbol{\delta u}$ and $\boldsymbol{\tau} = \boldsymbol{\delta\sigma}$ in (23) leads to the set of equations (21) which resulted from the first form of the Hellinger-Reissner functional. In this formulation the displacement boundary conditions are natural and the traction boundary conditions are essential.

If the second of the equations (22) is integrated by parts, the weak form becomes

$$\begin{aligned} \int_\Omega \left[\frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right) - \boldsymbol{D\sigma}\right] : \boldsymbol{\tau}\, d\Omega &= 0 \\ -\int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\nabla v})\; d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\,\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v}\, d\,\partial\Omega &= 0 \end{aligned} \tag{24}$$

Let $\boldsymbol{v} = \boldsymbol{\delta u}$ and $\boldsymbol{\tau} = \boldsymbol{\delta\sigma}$, where $\boldsymbol{\sigma}$ is a symmetric tensor. Because of the symmetry of the stress tensor, the following identity is true:

$$\frac{1}{2}\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right] : \boldsymbol{\delta\sigma} = (\boldsymbol{\nabla u}) : \boldsymbol{\delta\sigma} \tag{25}$$

The substitution of $\boldsymbol{v}$, $\boldsymbol{\tau}$ and (25) in (24) gives

$$\begin{aligned} \int_\Omega (\boldsymbol{\nabla u}) : \boldsymbol{\delta\sigma}\, d\Omega - \int_\Omega \boldsymbol{D\sigma} : \boldsymbol{\delta\sigma}\, d\Omega &= 0 \\ -\int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\nabla \delta u})\; d\,\Omega - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\delta u}\, d\,\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{\delta u}\, d\,\partial\Omega &= 0 \end{aligned} \tag{26}$$

Equations (26) represent the conditions for stationarity of the form of the Hellinger-Reissner functional expressed by (19). In this formulation the displacements are essential boundary conditions and the tractions are natural boundary conditions.

4.2 Displacement-Mean Stress Formulation for Nearly Incompressible and Incompressible Elasticity

For problems involving nearly incompressible and fully incompressible materials the work formulations are difficult to use. The functional of virtual work becomes indeterminate as Poisson ratio tends to 0.5, and the standard displacement-based method does not converge. The complementary virtual work principle also causes difficulties, since the stresses must be selected in such a way, that they satisfy the equilibrium equations a priori.

Herrmann [29] suggested the concept to *"utilize some function of the mean pressure as a primary independent variable, so that in the limit as Poisson ratio equals one half, the formulation specializes to the governing equations for an incompressible material"*. He introduced a mean stress function H, which is related to the mean stress $\sigma_m = \frac{1}{3}(\sigma_x+\sigma_y+\sigma_z)$ by the expression

$$H = \frac{3\sigma_m}{2\mu(1+\nu)} \tag{27}$$

Herrmann's formulation states that *"Among all admissible states of the variables u and H (displacement and mean stress variable) the actually occurring state is determined by the variational equation"*

$$\begin{aligned} \delta \int_\Omega \Big\{ \mu \big[(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + 2(\epsilon_{xy}^2 + \epsilon_{yz}^2 + \epsilon_{zx}^2) + 2\nu H(\epsilon_x + \epsilon_y + \epsilon_z) \;- \\ -\nu(1-2\nu)H^2 - 6\nu e_T H - 2(\epsilon_x + \epsilon_y + \epsilon_z)e_T\big] \\ -(f_x u_x + f_y u_y + f_z u_z)d\Omega - \int_{\partial\Omega} (t_x u_x + t_y u_y + t_z u_z) d\partial\Omega \Big\} = 0 \end{aligned} \tag{28}$$

where e_T is thermal expansion. According to Herrmann [29] *"An admissible displacement state is one that: 1) satisfies the prescribed displacement boundary conditions; 2) has continuous second derivatives within each region; 3) is continuous across all region interfaces. An admissible state of the mean stress variable H is one with continuous first derivatives within each region"*.

The functional

$$F(\boldsymbol{u}, p) \;=\; \int_\Omega \mu\,\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) \;-p\;(\nabla\cdot\boldsymbol{u}) \;+ \frac{1}{2\lambda}p^2 - \boldsymbol{f}\cdot\boldsymbol{u}\, d\Omega - \int_{\partial\Omega_t} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{u}\, d\Omega \tag{29}$$

is similar to Herrmann's principle for $e_T = 0$ (no thermal expansion), provided that $p = \nu\, H$.

The condition for stationarity of a two-field functional $F(\boldsymbol{u}, p)$ is given by the variational equations

$$\begin{aligned} \delta F(\boldsymbol{u}, p)\,|_{p=const} &= 0 \\ \delta F(\boldsymbol{u}, p)\,|_{\boldsymbol{u}=const} &= 0 \end{aligned} \tag{30}$$

The application of (30) to Herrmann's functional yields the set of equations

$$\begin{aligned} 2\int_\Omega \mu\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\boldsymbol{\delta u})\; d\Omega - \int_\Omega p\;(\nabla\cdot\boldsymbol{\delta u})\; d\Omega &= \int_{\partial\Omega_t} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{\delta u}\; d\Omega + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\delta u}\; d\Omega \\ \int_\Omega \frac{1}{\lambda} p\,\delta p\; d\Omega &= \int_\Omega (\nabla\cdot\boldsymbol{u})\;\delta p\; d\Omega \end{aligned} \tag{31}$$

Next, we derive a mixed weak form involving displacements and mean stress as separate unknowns. The incompressible material involves zero volumetric strain described by

$$\nabla\cdot\boldsymbol{u} \;=\; \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = 0 \tag{32}$$

The volumetric strain can be expressed in terms of mean stress p as

$$-(\lambda + 2\,\frac{\mu}{3})\,(\nabla\cdot\boldsymbol{u}) \;=\; p = -\frac{1}{3}\,(\sigma_x + \sigma_y + \sigma_z) \tag{33}$$

In the nearly incompressible limit $\mu \ll \lambda$, equation (33) can be approximated by

$$\lambda\,(\nabla\cdot\boldsymbol{u}) + p = 0 \tag{34}$$

After substitution of the constitutive relationships

$$\boldsymbol{\sigma} = \lambda\,(\nabla\cdot\mathbf{u})\;\mathbf{I} + 2\,\mu\boldsymbol{\epsilon}(\boldsymbol{u}) = -p\mathbf{I} + 2\,\mu\boldsymbol{\epsilon}(\boldsymbol{u}), \tag{35}$$

the virtual displacement principle takes the form

$$2\int_\Omega \mu\,\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\; d\Omega - \int_\Omega p\;(\nabla\cdot\boldsymbol{v})\; d\Omega = \int_{\partial\Omega_t} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{v}\; d\Omega + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\; d\Omega. \tag{36}$$

The incompressibility constraint is introduced in a weak sense by multiplication of equation (34) by a test scalar function q and integration over the domain Ω:

$$\int_\Omega \frac{1}{\lambda}\,p\,q\; d\Omega = \int_\Omega (\nabla\cdot\boldsymbol{u})\; q\; d\Omega \tag{37}$$

Equations (36) and (37) represent a weak formulation of the problem involving nearly incompressible and incompressible linear elastic materials. The unknowns are the displacement vector $\mathbf{u}$ and the scalar function mean stress p. The mean stress is equivalent to pressure, when Poisson's ratio is exactly equal to one half. When the material is nearly incompressible, $\frac{1}{\lambda}$ is not equal to zero, and p can be eliminated from equation (37) and substituted in (36). Then the mixed form is reduced to a modified displacement based form. In the case when the test functions $\boldsymbol{v}$ are equal to a variation of displacements $\boldsymbol{\delta u}$, and test functions q are equal to a variation of mean stress δp, the weak form is equivalent to the stationarity condition for Herrmann's principle. It should be noted that the requirements for admissible displacements and mean stress in the original formulation of Herrmann seem to be too strong. The weak formulation requires that displacements be in the space H^1 (i.e. the displacement functions and their first derivatives must be square integrable) and mean stresses be square integrable functions.

4.3 Hu-Washizu Formulation

The Hu-Washizu formulation involves three independent types of unknowns: displacements, strains and stresses. In the solid mechanics literature it is known as the Hu-Washizu principle [68]. The displacement-strain-stress mixed formulation can be derived from the potential energy functional. The relationships between displacements and strains can be imposed via an additional term having the physical meaning of energy. The augmented potential energy functional takes the form

$$F = \int_\Omega \left\{ \frac{1}{2}\boldsymbol{C\epsilon} : \boldsymbol{\epsilon} + \left[\boldsymbol{\epsilon} - \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T \right) \right] : \boldsymbol{\sigma} \quad - \quad \boldsymbol{f} \cdot \boldsymbol{u} \right\} d\,\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{u}\, d\,\partial\Omega \qquad (38)$$

The functional (38) is equivalent to the original Hu-Washizu functional expressed by equality (67) in the Appendix. Under the assumptions of small strains, conservative body and surface forces and the existence of a positive definite function of strains, the Hu-Washizu formulation is equivalent to a problem for minimimization of total potential energy under the constraints expressed by the stress-strain relationships. The constraints are imposed by Lagrange multipliers, which have the physical meaning of stresses. In general, the conditions for stationarity of a three-field functional $F(\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma})$ are given by the variational equations:

$$\begin{aligned} \delta F(\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma})\ |_{\boldsymbol{u}=const, \boldsymbol{\sigma}=const} &= 0 \\ \delta F(\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma})\ |_{\boldsymbol{u}=const, \boldsymbol{\epsilon}=const} &= 0 \\ \delta F(\boldsymbol{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma})\ |_{\boldsymbol{\epsilon}=const, \boldsymbol{\sigma}=const} &= 0 \end{aligned} \qquad (39)$$

Next, we show that the conditions for a saddle point of the Hu-Washizu functional are equivalent to a weak formulation of the equations of linear elasticity.

Let $\mathbf{e}$ and $\boldsymbol{\tau}$ be symmetric test tensors and their components possess first derivatives with respect to coordinates x, y and z. Let $\mathbf{v}$ be a vector of test functions satisfying homogeneous displacement boundary conditions on $\partial\Omega_u$. The three-field mixed formulation is derived based on the constitutive relations, stress-strain relationships and the equations of equilibrium. The constitutive relations are contracted with the test tensor $\mathbf{e}$ and integrated over the domain Ω. The strain-displacement relationships are contracted with the test tensor $\boldsymbol{\tau}$ and are also integrated over Ω. The equations of equilibrium are multiplied by a vector $\mathbf{v}$ and integrated over Ω. As a result, the following equations are obtained

$$\begin{aligned} \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{C}\boldsymbol{\epsilon}) : \boldsymbol{e}\, d\Omega &= 0 \\ \int_\Omega \left[\boldsymbol{\epsilon} - \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T \right) \right] : \boldsymbol{\tau}\, d\Omega &= 0 \\ \int_\Omega (\nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}) \cdot \boldsymbol{v}\, d\Omega &= 0 \end{aligned} \tag{40}$$

After integration by parts the third equation in (40) and taking into account that the test functions $\mathbf{v}$ satisfy homogeneous conditions on the part of the boundary Ω_u, where displacements are specified, the three field mixed formulation takes the form

$$\begin{aligned} \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{C}\boldsymbol{\epsilon}) : \boldsymbol{e} d\Omega &= 0 \\ \int_\Omega \left[\boldsymbol{\epsilon} - \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T \right) \right] : \boldsymbol{\tau} d\Omega &= 0 \\ -\int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\nabla v})\ d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\,\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{v}\, d\,\partial\Omega &= 0 \end{aligned} \tag{41}$$

Equations (41) are valid for each set of test functions $\boldsymbol{e}, \boldsymbol{\tau}, \boldsymbol{v}$. Let the functions $\boldsymbol{e}$ be variations of strains $\boldsymbol{e} = \boldsymbol{\delta\epsilon}$, the functions $\boldsymbol{\tau}$ be variations of stresses, $\boldsymbol{\tau} = \boldsymbol{\delta\sigma}$ and the functions $\boldsymbol{v}$ be variations of displacements, $\boldsymbol{v} = \boldsymbol{\delta u}$. The substitution of $\boldsymbol{e}, \boldsymbol{\tau}$ and $\boldsymbol{v}$ in (41) leads to the set of equations

$$\begin{aligned} \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{C}\boldsymbol{\epsilon}) : \boldsymbol{\delta\epsilon}\, d\Omega &= 0 \\ \int_\Omega \left[\boldsymbol{\epsilon} - \frac{1}{2}\left(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T \right) \right] : \boldsymbol{\delta\sigma}\, d\Omega &= 0 \\ -\int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\nabla \delta u})\ d\,\Omega - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\delta u}\, d\,\Omega + \int_{\partial\Omega_t} (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{\delta u}\, d\,\partial\Omega &= 0 \end{aligned} \tag{42}$$

which are the conditions for stationarity of the Hu-Washizu functional.

5 CONDITIONS FOR STABILITY, UNIQUENESS AND CONVERGENCE OF THE STANDARD MIXED FORMULATIONS

5.1 The Limitation Principle

At the continuous level, without any discretization into finite elements, the weak formulations are equivalent. They represent one or another integral or weak form of the linear elasticity equations. However, at the discrete level, they are not equivalent and might lead to different approximate solutions. It has been observed that there is a restriction on the combinations of interpolation functions, which can be used in the discretization of the mixed formulations.

De Veubeke [19] discussed the limitation principle related to the theoretical possibility of using separate approximations of displacements and stresses in the Hellinger-Reissner principle. If stresses are expressed in terms of displacements and substituted in the Hellinger-Reissner principle, then it reduces to the principle of virtual work. If no restrictive assumptions are made on the stresses, then *"if a net of finite elements is analyzed by compatible displacement modes and the stresses left free to be determined by energy considerations, the best stresses are those associated with the strains derived from the displacements and the* *degrees of freedom in the displacement modes are governed by the ordinary principle of variation of displacements. In other words, it is useless to look for a better solution by injecting additional degrees of freedom in the stresses, although the stresses obtained as a rule, will not satisfy the detailed equilibrium conditions".* If displacements are expressed in terms of stresses and substituted in the Hellinger-Reissner principle, it reduces to the complementary energy principle. *"If no restrictive assumptions are made on the displacements"*, then "*if the stresses form an equilibrium field a priori, their degrees of freedom are governed by the ordinary complementary energy principle. In as much as the resulting strains are not, as a rule, integrable the principle gives no indication concerning a best associated displacement field".* The limitation principle of De Veubeke was a prelude to the LBB condition, which is discussed next.

5.2 The LBB Condition

The displacement-mean stress formulation of Herrmann, the Hu-Washizu and Hellinger-Reissner principles represent saddle points, or min-max problems, i.e. problems, where a minimum is searched with regard to some of the field variables and a maximum must be found for the rest of the variables. The mathematical basis for mixed formulations was developed by Ladyzhenskaya [36], Brezzi [10] and Babuška [4]. They established the conditions for uniqueness, stability and convergence of saddle point problems. Their theorems are known as the celebrated "Ladyzhenskaya - Babuška - Brezzi" condition. In general, a saddle point problem can be written in an abstract form as follows:

Given the continuous linear functionals $f \in W'$ and $g \in V'$, find $\{u, p\} \in W \times V$ such that the following equations are satisfied:

$$\begin{aligned} a(u,v) + b(v,p) &= f(v) \;\; \forall v \in W \\ b(u,q) &= g(q) \;\; \forall q \in V, \end{aligned} \tag{43}$$

where W and V are the definition spaces for variables u and p correspondingly, a and b are continuous bilinear forms. The notation (x, y) is the L^2 inner product of two scalar, vector or second order tensor functions x and y, $(x,y) = \int_\Omega x \cdot y d\Omega$ where '$\cdot$' is the appropriate inner product.

The existence and uniqueness of the solution of the saddle point problem (43) is governed by the following theorems:

- **Continuity of a and b.**

  There exist positive constants c_1, $c_2 < \infty$ such that:

$$\begin{aligned} a(u,v) &\leq c_1 \|u\|_W \|v\|_W, \;\; \forall u, v \in W \\ b(u,q) &\leq c_2 \|u\|_W \|q\|_V, \;\; \forall u \in W, \forall q \in V \end{aligned} \tag{44}$$

- **Stability Condition (K-ellipticity of a).**

  There exists a constant $\alpha > 0$ such that:

$$|a(v,v)| \geq \alpha \|v\|_W^2 \;\; \forall v \in K \tag{45}$$

  where $K = \{v \in W | \;\; b(v,q) = 0 \;\; \forall q \in V\}$.

- **Ladyzhenskaya - Babuška - Brezzi condition.**

  There exists a constant $\beta > 0$ such that:

$$\sup_{v \in W} \frac{b(v,q)}{\|v\|_W} \geq \beta \|q\|_V \quad \forall q \in V \tag{46}$$

  Then the problem (43) has a unique solution $\{u, p\} \in W \times V$.

If the left-hand side of (46) is divided by $\|q\|_V$, then the condition is written as

$$\inf_{q \in V} \sup_{v \in W} \frac{b(v,q)}{\|v\|_W \|q\|_V} \geq \beta > 0. \tag{47}$$

In the last form the LBB condition is known as "$\inf - \sup$" condition. If (44) and (45) are satisfied, then (46) is a necessary and sufficient condition for existence, uniqueness and stability of the saddle point problem.

The discretization of problem (43) involves the selection of finite dimensional subspaces $W_h \subset W$ and $V_h \subset V$, spanned by finite element functions, say piece-wise polynomials. The classical Galerkin method is then formulated as follows:

Given $f \in W'$ and $g \in V'$, find $(u_h, p_h) \in W_h \times V_h$ such that:

$$\begin{array}{rcl} a(u_h, v_h) + b(v_h, p_h) & = & f(v_h) \;\; \forall v_h \in W_h \\ b(u_h, q_h) & = & g(q_h) \;\; \forall q_h \in V_h, \end{array} \tag{48}$$

The discrete analogue of the conditions (44), (45) and (47) holds for problem (48), if (u, v, p, q, w, v) are replaced by their discrete counterparts $(u_h, v_h, p_h, q_h, w_h, v_h)$, and K is replaced by $K_h \subset W_h$, where

$K_h = \{v_h \in W_h | \;\; b(v_h, q_h) = 0 \;\; \forall q_h \in V_h\}$.

Since $K_h \not\subset K$ in general, the existence, uniqueness and stability conditions must be verified for each combination of W_h and V_h. For many problems, combinations of functions, which seem "natural" (such as equal order approximations) do not satisfy the stability conditions with α and β independent of the mesh size h. Spurious oscillations and locking (lack of approximation when the mesh is refined) occur when working with combinations of W_h and V_h which violate either (45) or (47). Within the classical Galerkin formulation one needs to use combinations of functions which satisfy the LBB condition.

The existence of a positive constant β guarantees that the mixed method results in a nonsingular matrix. This does not always mean that the method is stable (i.e. for small changes in input it gives small variations of the solution). The stability depends on β. If it can be proved that β is exactly a constant and does not depend on the size of discretization h, then the method is stable and the numerical solution converges to the exact solution. If β depends on h, then even if the resulting matrix is non-singular, the method might not be stable and the numerical solution might not converge to the exact solution. The proof that β is a constant must be performed for all combinations of the approximation functions. The theoretical proof is a formidable task, but one might perform computational experiments and see how the solution behaves with respect to the mesh size. If β is equal to zero, then the resulting matrix is singular. There are two ways to handle this obstacle. First, one can impose appropriate boundary conditions, which eliminate the incompatible functions. Then the resulting linear system will become invertible. Second, an iterative method can be used to solve the singular problem, and, after a satisfactory number of iterations, in terms of post-processing, the "bad" interpolation functions can in principle be eliminated.

6 RECENT FORMULATIONS FOR LINEAR ELASTICITY

6.1 Galerkin Least Squares

Franca [20] and Franca and Hughes [21] proposed a modification of the general abstract form of the saddle point problems (see Appendix). For Hellinger-Reissner formulation, the modification can be written as follows:

Given $\boldsymbol{f} \in V'$, find $\left(\boldsymbol{u}^h, \boldsymbol{\sigma}^h \boldsymbol{p}^h\right) \quad \in \quad V_h \times W_h$ such that

$$\begin{aligned} a_h^*(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_h(\boldsymbol{u}_h, \boldsymbol{\tau}_h) &= f_h(\boldsymbol{\tau}_h), \quad \forall \boldsymbol{\tau}_h \in V_h \\ b_h(\boldsymbol{v}_h, \boldsymbol{\sigma}_h) + c_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= g_h(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in W_h \end{aligned}$$

where

$$\begin{aligned} a_h^*(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) &= -\int_\Omega (\boldsymbol{D}\boldsymbol{\sigma}_h) : (\boldsymbol{\tau}_h)\, d\,\Omega + \delta_1 \gamma_1 \int_{\Omega_h} (\boldsymbol{D}\,\boldsymbol{\sigma_h}) : (\boldsymbol{D}\,\boldsymbol{\tau_h})\, d\,\Omega_h \\ &\quad -\delta_2 \frac{h^2}{\gamma_1} \int_{\Omega_h} (\nabla \cdot \boldsymbol{\sigma}_h) : (\nabla \cdot \boldsymbol{\tau}_h)\, d\,\Omega_h \\ b_h(\boldsymbol{u}_h, \boldsymbol{\tau}_h) &= \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{u}_h) : \boldsymbol{\tau}_h\, d\,\Omega - \delta_1 \gamma_1 \int_{\Omega_h} \boldsymbol{\epsilon}(\boldsymbol{u}_h) : \boldsymbol{D}\boldsymbol{\tau}_h\, d\,\Omega_h \\ c_h(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \delta_1 \gamma_1 \int_{\Omega_h} \boldsymbol{\epsilon}(\boldsymbol{u}_h) : \boldsymbol{\epsilon}(\boldsymbol{v}_h)\, d\,\Omega_h \\ f_h(\boldsymbol{\tau}_h) &= \delta_2 h^2 \int_{\Omega_h} \boldsymbol{f} \cdot (\nabla \cdot \boldsymbol{\tau}_h)\, d\,\Omega \\ g_h(\boldsymbol{v}_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, d\,\Omega \end{aligned} \tag{49}$$

The equations (49) represent a stabilization of the original Galerkin formulation by adding least squares-like terms. Franca [20] pointed out that the difference between Galerkin least squares and the least squares as a methods for solving partial differential equations is that the least squares terms are evaluated only in the interior of each element and pre-multiplied by a coefficient, which depends on the mesh size h. The δ_1 term adds stability to the Lagrange multiplier (displacements in the Hellinger-Reissner formulation). The δ_2 term stabilizes the primary variable (stresses in the Hellinger-Reissner formulation). Franca [20] observed that the general formulation (68) expresses two classes of finite elements depending on the nature of the governing stability equations:

- Circumventing the LBB condition, when $\delta_1 > 0, \delta_2 \geq 0$.
- Satisfying the LBB condition, when $\delta_1 = 0, \delta_2 \geq 0$.

The theoretical investigations by Franca [20] and Franca and Hughes [21] on circumventing the BB were applied to compressible elasticity. The computational results confirmed the predictions of the theoretical analysis.

6.2 First-Order System Least Squares Methods

The general concept involves minimization of least squares functionals formed by representation of the higher order partial differential equations as a system of first order equations. Usually, the least squares method is associated with squaring the residual of the higher order partial differential equation. It is well-known that this leads to squaring of the condition number of the resulting linear system and makes the standard version of the least squares method difficult to use. This problem does not exist in FOSLS methods, where the squaring involves low (first) order partial differential equations.

Carey and Chen [14] applied a least squares numerical technique to a second-order ordinary differential equation. They represented the higher order ordinary differential equation in terms of a system of first-order ordinary differential equations. Their approach involved minimization of a least squares functional consisting of the sum of the squares of the residuals of the first-order equations. The numerical experiments performed by Carey and Chen seemed very encouraging and were partially explained in the theoretical work of Pehlivanov, Carey, Lazarov and Chen [45]. Recently the least squares methods became the object of a renewed interest. The least squares methods have the advantage that the resulting linear system is symmetric and that the compatibility condition for the spaces used for the different field variables can be circumvented. A least squares method for the Stokes equations, with application to linear elasticity was introduced in the mathematical literature by Cai, McCormick and Manteuffel [12]. The original formulation used displacements and displacement gradients as independent unknowns. Later, Cai, Manteuffel, McCormick and Parter extended and analyzed the original first-order least squares formulation for the linear elasticity problem with pure traction boundary conditions [13]. The linear elasticity problem is represented by a first-order system of partial differential equations

$$\begin{array}{rcll} \boldsymbol{U} - \nabla \cdot \boldsymbol{u} & = & \mathbf{0} & \text{in } \Omega \\ -\nabla \cdot A\boldsymbol{U} & = & \boldsymbol{f} & \text{in } \Omega \\ \nabla \boldsymbol{U} & = & \mathbf{0} & \text{in } \Omega \\ \boldsymbol{n} \cdot A\boldsymbol{U} & = & \mathbf{0} & \text{on } \partial\Omega_t \end{array} \tag{50}$$

where $\boldsymbol{U}$ is the gradient of displacement vector, $\boldsymbol{U} = \left(\frac{\partial u_x}{\partial x}, \frac{\partial u_x}{\partial y}, \frac{\partial u_y}{\partial x}, \frac{\partial u_y}{\partial y}\right)$, $\boldsymbol{n}$ is the unit normal to the boundary, $\boldsymbol{n} = (n_x, n_y)$, and $A = \lambda A_1 + 2\mu A_2$ is a (4 by 4) matrix defined as follows

$A_1 = \boldsymbol{b}\boldsymbol{b}^T$, where $\boldsymbol{b} = (1, 0, 0, 1)^T$ and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{51}$$

One of the least squares approaches involves minimization of the functional

$$G(\boldsymbol{U}, \boldsymbol{u}; \boldsymbol{f}) = \| \boldsymbol{f} + \nabla \cdot A\boldsymbol{U} \|^2 + \| \nabla \boldsymbol{U} \|^2 + (A(\boldsymbol{U} - \nabla \boldsymbol{u}), \boldsymbol{U} - \nabla \boldsymbol{u}) . \tag{52}$$

with regard to displacements and displacement gradients.

7 A NEW MIXED LEAST SQUARES FORMULATION FOR LINEAR ELASTICITY

Recently we developed an original mixed least squares formulation for solving problems in linear elasticity [60], [65], [64]. It involves separate approximations for stresses and displacements, allows continuous or discontinuous displacement approximation and results in a positive definite coefficient matrix, which is suitable to be solved by using multilevel iterative solvers. The approximate solution of the linear elasticity problem is obtained via a minimization of a least squares functional F depending on displacements and stresses. The mathematical formulation is presented next.

7.1 Strong Form of the Linear Elasticity Problem

Consider the domain of an elastic body $\Omega \subset R^d$, with sufficiently smooth boundary $\partial\Omega = \partial\Omega_u \cup \partial\Omega_t$ where $d = 2$ or 3 is the number of space dimensions (see Figure 1). Given body forces $\boldsymbol{f}$ in Ω, tractions $\boldsymbol{t}$ on traction part of the boundary $\partial\,\Omega_t$ and displacements $\boldsymbol{g}$ on the displacement part of the boundary $\partial\,\Omega_u$, find a pair of stresses and displacements $(\boldsymbol{\sigma},\ \boldsymbol{u})$ satisfying the first order system of partial differential equations:

$$\begin{aligned} 2\,\boldsymbol{D}\boldsymbol{\sigma} &= \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T && \text{in } \Omega \\ \nabla\cdot\boldsymbol{\sigma} &= \boldsymbol{f} && \text{in } \Omega \end{aligned} \tag{53}$$

and the boundary conditions:

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{g} && \text{on } \partial\,\Omega_u \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \boldsymbol{t} && \text{on } \partial\,\Omega_t \end{aligned} \tag{54}$$

where $\boldsymbol{D}$ is the fourth order compliance tensor of the elastic constants and $\boldsymbol{n}$ is the unit outward normal to the boundary surface.

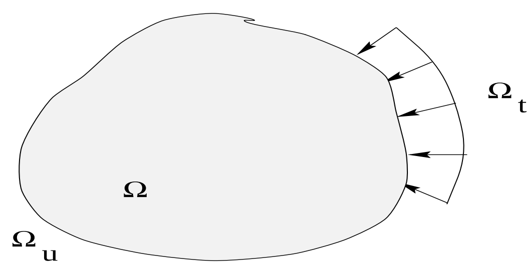


Figure 1. Elastic body occupying domain Ω bounded by a surface $\partial\Omega = \partial\Omega_u \cup \partial\Omega_t$

The equations denoted by the first part of expression (53) are the constitutive relationships for a linear elastic medium, and the equations in the second part are the equations of equilibrium.

7.2 Weak Formulation

Let $L_2(\Omega)$ be the space of square-integrable functions on Ω, and $H^S(\boldsymbol{div}, \Omega)$ be a space of symmetric tensor functions, which together with their divergence are square integrable on Ω. Denote by $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$ the definition spaces for the boundary displacement function $\boldsymbol{g}$ and boundary traction function $\boldsymbol{t}$; Given $\boldsymbol{f} \in \left[L^2(\Omega)\right]^d$, $\boldsymbol{g} \in \left[H^{\frac{1}{2}}(\partial\,\Omega_u)\right]^d$ and $\boldsymbol{t} \in \left[H^{-\frac{1}{2}}(\partial\,\Omega_t)\right]^d$, find $(\boldsymbol{\sigma},\ \boldsymbol{u})$, where $\boldsymbol{\sigma} \in H^S(\boldsymbol{div}, \Omega)$, $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{t}$ on $\partial\Omega_t$, $\boldsymbol{u} \in \left[L^2(\Omega)\right]^d$, such that:

$$\begin{aligned} 2\int_\Omega \boldsymbol{D}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) - \int_{\partial\Omega_u} \boldsymbol{g}\cdot(\boldsymbol{\tau}\cdot\boldsymbol{n}) &= 0 \\ \int_\Omega (\nabla\cdot\boldsymbol{\sigma} - \boldsymbol{f})\cdot(\nabla\cdot\boldsymbol{\tau}) &= 0 \end{aligned} \tag{55}$$

for every $\boldsymbol{\tau} \in H^S(\boldsymbol{div}, \Omega) \ \ | \, \boldsymbol{\tau} \cdot \boldsymbol{n} = \mathbf{0}$ on $\partial\,\Omega_t$.

Note that in (55) there are no derivatives on the displacement vector, which allows the use of discontinuous approximation functions for the displacement variable.

7.3 The Least Squares Functional

The problem domain is discretized into triangles or quadrilaterals in the two-dimensional space and into tetrahedra or hexahedra in the three-dimensional space. For simplicity, we assume that the domain is polygonal.

The numerical technique proposed in this study involves minimization of a least squares functional F which consists of two parts. The first part F_1 is formed from the first of equations (55) by substitution of $\boldsymbol{\tau}$ equal to the test functions $\boldsymbol{\phi}_i$ and summation of the squares of the obtained residuals.

It is defined on $H^S(\boldsymbol{div}, \Omega) \times \left[L^2(\Omega)\right]^d$ and has the following form

$$F_1\,(\boldsymbol{\sigma},\ \boldsymbol{u},\ \boldsymbol{g}) \ \stackrel{\text{def}}{=} \sum_{i=1}^{m} \left[2 \int_\Omega (\boldsymbol{D}\,\boldsymbol{\sigma} \ : \boldsymbol{\phi}_i + \int_\Omega \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\phi}_i) - \int_{\partial\Omega_u} \boldsymbol{g} \cdot (\boldsymbol{\phi}_i \ \cdot \ \boldsymbol{n}) \right]^2 \tag{56}$$

where $\boldsymbol{\phi}_i$ are test tensor functions, which satisfy the homogeneous traction boundary condition $\boldsymbol{\phi}_i \cdot \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega_t$ and form a basis for the test space $T^h_{\mathbf{0}} \subset \ H^S(\boldsymbol{div}, \Omega)$.

The second part of the functional F_2 is obtained from the second of equations (55) by substitution of $(\nabla \cdot \boldsymbol{\tau})$ by $(\nabla \cdot \boldsymbol{\sigma} - f)$. It is defined on $H^S(\boldsymbol{div}, \Omega)$ and is given by

$$F_2\,(\boldsymbol{\sigma},\ \boldsymbol{f}) \ \stackrel{\text{def}}{=} \ \int_\Omega |\ \nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}\ |^2 \equiv ||\ \nabla \cdot \boldsymbol{\sigma} - \boldsymbol{f}\ ||\ ^2_{[L_2(\Omega)]^d} \tag{57}$$

The functional to be minimized is defined on $H^S(\boldsymbol{div}, \Omega) \times \left[L^2(\Omega)\right]^d$ and takes the form

$$F\,(\boldsymbol{\sigma}, \boldsymbol{u},\ \boldsymbol{f},\ \boldsymbol{g}) \ \stackrel{\text{def}}{=} \ F_1\,(\boldsymbol{\sigma},\ \boldsymbol{u},\ \boldsymbol{g}) + F_2\,(\boldsymbol{\sigma},\ \boldsymbol{f}) \tag{58}$$

An approximate solution to the problem (55) will be any pair $\left(\boldsymbol{\sigma}^h, \boldsymbol{u}^h\right), \boldsymbol{\sigma}^h \in T^h_{\boldsymbol{t}},\ \boldsymbol{u}^h \in U^h_{\boldsymbol{g}}$ or $\boldsymbol{u}^h \in U^h_{disc}$, which minimizes $F\,(\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{f}, \boldsymbol{g})$, where $T^h_{\boldsymbol{t}}$ is the finite element space for stresses, $U^h_{\boldsymbol{g}}$ and U^h_{disc} are the finite element spaces for displacements in the case of continuous and discontinuous approximation respectively.

The trial stress space $T^h_{\boldsymbol{t}}$ consists of symmetric tensors over Ω which satisfy the traction boundary conditions $\boldsymbol{\sigma}^h \cdot \boldsymbol{n} = \boldsymbol{t}$ on $\partial\Omega_t$. The test tensor functions $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \cdots, \boldsymbol{\phi}_m$ satisfy zero traction boundary conditions and span the space $T^h_{\mathbf{0}}$.

Displacements can be approximated by continuous or discontinuous functions. Let the finite element space for displacements be spanned by the functions $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_n$. In the case of continuous approximation, $\boldsymbol{\eta}_i$ are piece-wise continuous functions, satisfying the displacement boundary condition $\boldsymbol{u}, \boldsymbol{g}$ on $\partial\Omega_u$ and belong to the space $U^h_{\boldsymbol{g}}$. In the case of discontinuous displacement approximation, the space for displacements is denoted by U^h_{disc} and consists of square integrable vector functions. No boundary conditions are imposed on the functions $\boldsymbol{\eta}_i$.

After applying the standard minimization procedure, that is equating to zero all partial derivatives of the functional F with respect to the unknown components of displacement vector and stress tensor, we obtain a discrete form of the problem (58) which has the form $\mathbf{K}\,\mathbf{V} = \mathbf{R}$, where $\mathbf{V}$ is the unknown of the displacement vector and stress tensor components, $\mathbf{R}$ is the right-hand side, and $\mathbf{K}$ is the coefficient matrix. The coefficient matrix $\mathbf{K}$ is symmetric and is represented by the sum $(\mathbf{A}^{\mathbf{T}}\mathbf{A}+\mathbf{B})$, where $\mathbf{A}$ is a rectangular matrix determined by differentiation of the first part of the functional. The number of its columns is equal to the number of unknown displacement and stress components and the number of its rows is determined by the number of the stress test functions. The matrix $\mathbf{B}$ is symmetric and is obtained by differentiation of the second part of functional.

The new formulation involves separate approximations for displacements and stresses, allows for discontinuous or continuous approximation of displacements, and results in a positive definite coefficient matrix and does not require compatibility between approximation spaces for displacements and stresses. A condition for existence and uniqueness of the solution of the discrete problem was established and verified analytically and numerically for two low order piece-wise polynomial FEM spaces [60]. The condition for uniqueness of the solution via the mixed least squares method is given by expression similar to the LBB condition for the Hellinger-Reissner formulation. The difference is, that in the proposed least squares formulation the compatibility is between the approximation spaces for displacements and the spaces for test functions, while in the Hellinger-Reissner formulation the compatibility condition relates the approximation spaces for displacements and stresses.

7.4 Numerical Example

Bending of a Beam Loaded by Uniform Load

This example is described by Timoshenko and Goodier [66]. A beam of a rectangular cross section of unit width, simply supported at the ends is loaded by a uniformly distributed load of intensity q (see Figure 2). Because of the symmetry in geometry and load and linearity of the material properties, only one half of the beam was modeled. The following boundary conditions were prescribed: on the lower boundary of the beam the components of traction vector were taken equal to zero; on the right boundary the traction components σ_x and τ_{xy} were specified as computed by Timoshenko and Goodier [66]; on the upper boundary traction components $\tau_{xy} = 0$ and $\sigma_y = -q$ were specified. On the left boundary the conditions of symmetry $\tau_{xy} = 0$ and $u_x = 0$ were imposed. The following data for material properties and beam geometry were used: Young's modulus: $E = 0.1\ GPa$, Poisson ratio $\nu = 0.2$, half-span of beam $l = 2\ m$, half-width of beam $c = 0.1\ m$, uniform load density $q = 10\ kPa$.

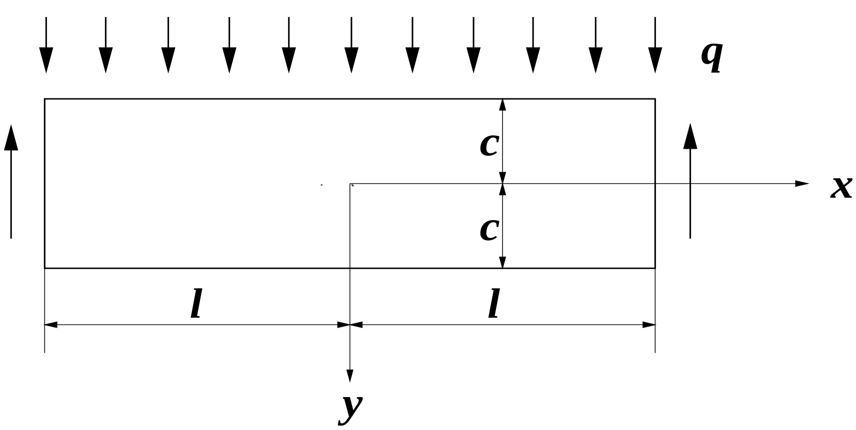


Figure 2. Beam loaded by uniform load

The mixed least squares formulation was implemented in an original computer program, written from scratch in C programming language, using object-oriented logic. The computer code was first applied to the state of plane stress. The initial results were obtained with the use of an element which involves piece-wise constant interpolations for displacements and bilinear interpolations for stresses. It is well known that such an approximation of displacements is not possible in the standard finite element formulation.

The computer simulation was performed for rectangular grids, consisting of different number of elements in the x and y directions. Figures 3, 4 and 5 illustrate a comparison between the exact and approximate solutions for the three stress tensor components corresponding to a pointing upwards coordinate system with origin in the middle of the lower wall of the beam). The approximate stress tensor components were obtained with a rectangular grid of 10 elements in x direction and 49 elements in y direction. Smooth stresses were computed at the points of the grid, without any post-processing of the solution. The low order interpolation polynomials required more elements to approach the exact solution. The 'constant displacement-bilinear stress element' resulted in slower convergence for displacements compared to stresses. In the numerical simulation the displacement boundary conditions were considered to be essential. When they are approximated by piece-wise constants, the boundary value is imposed over the entire element. When the size of elements, where displacement boundary conditions were specified, decreased, the error in displacements also decreased.

The displacements improved when a 'bilinear displacement-bilinear stress' element was used. The results for both stresses and displacements were close to the exact solution. It was noticed that with the decrease in mesh size, the approximate solution for every component of displacement vector and stress tensor converged to the exact solution. The obtained rate of convergence for displacements was similar to the rate of convergence for stresses. Numerous analyzes were done for rectangular meshes and different number of elements. In all cases the approximate solution converged to the exact solution. No "spurious modes" or oscillations were noticed. The results for displacements converged faster than those obtained with the 'constant displacement- bilinear stress' element because the boundary conditions were imposed correctly and because of the higher order of approximation. The comparison between the results obtained with both elements showed that good stress distributions can be computed even when the displacements are not so good. It seems that in the proposed method the displacements and stresses converge independently of each other. The obtained displacements with the 'bilinear displacement-bilinear stress' element are shown in Figure 6. For a regular mesh consisting of 10x50 elements the exact solution was very close to the solution obtained with least squares mixed method.

The convergence of the proposed numerical technique for an incompressible material in the plane strain state (Poisson's ratio equal to 0.5) was also studied. The same boundary conditions were applied. For the 'bilinear displacement-bilinear stress' element, regular rectangular meshes involving 6 by 6, 12 by 12 and 24 by 24 elements were used. The computations with the 'constant displacement-bilinear stress' element were performed on meshes involving 6 by 7, 12 by 13 and 24 by 25 elements. For the results from every numerical experiment, we computed the L_2 norms of the absolute errors in stresses and displacements. In order to illustrate the rate of convergence, we recorded graphically the logarithm of the L_2 norms versus the logarithm of the reciprocal of the largest diameter of element, which occurred in the computational mesh, in terms of Log of $1/h$.

To make the graphical illustration more transparent, we "normalized" the absolute error. In our presentation of the stress convergence we took the logarithm of the computed absolute error of the stress divided by 100,000. Also, instead of taking Log of the computed absolute error of displacements, we calculated the Log of the absolute displacement error divided by 0.01.

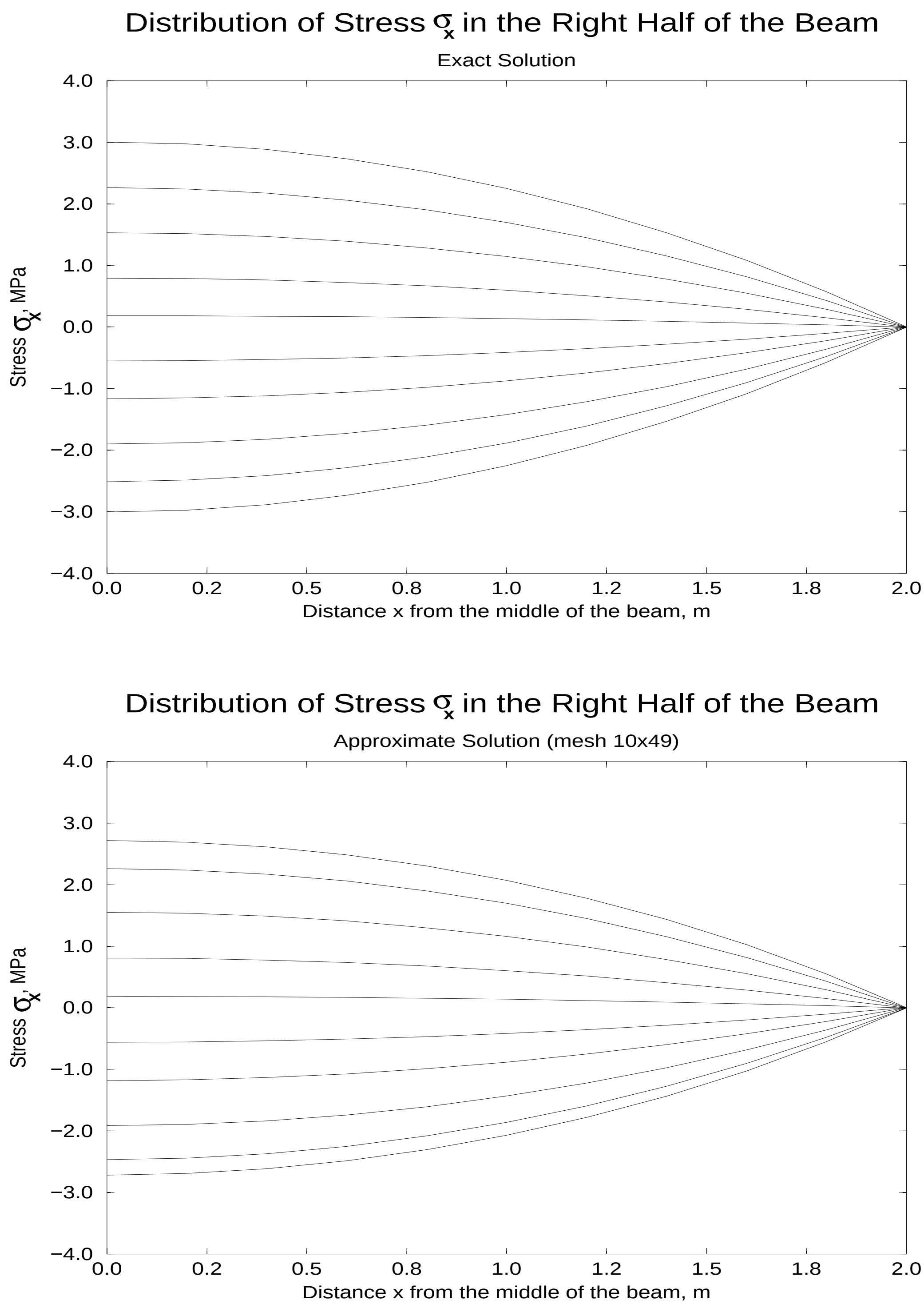
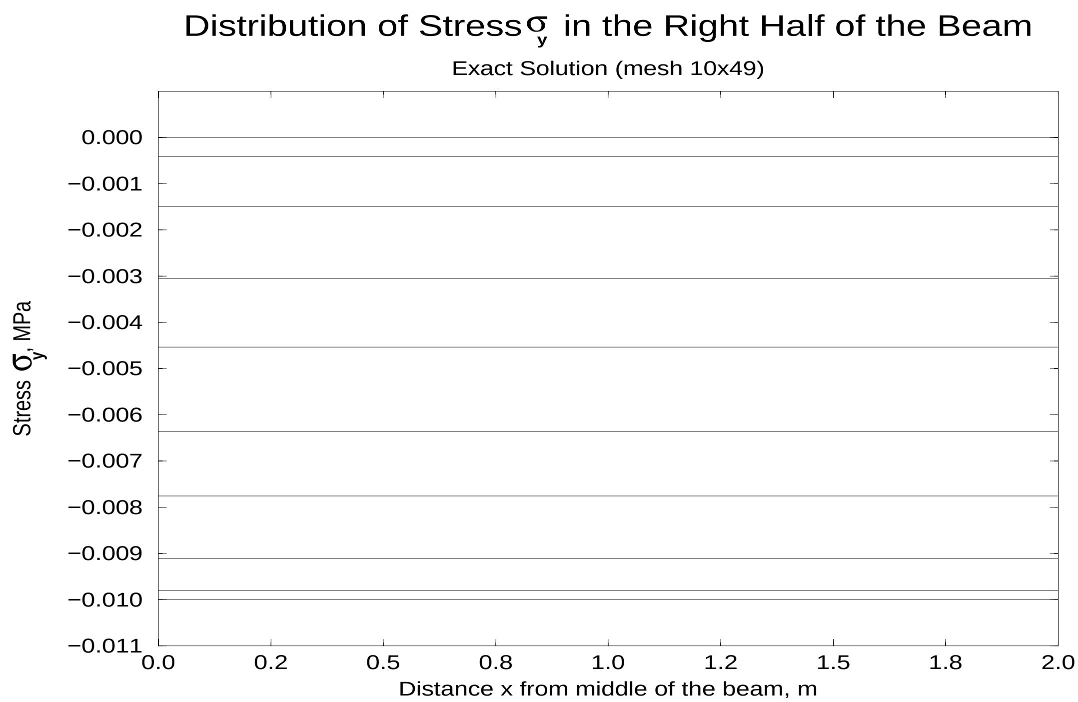


Figure 3. Exact and approximate solutions for σ_x in the beam, 'constant displacement-bilinear stress element'



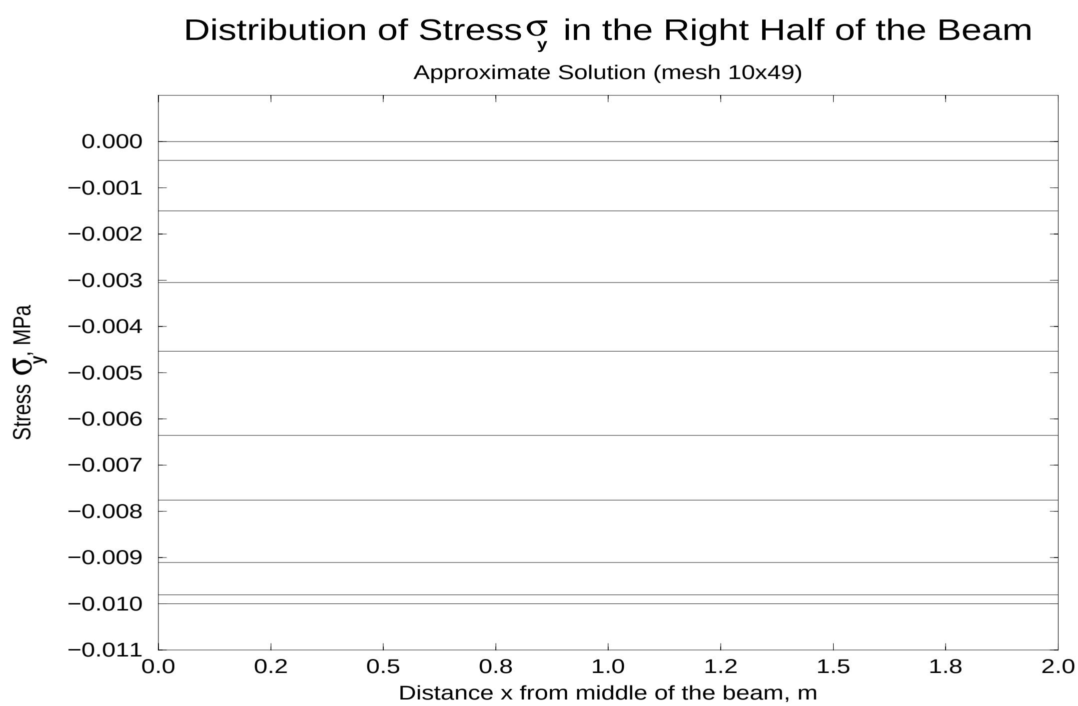
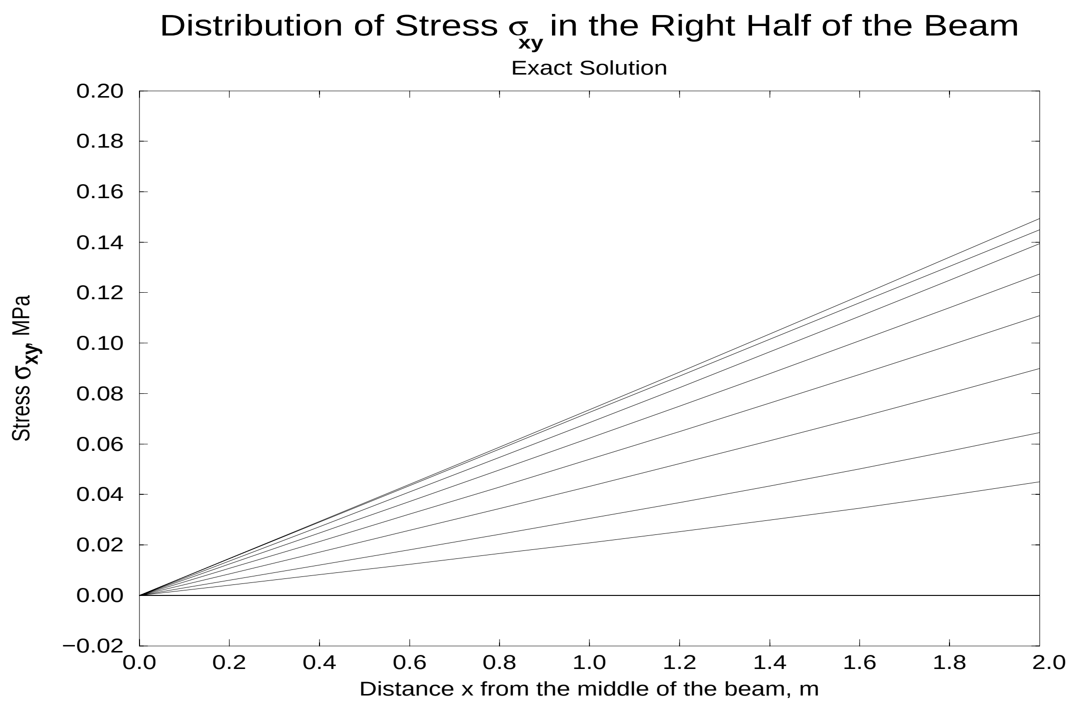


Figure 4. Exact and approximate solutions for σ_y in the beam, 'constant displacement-bilinear stress' element



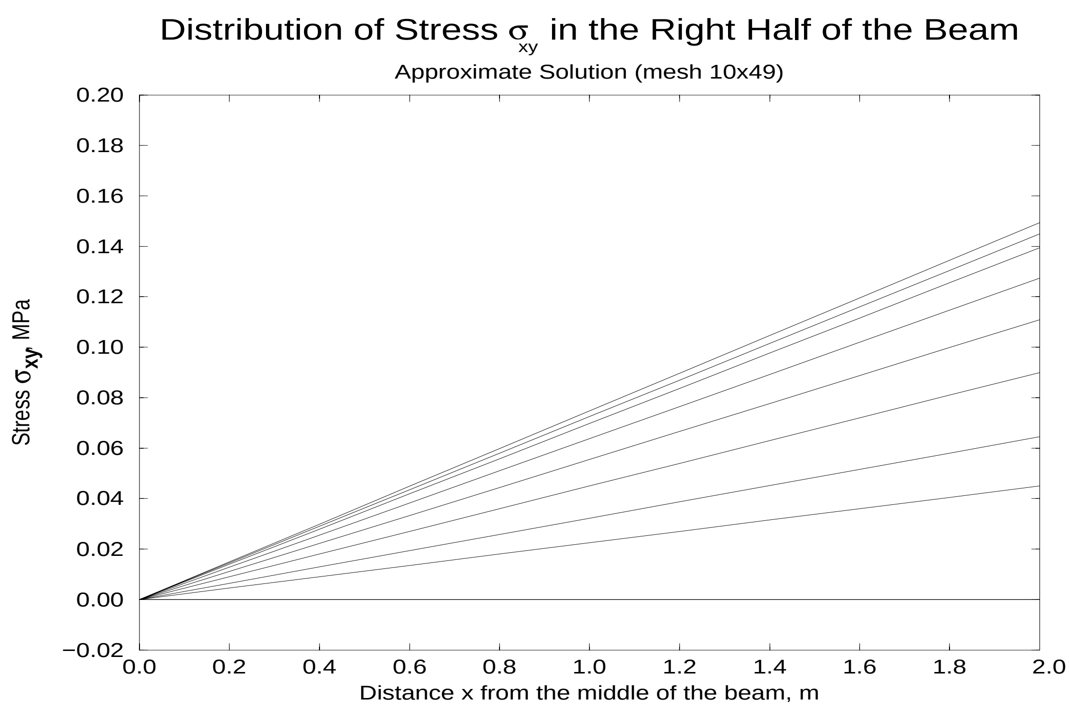
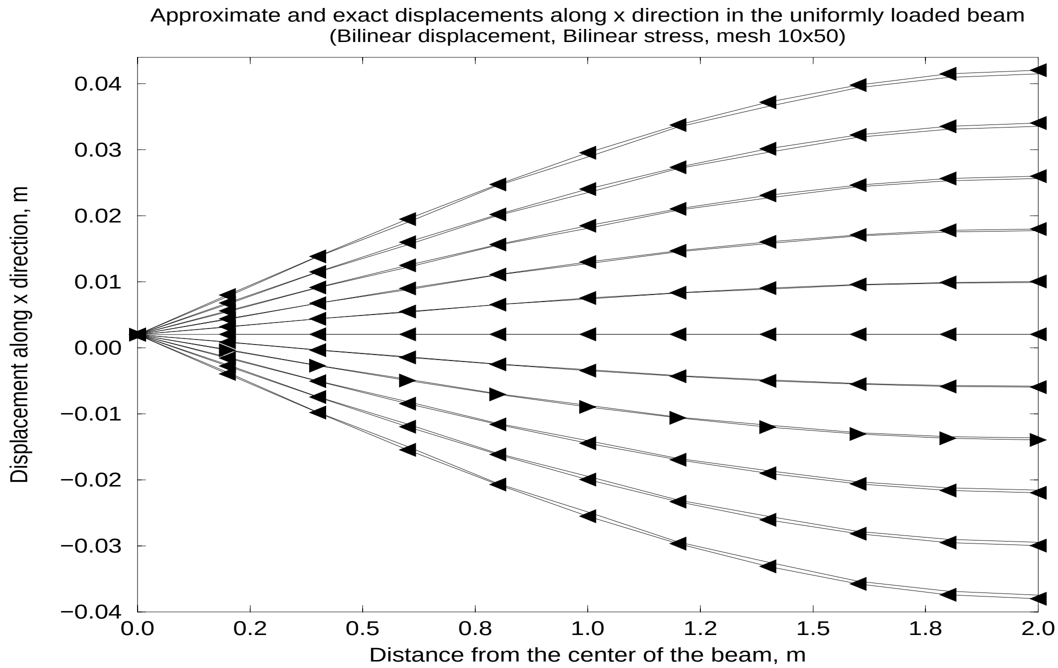


Figure 5. Exact and approximate solutions for σ_{xy} in the beam, 'constant displacement-bilinear stress' element



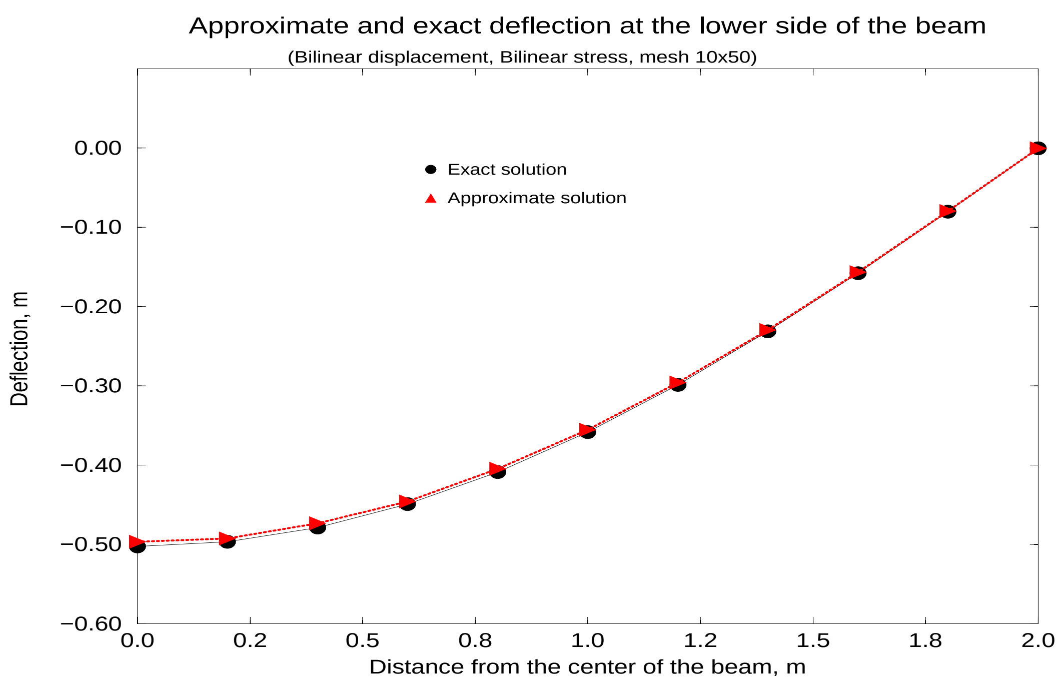


Figure 6. Displacements along x direction and deflection for the right half of the beam, loaded by uniformly distributed load, m, 'bilinear displacement-bilinear stress' element

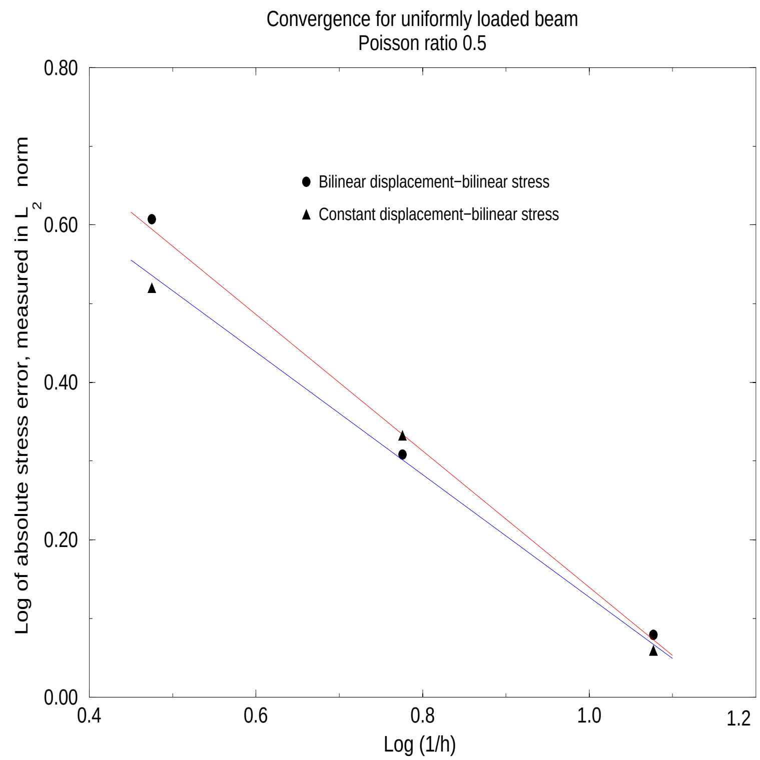


Figure 7. Stress convergence of the 'bilinear displacement-bilinear stress' element and 'constant displacement-bilinear stress' element for the beam loaded by a uniformly distributed load, Poisson ratio 0.5, plane stress and plane strain states

Figure 7 shows a comparison between the convergence rate of the two elements for Poisson's ratio equal to 0.5. It is seen that the rate of stress convergence is very close for both elements. The displacements obtained with 'bilinear displacement-bilinear stress' element were better than those obtained with 'constant displacement-bilinear stress' element. The results for stresses did not seem to be affected by the results for displacements. Figure 8 shows a comparison between the rates of convergence of stresses when the 'bilinear displacement-bilinear stress' element was used, for Poisson's ratios 0.5 and 0.2. In this problem the stresses do not depend on Poisson's ratio and both plane stress and plane strain states provided identical results. Figures 9 and 10 illustrate the displacement convergence for plane stress and plane strain respectively. It is seen that for both Poisson ratios the method exhibits a stable rate of convergence. The same stability was observed, when the 'constant displacement-bilinear stress' was used.

The initial computational and theoretical results from the new mixed least squares method seem to be very encouraging and stimulating for numerous future developments. We see the immediate continuation of the method in several major directions. An important feature of the mixed least squares method is that it allows selective continuous or discontinuous approximation of displacements over different parts of the problem domain. This makes it suitable for further extension and application to problems involving discontinuity in displacements, such as joints, interfaces and cracks. The excellent results for stresses at the inter-element contacts, as well as the capability of the method to work in the incompressible limit makes it attractive for future extension to nonlinear problems in-

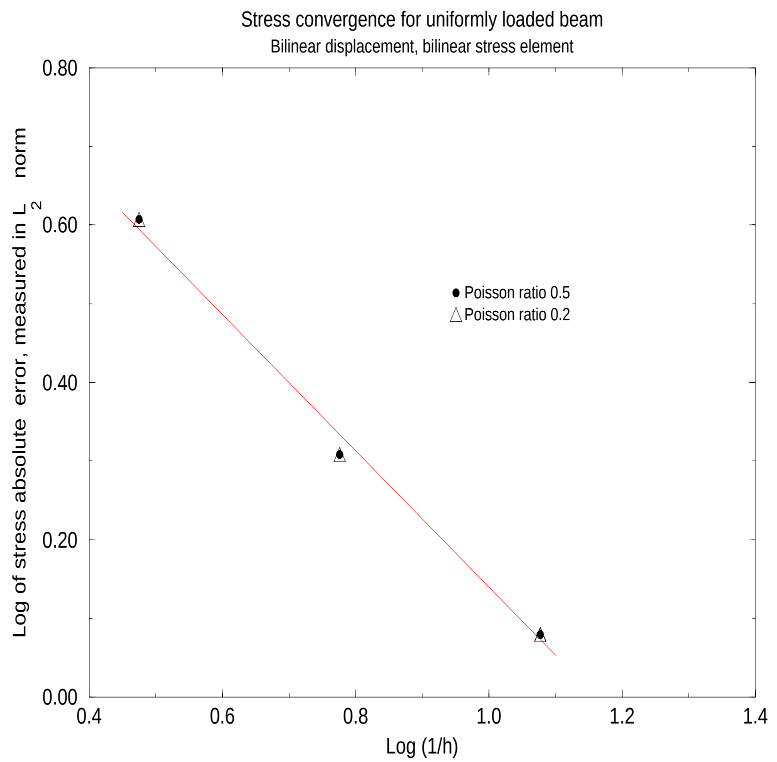


Figure 8. Stress convergence of the 'bilinear displacement-bilinear stress' element for the beam loaded by a uniformly distributed load for Poisson ratio 0.5 and 0.2, plane stress and plane strain states

volving plasticity and contact interfaces. The method also represents a theoretical interest for a strict mathematical proof of convergence. It is important to note that it allows the use of an innovative approach for numerical stabilization. The method can be stabilized by the addition of more test functions. The number and type of the test functions can be different from that of the trial stress functions. The addition of more test functions will not destroy the symmetry of the coefficient matrix and the consistency of the method will be preserved. Also, the addition of new constraints will not increase the number of equations in the resulting linear system. The mixed least squares method can be used to solve problems involving nonlinear constitutive relationships between stresses and strains. The standard linearization techniques, such as fixed point or Newton-Raphson methods can be applied. In the case of hyper-elasticity the second part of the functional remains the same, and the first part takes the form:

$$F_1\left(\boldsymbol{\sigma},\, \boldsymbol{u},\, \boldsymbol{g}\right) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \left[\int_{\Omega} (\frac{\partial W}{\partial \boldsymbol{\sigma}} \;:\; \boldsymbol{\phi}_i + \int_{\Omega} \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\phi}_i) - \int_{\partial \Omega_u} \boldsymbol{g} \cdot (\boldsymbol{\phi}_i \cdot \boldsymbol{n}) \right]^2 \tag{59}$$

For geometrically nonlinear problems, the mixed least squares formulation is still valid, but for a general problem domain, it seems that the trial functions for displacements must be continuous. This topic, as well as the issues discussed previously, represent an interesting field for future research undertakings.

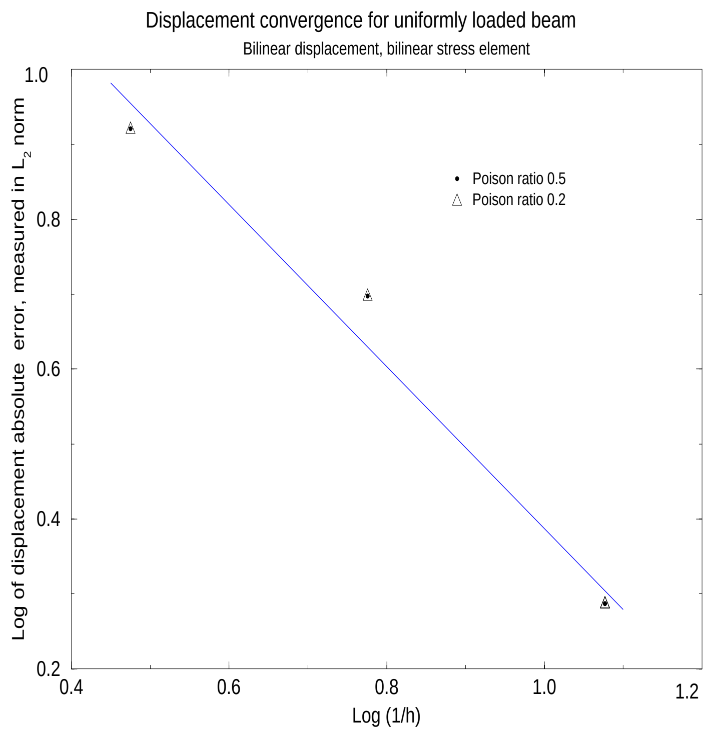


Figure 9. Displacement convergence of the 'bilinear displacement-bilinear stress' element for the beam loaded by a uniformly distributed load for Poisson ratio 0.5 and 0.2, plane stress state

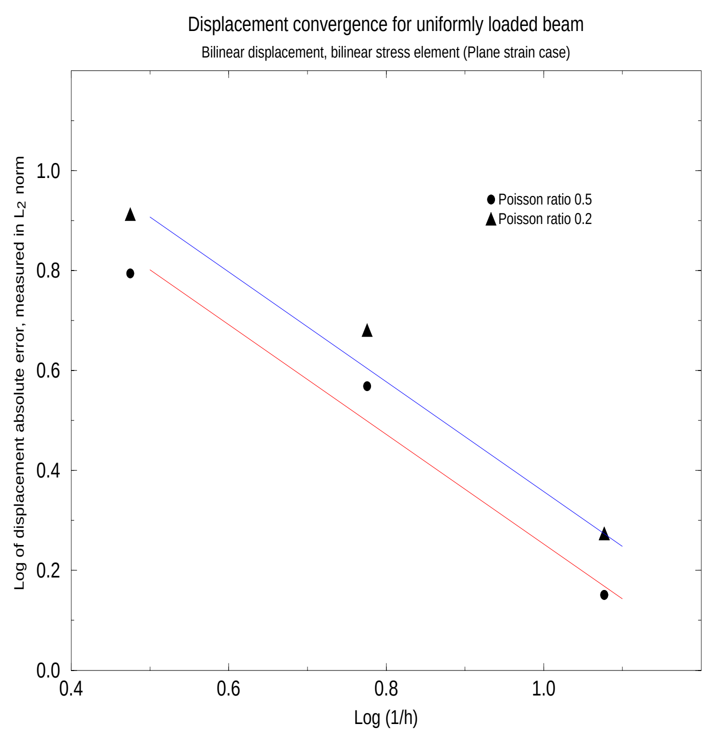


Figure 10. Displacement convergence of the 'bilinear displacement-bilinear stress' element for the beam loaded by a uniformly distributed load for Poisson ratio 0.5 and 0.2, plane strain state

Formulation	Type of Problem	Coefficient matrix	Boundary conditions
Virtual work	Minimization problem	Positive definite	Displacements-essential, Tractions-natural
Virtual Forces	Minimization problem	Positive definite Stresses must satisfy equilibrium apriori	Tractions-essential, Displacements-natural
Displacement-Mean Stress	Saddle Point Problem or minimization of virtual work under incompressibility constraint	Indefinite	Displacements-essential, Tractions-natural
Hellinger-Reissner	Saddle Point Problem or minimization of virtual forces under equilibrium constraint	Indefinite	1. Displacements-essential, Tractions-natural 2. Displacements-natural, Tractions-essential
Hu-Washizu	Saddle Point Problem or minimization of virtual work under displacement-strain constraint	Indefinite	Displacements-essential, Tractions-natural
Mixed Least Squares	Minimization Problem	Positive definite	Displacements-natural or essential, Tractions-essential

Table 1. Classification of weak formulations for elasticity

8 CONCLUSIONS

In this paper a detailed review of the classical and some newly developed formulations for solving problems in linear elasticity is presented. The derivation of the different formulations, based on the partial differential equations of elasticity, is emphasized. Table 1 represents a general classification of the different weak formulations in terms of type of mathematical problem, type of the resulting coefficient matrix and way in which the boundary conditions are imposed. It should be noted that at the continuous level all the formulations are equivalent. However, at the discrete level, their behavior is different. The classical virtual work and virtual force formulations are pure minimization problems and the resulting coefficient matrix is positive definite. The common feature of the classical mixed formulations is that they are saddle-point problems, the coefficient matrix is indefinite and the spaces used for different variables must satisfy the LBB condition in order to guarantee stability. Those methods can be stabilized at the discrete level by modifying the coefficient matrix and the right hand side of the linear system. In the past several years, different least squares types of formulations for linear elasticity have been developed. They are known as first-order least squares system [12], mixed least squares [60], etc. A common feature of these methods is that in some of cases the LBB condition can be circumvented, and others can be stabilized by the addition of more least squares terms in the functional, without destroying the symmetry of the resulting matrix and without increasing the number of the unknowns. The major goal in the least squares development is to combine the capability of the mixed formulations with obtaining a linear system with a positive definite matrix, which is easier to solve iteratively. We believe that in the future these methods will represent not only theoretical interest, but will find their appropriate area of application. The authors hope that the review presented in this paper will be useful for those who make their first steps in the field, as well as for those who might refresh their concepts in computational elasticity.

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APPENDIX

Strain Energy Density

For linear isotropic elasticity the strain energy density is

$$A = \frac{E\,\nu}{2(1+\nu)(1-2\nu)}\left(\epsilon_x+\epsilon_y+\epsilon_z\right)^2 + G\left(\epsilon_x^2+\epsilon_y^2+\epsilon_z^2\right) + \frac{G}{2}\left(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{xz}^2\right). \tag{60}$$

Complementary Energy Density

For a linear elastic isotropic material the complementary energy density is equal to

$$B \;=\; \frac{1}{2E}\left[(\sigma_x+\sigma_y+\sigma_z)^2 + 2(1+\nu)(\sigma_{yz}^2+\sigma_{zx}^2+\sigma_{xy}^2-\sigma_y\sigma_z-\sigma_z\sigma_x-\sigma_x\sigma_y\right] \tag{61}$$

Detailed Form of the Original Reissner Principle

$$\begin{aligned}
&\int_\Omega \Bigg[\left(\frac{\partial u_x}{\partial x}-\frac{\partial W}{\partial \sigma_x}\right)\delta\sigma_x + \left(\frac{\partial u_y}{\partial y}-\frac{\partial W}{\partial \sigma_y}\right)\delta\sigma_y + \left(\frac{\partial u_z}{\partial z}-\frac{\partial W}{\partial \sigma_z}\right)\delta\sigma_z \\
&\quad + \left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}-\frac{\partial W}{\partial \sigma_{xy}}\right)\delta\sigma_{xy} + \left(\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}-\frac{\partial W}{\partial \sigma_{xz}}\right)\delta\sigma_{xz} \\
&\quad + \left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}-\frac{\partial W}{\partial \sigma_{yz}}\right)\delta\sigma_{yz} - \left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right)\delta u_x \\
&\quad - \left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+f_y\right)\delta u_y \\
&\quad - \left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+f_z\right)\delta u_z\Bigg]\, d\Omega \\
&\quad + \int_{\partial\Omega_u}\left(t_x\,\delta u_x + t_y\,\delta u_y + t_z\,\delta u_z\right) d\partial\Omega \\
&\quad + \int_{\partial\Omega_t}\left[(t_x-\tilde{t}_x)\delta u_x + (t_y-\tilde{t}_y)\delta u_y + (t_z-\tilde{t}_z)\delta u_z\right] d\partial\Omega \;=\; 0
\end{aligned} \tag{62}$$

where $W(\sigma_x,\sigma_y,\sigma_z,\sigma_{xy},\sigma_{yz},\sigma_{xz})$ is a given function such that $\boldsymbol{\epsilon} = \dfrac{\partial W}{\partial \boldsymbol{\sigma}}$.

First Form of Hellinger-Reissner Principle

The first form of the principle is formulated as

$$\begin{aligned}
\Pi_{HR} \;=\; &\int_\Omega \Bigg[B(\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz}) + \left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+f_x\right) v_x \\
&+ \left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+f_y\right) v_y \\
&+ \left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+f_z\right) v_z\Bigg] d\Omega \\
&+ \int_{\partial\Omega_t}\left[(t_x-\tilde{t}_x)\,v_x + (t_y-\tilde{t}_y)\,v_y + (t_z-\tilde{t}_z)\,v_z\right] d\,\partial\Omega \\
&+ \int_{\partial\Omega_u}(t_x\,g_x + t_y\,g_y + t_z\,g_z)\, d\,\partial\Omega
\end{aligned} \tag{63}$$

where $B(\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz})$ is the complementary energy density.

Second Form of Hellinger-Reissner Principle

The second form of the Hellinger-Reissner formulation discussed by Washizu [68] was

$$\begin{aligned}\Pi_{HR} \;=\; & \int_\Omega \Big[\sigma_x \frac{\partial u_x}{\partial x} + \sigma_y \frac{\partial u_y}{\partial y} + \sigma_z \frac{\partial u_z}{\partial z} + \tau_{xy}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \\ & + \tau_{xz}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) + \tau_{yz}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ & - B(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}) + (f_x\, u_x + f_y\, u_y + f_z\, u_z)\Big]\, d\Omega \\ & - \int_{\partial\Omega_t} (\tilde{t}_x\, u_x + \tilde{t}_y\, u_y + \tilde{t}_z\, u_z) d\partial\Omega \\ & - \int_{\partial\Omega_u} [(u_x - g_x)t_x + (u_y - g_y)t_y + (u_z - g_z)t_z]\, d\partial\Omega\end{aligned} \tag{64}$$

Derivations related to Herrmann's Principle

Following the notation of Herrmann's principle, we can derive:

$$\nu\, H = \frac{\nu}{1+\nu}\,\frac{1}{2\mu}\,3\sigma_m \;=\; \frac{\lambda}{3\lambda + 2\mu}\frac{1}{2\mu}\,3\sigma_m = \frac{1}{3 + \dfrac{2\mu}{\lambda}}\,\frac{1}{2\mu}\,3\sigma_m \tag{65}$$

For the nearly incompressible and incompressible cases

$$\lim_{\lambda\to\infty} (\nu\, H) = \lim_{\lambda\to\infty} \left[\frac{1}{3 + \dfrac{2\mu}{\lambda}}\,\frac{3}{2\mu}\,\sigma_m\right] = \frac{1}{2\mu}\,\sigma_m = \frac{1}{2\mu} p.$$

The substitution of p by $(2\mu\,\nu\, H)$ in equation (29) leads to

$$\begin{aligned}F(\boldsymbol{u}, p) \;=\; & \int_\Omega \left\{\mu\boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) \;-2\mu\nu H\;(\nabla\cdot\boldsymbol{u})\; - \frac{2\mu^2\,\nu^2\,H^2}{\lambda} - \boldsymbol{f}\cdot\boldsymbol{u}\right\}\, d\Omega \\ & - \int_{\partial\Omega_t} (\boldsymbol{\sigma}\cdot\boldsymbol{n})\cdot\boldsymbol{u}\, d\Omega\end{aligned} \tag{66}$$

When expanded, equation (66) becomes Herrmann's principle expressed by (28) for zero thermal expansion ($e_T = 0$).

Hu-Washizu Principle

Hu-Washizu principle states that the solution to the linear elasticity problem can be determined from the conditions for stationarity of the following functional

$$\begin{aligned}\Pi_{HW} \;=\; & -\int_\Omega A\,(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}) + \int_\Omega (f_x\, u_x + f_y\, u_y + f_z\, u_z)\; d\,\Omega \\ & + \int_\Omega \Big[\left(\epsilon_x - \frac{\partial u_x}{\partial x}\right)\sigma_x + \left(\epsilon_y - \frac{\partial u_y}{\partial y}\right)\sigma_y + \left(\epsilon_z - \frac{\partial u_z}{\partial z}\right)\sigma_z \\ & + \left(\gamma_{xy} - \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\tau_{xy} + \left(\gamma_{yz} - \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right)\tau_{yz} \\ & + \left(\gamma_{xz} - \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\tau_{xz}\Big]\; d\,\Omega + \int_{\partial\Omega_t} \left(\tilde{t}_x\, u_x + \tilde{t}_y\, u_y + \tilde{t}_z\, u_z\right)\; d\,\partial\Omega \\ & + \int_{\partial\Omega_u} [(u_x - g_x)t_x + (u_y - g_y)t_y + (u_z - g_z)t_z]\; d\,\partial\Omega\end{aligned} \tag{67}$$

where A is the strain energy density expressed in terms of strain components and $\tilde{t}_x, \tilde{t}_y, \tilde{t}_z$ are the prescribed surface tractions.

Abstract Form of Saddle Point Problems (Franca and Hughes)

Franca [20] and Franca and Hughes [21] proposed the following modification of the general abstract form of the saddle point problems:

Given $\boldsymbol{f} \in V'$ and $\boldsymbol{g} \in W'$, find $\left(\boldsymbol{u}^h, \boldsymbol{p}^h\right) \in V_h \times W_h$ such that

$$\begin{aligned} a_h^*(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) &= f_h(\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in V_h \\ b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) + c_h(\boldsymbol{p}_h, \boldsymbol{q}_h) &= g_h(\boldsymbol{q}_h), \quad \boldsymbol{q}_h \in W_h \end{aligned}$$

where

$$\begin{aligned} a_h^*(\boldsymbol{u}_h, \boldsymbol{v}_h) &= a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \delta_1 h^{2r_1}(A\boldsymbol{u}_h, A\boldsymbol{v}_h)_h - \delta_2 h^{2r_2}(B\boldsymbol{u}_h, B\boldsymbol{v}_h)_h \\ b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) &= b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) + \delta_1 h^{2r_1}(B^*\boldsymbol{p}_h, A\boldsymbol{v}_h)_h \\ c_h(\boldsymbol{p}_h, \boldsymbol{q}_h) &= \delta_1 h^{2r_1}(B^*\boldsymbol{p}_h, B^*\boldsymbol{q}_h)_h \\ f_h(\boldsymbol{v}_h) &= (\boldsymbol{f}, \boldsymbol{v}_h) + \delta_1 h^{2r_1}(\boldsymbol{f}, A\boldsymbol{v}_h)_h - \delta_2 h^{2r_2}(\boldsymbol{g}, B\boldsymbol{v}_h)_h \\ g_h(\boldsymbol{q}_h) &= (\boldsymbol{g}, \boldsymbol{q}_h) + \delta_1 h^{2r_1}(\boldsymbol{f}, B^*\boldsymbol{q}_h)_h \end{aligned} \tag{68}$$

In (68) δ_1 and δ_2 are non-negative stability constants and r_1 and r_2 are non-negative exponents. For the mixed Hellinger-Reissner formulation the following notation is used:

$$\begin{aligned} a(\boldsymbol{u}_h, \, \boldsymbol{v}_h) &= -(\boldsymbol{C}^{-\mathbf{1}}\, \boldsymbol{\sigma}_h, \, \boldsymbol{\tau}_h) \\ b(\boldsymbol{p}_h, \, \boldsymbol{v}_h) &= (\boldsymbol{\epsilon}(\boldsymbol{v}_h), \, \boldsymbol{\sigma}_h) \\ f(\boldsymbol{v}_h) &= \mathbf{0} \\ A\, \boldsymbol{u}_h &= -\boldsymbol{C}^{-1}(\boldsymbol{\sigma}_h) \\ B\, \boldsymbol{u}_h &= -div\, \boldsymbol{\sigma}_h \\ B^*\, \boldsymbol{p}_h &= \boldsymbol{\epsilon}(\boldsymbol{u}_h) \\ \boldsymbol{g} &= \boldsymbol{f} \end{aligned} \tag{69}$$