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A Modified Navier-Stokes Equation, and its Consequences on Sound Dispersion (*).

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Summary. — Taking into account some considerations developed in the past by Cattaneo we suggest a modified Navier-Stokes equation which includes the inertial property of the momentum flux. The linearized system of hydrodynamic equations where the inertial properties of both momentum and heat flux are taken into account leads to a new dispersion equation which does not exhibit the theoretical disadvantages of the Kirchhoff equation. Also the consistency with the experimental results about sound dispersion in rarefied monoatomic gases is definitely improved.

1. – Introduction.

Several years ago CATTANEO (1) suggested a modification of the heat diffusion equation in order to overcome the well-known paradox, due to the parabolic character of the equation, according to which heat diffuses with infinite velocity. Ten years later, VERNOTTE (2), re-examining the difficulties related to the classical theory of heat diffusion, proposed a similar change of the heat diffusion equation. Apparently, the subject was not pursued in the following years although LUIKOV (3), in his recent textbook on heat diffusion theory, quotes the conclusions of CATTANEO and VERNOTTE as an established result. It is worth-while to recall the main aspects of the previously

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(1) C. CATTANEO: *Atti Sem. Mat. Fis. Univ. Modena*, **3**, 83 (1948).

(2) P. VERNOTTE: *Compt. Rend.*, **246**, 3154 (1958). After this work the same authors published the following papers: C. CATTANEO: *Compt. Rend.*, **247**, 431 (1958); P. VERNOTTE: *Compt. Rend.*, **247**, 2103 (1958).

(3) A. V. LUIKOV: *Analytical Heat Diffusion Theory* (New York, 1968).

mentioned equation. If we consider a homogeneous isotropic medium in which thermal expansion can be disregarded, the first law of the thermodynamics can be written

$$\varrho C_V \frac{\partial T}{\partial t} = -\operatorname{div} \boldsymbol{J}_q , \tag{1.1}$$

where ϱ is the (constant, uniform) density, C_V is the specific heat at constant volume and $\boldsymbol{J}_q$ is the heat flow per unit time across the unit surface. Coupling the equation (1.1) with Fourier's phenomenological law of heat conduction

$$\boldsymbol{J}_q = -k \operatorname{grad} T , \tag{1.2}$$

the well-known heat diffusion equation is obtained if the thermal conductivity k is temperature independent

$$\frac{\partial T}{\partial t} = \chi_V \Delta T . \tag{1.3}$$

Here $\chi_V = k/\varrho C_V$ is the so-called thermometric conductivity and Δ is the usual Laplace operator. Nevertheless, the coupling of the two equations (1.2) and (1.3) does not seem completely legitimate because Fourier's law is experimentally established only in stationary conditions. It appears reasonable to modify eq. (1.2) by adding a term which describes, at least in a transitory phase, the inertial property of the thermal flow with respect to the origin of the flow itself. The proposed change is

$$\boldsymbol{J}_q = -k \operatorname{grad} T - \tau_q \frac{\partial \boldsymbol{J}_q}{\partial t} , \tag{1.2'}$$

where τ_q is a constant with the dimension of time. We note that the law (1.2') is justified by CATTANEO on the basis of a revision of the transport equation in the kinetic theory of gases; in this way one is able to evaluate the order of magnitude of τ_q in terms of microscopic parameters. The order of magnitude of τ_q for gases is $\Lambda/\bar{v} \simeq 10^{-9}$ s, where Λ and $\bar{v}$ are the mean free path and the mean velocity, respectively. Coupling (1.2') with (1.1) we obtain the following equation instead of (1.3):

$$\frac{\partial T}{\partial t} = \chi_V \Delta T - \tau_q \frac{\partial^2 T}{\partial t^2} . \tag{1.4}$$

Equation (1.4) is of the hyperbolic type known as « the telegraph equation ». So, if (1.4) is the governing equation of heat diffusion, the thermal energy automatically assumes a finite propagation velocity without losing the essential feature of a diffusive phenomenon (*). In this paper we suggest a modifi-

(*) See C. CATTANEO: ref. (1).

cation of the hydrodynamic equations of viscous fluids following the same line of thought as that which leads to eq. (1.4). As is well known, the equation of motion of a fluid particle, in the absence of external forces, can be written (*)

$$\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = -\partial^\alpha P - \partial_\beta \pi^{\alpha\beta}, \tag{1.5}$$

where we have split the total pressure tensor into a scalar hydrostatic part P and a viscous pressure tensor $\pi^{\alpha\beta}$. We shall assume here that the pressure tensor is symmetric. Equation (1.5) must be coupled with the continuity equation, the first law of thermodynamics and two other functions of state if the internal energy u and temperature T, as well as ϱ and P, are unknown functions. However, the system of nonlinear equations becomes a definite one only if the phenomenological equations governing heat flow (Fourier law) and momentum flow are given.

The latter one is given by a generalized Newton law of friction

$$\pi^{\alpha\beta} = -2\mu U^{\alpha\beta} - \lambda(\partial^\gamma v_\gamma) g^{\alpha\beta}, \tag{1.6}$$

where $U^{\alpha\beta} = \frac{1}{2}(\partial^\alpha v^\beta + \partial^\beta v^\alpha)$ is the rate of strain tensor. From eq. (1.5), using (1.6), the well-known Navier-Stokes equation follows. We shall confine ourselves to an analysis of the linearized system of hydrodynamic equations, using the same restriction as that applied to the heat diffusion equation. Such a restriction leads to the study of sound propagation in a viscous fluid where the state parameters, as well as the velocity, are considered small perturbations with respect to the equilibrium values. Therefore, referring to such a problem, the analysis of the system of the hydrodynamic equations leads to the well-known Kirchhoff equation which links the absorption length and the wave number with the frequency ω of the different elementary waves, whose superposition describes the acoustical perturbation. Nevertheless, the governing equations of the dynamics of a viscous fluid do not allow the definition of a wave front as a characteristic surface ([4]) of the system of differential equations. This feature of the whole system is also found with the linearized system, against all experimental evidence. In several respects this difficulty resembles the one met in the heat equation, although here we should talk about « impossibility of wave propagation » ([5]) rather than propagation at infinite velocity.

(*) Here the covariant notation for spatial co-ordinates x^1, x^2, x^3 is used for the sake of convenience. The metric being Euclidean $g^{\alpha\beta} = g_{\alpha\beta} = \delta_{\alpha\beta}$, $v^\alpha = v_\alpha$ and $\partial^\alpha = \partial_\alpha = \partial/\partial x^\alpha$. Later on we shall denote the local temporal derivative by ∂_t and the substantial one by $\mathrm{d}/\mathrm{d}t = \partial_t + v^\alpha \partial/\partial x^\alpha$. The index notation with the usual summation convention from 1 to 3 is used.

([4]) T. LEVI-CIVITA: *Caratteristiche dei sistemi differenziali e propagazione ondosa* (Bologna, 1931).

([5]) G. LAMPARIELLO: *Rend. Accad. Lincei*, **13**, 688 (1931).

The afore-mentioned anomalous result may be related to the behaviour of the phase velocity of elementary waves which describe the acoustical perturbation. In fact, if the phase velocity tends to a finite limiting value $v_{p\infty}$ for very high values of the frequency, then, as shown in Appendix A, there is wave propagation and the characteristic surface moves with velocity $v_{p\infty}$. Thus we can state that from the Navier-Stokes equation it follows that the phase velocity is not finite but increases towards infinity when the frequency ω increases. In this work we will show that, substituting the phenomenological law (1.6) with the following one:

$$\pi^{\alpha\beta} = -2\mu U^{\alpha\beta} - \lambda(\partial^\gamma v_\gamma) g^{\alpha\beta} - \tau_V \frac{\partial \pi^{\alpha\beta}}{\partial t}\,, \tag{1.6'}$$

there do not exist any harmonic components of the solution, however high the frequency may be, for which the phase velocity is unlimited; this implies the existence of a wave front in all cases. It is clear that the corrective term in eq. (1.6') can be justified on the basis of the kinetic theory of gases by means of considerations analogous to the ones of CATTANEO: in fact, they refer to general transport phenomena, and therefore they apply to momentum transfer as well as to energy transfer. In this work we will study essentially the consequences on sound propagation resulting from eqs. (1.6'), (1.2'). For this purpose, in Sect. **2** we will review the standard theory of sound dispersion showing the difficulties mentioned above. In Sect. **3** we will derive the new dispersion relation by introducing eqs. (1.2') and (1.6') in the linearized system and will obtain the limited value of phase velocity. In Sect. **4** we will consider the case of rarefied monoatomic gases and the results obtained will be compared with experimental data on sound dispersion. It will be shown that there is good agreement even in the spectral region where the Kirchhoff formula fails.

2. – The theory of acustical waves and the Kirchhoff dispersion equation.

The starting point of the standard theory of sound is the hydrodynamic system of equations which we report here:

$$\partial_t \varrho + \partial_\beta(\varrho v^\beta) = 0\,, \tag{2.1}$$

$$\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = -\partial_\beta \pi^{\alpha\beta} - \partial^\alpha P\,, \qquad (\alpha, \beta = 1, 2, 3)\,, \tag{2.2}$$

$$\varrho \frac{\mathrm{d}u}{\mathrm{d}t} = -P\partial_\beta v^\beta - \pi^{\alpha\beta} U_{\alpha\beta} - \partial_\beta J_q^\beta\,, \tag{2.3}$$

$$\varrho = \varrho(T, P), \qquad u = u(T, P)\,. \tag{2.4}$$

Equations (2.1) and (2.2) are the law of mass conservation and the momentum balance equation for a fluid when external forces are absent and $\pi^{\alpha\beta} = P^{\alpha\beta} - Pg^{\alpha\beta}$

is the viscous pressure tensor of the fluid. Equation (2.3) follows from the conservation of the total energy and represents the first law of thermodynamics where u is the specific internal energy; the last two terms on the right-hand side of (2.3) are the heat amount absorbed by the fluid particle on account of the energy lost by viscous friction from the surrounding fluid and the heat amount absorbed on account of temperature gradients, respectively. Equation (2.4) contains functions of state of the fluid which must be considered functions explicitly known by means of the thermal properties of the fluid itself, *i.e.* specific heat at constant pressure and at constant volume, isothermal and adiabatic compressibility and the coefficient of thermal expansion. The system of the seven equations (2.1)-(2.4) with the seven unknown function v^α, ϱ, u, P, T can be considered a defined one if the phenomenological laws are specified:

$$\begin{cases} J_q^\alpha = -k\partial^\alpha T\,, \\ \pi^{\alpha\beta} = -2\mu U^{\alpha\beta} - (\eta - \tfrac{2}{3}\mu)(\partial_\gamma v^\gamma) g^{\alpha\beta}\,, \end{cases} \tag{2.5}$$

where k is the thermal conductivity of the fluid, μ the coefficient of viscosity and η the coefficient of bulk viscosity (*). Furthermore, if we choose P and T as independent parameters, we can insert the specific enthalpy (**) $h = u + P/\varrho$ in eq. (2.3) using the relation

$$\varrho \frac{\mathrm{d}u}{\mathrm{d}t} = \varrho \frac{\mathrm{d}h}{\mathrm{d}t} - \frac{\mathrm{d}P}{\mathrm{d}t} - P(\partial_\gamma v^\gamma)\,. \tag{2.6}$$

By introducing (2.5) in the system of equations (2.1)-(2.4), assuming k, μ and η to be constant and using (2.6) we obtain

$$\partial_t \varrho + \partial_\beta(\varrho v^\beta) = 0\,, \tag{2.1'}$$

$$\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = -\partial^\alpha P + \mu \Delta v^\alpha + \left(\eta + \frac{1}{3}\mu\right) \partial^\alpha(\partial_\gamma v^\gamma)\,, \tag{2.2'}$$

$$\varrho \frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}P}{\mathrm{d}t} + \pi + k\Delta T\,, \tag{2.3'}$$

$$\varrho = \varrho(P, T)\,, \qquad h = h(P, T)\,. \tag{2.4'}$$

(*) The coefficient η is linked to the so-called second coefficient of viscosity λ, used in eq. (1.6), by the relation $\eta = \lambda + \frac{2}{3}\mu$. We prefer using the coefficient of bulk viscosity because it is usually negligible in gases.

(**) Some authors ([6]), in connection with the thermodynamics of irreversible processes, prefer to introduce the specific entropy s; using this state function, the energy equation becomes $\varrho(\mathrm{d}s/\mathrm{d}t) = -\partial_\beta(J_q^\beta/T) - (1/T^2) J_q^\beta \partial_\beta T - (1/T)\pi^{\alpha\beta} U_{\alpha\beta}$. In this way one makes clear the entropy production and the irreversibility of the process, but we are not interested in the following to emphasize these well-known features of the process.

([6]) See, *e.g.*, S. R. DE GROOT and P. MAZUR: *Nonequilibrium Thermodynamics* (Amsterdam, 1962).

Equation (2.2′) is the well-known Navier-Stokes equation, whereas eq. (2.3′) restates the first law of thermodynamics, and the scalar invariant quantity π turns out to be everywhere positive as it must be since it has the meaning of heat produced by friction (per unit of time and volume). Explicitly we have

$$\pi = -\pi^{\alpha\beta} U_{\alpha\beta} = (\eta + \tfrac{4}{3}\mu)(\partial_\gamma v^\gamma)^2 + \\ + 2\mu[(\partial_1 v_2 + \partial_2 v_1)^2 + (\partial_2 v_3 + \partial_3 v_2)^2 + (\partial_1 v_3 + \partial_3 v_1)^2] . \tag{2.7}$$

In the theory of sound one considers the fluid motion corresponding to a small disturbance from the equilibrium state of the fluid where all thermodynamic quantities are constant in space and time. Hence it is appropriate to linearize the hydrodynamic equations by replacing each of these quantities by their equilibrium values plus a small fluctuation, ignoring terms which are of second order (in the fluctuations). So, if P_0 and T_0 are the equilibrium values, we put

$$\begin{cases} P(\boldsymbol{r}, t) = P_0 + P'(\boldsymbol{r}, t) , \\ T(\boldsymbol{r}, t) = T_0 + T'(\boldsymbol{r}, t) . \end{cases} \tag{2.8}$$

Furthermore we assume, for simplicity, that the equilibrium flow velocity is zero, so that

$$\boldsymbol{v}(\boldsymbol{r}, t) = \boldsymbol{v}'(\boldsymbol{r}, t) . \tag{2.9}$$

In the following part, simplifying the notations, we will denote the perturbation or the primed quantities P', T' and $\boldsymbol{v}'$ by p, τ and $\boldsymbol{v}$, respectively, according to the notation of MORSE and INGARD ([7]). Using (2.8) we obtain

$$\begin{cases} \varrho = \varrho_0 + \delta = \varrho_0 + \left(\dfrac{\partial \varrho}{\partial T}\right)_P \tau + \left(\dfrac{\partial \varrho}{\partial P}\right)_T p , \\ h = h_0 + x = h_0 + \left(\dfrac{\partial h}{\partial T}\right)_P \tau + \left(\dfrac{\partial h}{\partial P}\right)_T p , \end{cases} \tag{2.10}$$

where the derivatives, computed for the equilibrium values P_0 and T_0, are constant with respect to p and τ. Recalling the definitions of isothermal compressibility and coefficient of thermal expansion

$$K_T = \frac{1}{\varrho}\left(\frac{\partial \varrho}{\partial P}\right)_T , \qquad \beta = -\frac{1}{\varrho}\left(\frac{\partial \varrho}{\partial T}\right)_P ,$$

([7]) P. M. MORSE and K. INGARD: *Theoretical Acoustics* (New York, 1968), p. 278.

and using the well-known thermodynamic relations

$$\left(\frac{\partial h}{\partial P}\right)_T = \frac{1}{\varrho} + \frac{T}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P, \qquad C_p - C_V = \frac{T\beta^2}{\varrho K_T},$$

we can write (2.10) in terms of the thermal coefficients characterizing the fluid. Finally, substituting the specific enthalpy h in the energy equation and taking into account that $\mathrm{d}/\mathrm{d}t \to \partial_t$, according to our hypotheses, one obtains the linearized system

$$\partial_t \delta + \varrho_0(\partial_\beta v^\beta) = 0\,, \tag{2.11}$$

$$\varrho_0 \partial_t v^\alpha = -\partial^\alpha p + \mu \Delta v^\alpha + \left(\eta + \frac{1}{3}\mu\right)\partial^\alpha(\partial_\beta v^\beta)\,, \tag{2.12}$$

$$\varrho_0 C_p \left(\partial_t \tau - \frac{\gamma - 1}{\alpha\gamma}\,\partial_t p\right) = k\Delta\tau\,, \tag{2.13}$$

$$\delta = \frac{\gamma}{c^2}(p - \alpha\tau)\,, \tag{2.14}$$

where (2.13) is again the first law of thermodynamics and (2.14) the equation of state of the fluid with $\gamma = C_p/C_V$, $\alpha = \beta/K_T$ and

$$c^2 = \left(\frac{\partial p}{\partial \varrho}\right)_s = \frac{1}{\varrho_0 K_s}$$

is the well-known Laplace sound velocity in an ideal fluid (*i.e.* without viscosity and heat conduction) in terms of adiabatic compressibility K_s. We note that any vector function of position, such as $\boldsymbol{v}$, can always be separated into a longitudinal (or lamellar) part $\boldsymbol{v}_\parallel$, for which $\operatorname{curl} \boldsymbol{v}_\parallel = 0$, and a transverse (or rotational) part $\boldsymbol{v}_\perp$, for which $\operatorname{div} \boldsymbol{v}_\perp = 0$. Therefore, if we use the identity

$$\operatorname{curl}\operatorname{curl} \boldsymbol{v} = \operatorname{grad}\operatorname{div} \boldsymbol{v} - \Delta \boldsymbol{v}\,,$$

the equation of motion can be split into two separate equations, one relating p to the longitudinal part of $\boldsymbol{v}$, the other giving the behaviour of the transverse part of $\boldsymbol{v}$, unrelated to pressure waves:

$$\varrho_0 \partial_t v_\parallel^\alpha = -\partial^\alpha p + (\eta + \tfrac{4}{3}\mu)\Delta v_\parallel^\alpha\,, \tag{2.12'}$$

$$\varrho_0 \partial_t v_\perp^\alpha = -\mu \Delta v_\perp^\alpha\,, \tag{2.13'}$$

thus the two parts of the velocity solution, $\boldsymbol{v}_\parallel$ and $\boldsymbol{v}_\perp$, can be solved separately and need not be combined until we come to satisfy the boundary conditions.

In the following considerations, since we are not interested in boundary conditions, we can limit ourselves, in eqs. (2.12) and (2.13), to consider only the longitudinal part of the velocity. Taking into account also the assumed isotropy of the medium, we will consider the one-dimensional case thus avoiding cumbersome formulae superfluous to the study of the general properties of the solution of the system in an unlimited medium. Eliminating $\delta(x, t)$ from the system (2.11)-(2.14) we obtain

$$\partial_{xx} p = \frac{\gamma}{c^2} [\partial_{tt}(p - \alpha\tau) - l_v c \partial_{xx} \partial_t (p - \alpha\tau)] , \tag{2.15}$$

$$\partial_{xx} \tau = \frac{1}{c l_q} \partial_t \left[\tau - \frac{\gamma - 1}{\alpha\gamma} p \right] , \tag{2.16}$$

$$\varrho_0 \partial_t v = - \partial_x \left[p + \frac{\gamma l_v}{c} \partial_t (p - \alpha\tau) \right] , \tag{2.17}$$

where l_v and l_q are two characteristic lengths defined by

$$l_v = \frac{\eta + \frac{4}{3}\mu}{\varrho_0 c} , \qquad l_q = \frac{k}{\varrho_0 c C_p} . \tag{2.18}$$

The system (2.15)-(2.17) consists of three linear equations with the unknowns p, τ and v. If we examine the first two, which contain only p and τ, we can see that, in the absence of viscosity and thermal conductivity ($l_v = l_q = 0$), the pressure fluctuation satisfies the well-known wave equation in an ideal fluid:

$$\partial_{xx} p - \frac{1}{c^2} \partial_{tt} p = 0 .$$

When viscosity and thermal conductivity are not negligible, as in real fluids, we still obtain equations which satisfy p and τ. One usually studies the properties of the general solution in an unlimited medium assuming that it can be expressed by the Fourier integral

$$p(x, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{p}(k, \omega) \exp [i(kx - \omega t)] \mathrm{d}k \, \mathrm{d}\omega , \tag{2.19}$$

where the amplitudes of the partial waves are given by the Fourier transform

$$\hat{p}(k, \omega) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, t) \exp [- i(kx - \omega t)] \mathrm{d}x \, \mathrm{d}t . \tag{2.20}$$

The definition of the Fourier transform (2.20) can be extended to complex values of k or, alternatively, ω provided that the imaginary parts of these quantities satisfy suitable conditions that ensure the convergence of the integral (2.20). For simplicity and for a direct reference to the experiment described in Sect. **4**, we will assume a forced perturbation so that we can consider ω real. In this way eq. (2.19) describes the solution by means of superposition of damped partial waves with amplitude $\hat{p}(k, \omega) \exp[-\mathrm{Im}\{k\}x]$, direction of propagation along the x-axis and phase velocity $v_p = \omega/\mathrm{Re}\{k\}$. The properties of such a solution can be derived directly by the analysis of the partial waves because of the linearity of eqs. (2.16) and (2.17). Therefore writing

$$\begin{cases} p(x, t) = \hat{p}(k, \omega) \exp[i(kx - \omega t)], \\ \tau(x, t) = \hat{\tau}(k, \omega) \exp[i(kx - \omega t)], \end{cases} \tag{2.21}$$

and substituting in (2.15) and (2.16), we obtain

$$\begin{cases} k^2 \hat{p} = \dfrac{\gamma}{c^2}[\omega^2(\hat{p} - \alpha\hat{\tau}) + i\omega l_\tau c k^2(\hat{p} - \alpha\hat{\tau})], \\ k^2 \hat{\tau} = \dfrac{i\omega}{c l_q}\left[\hat{\tau} - \dfrac{\gamma - 1}{\alpha\gamma}\hat{p}\right]. \end{cases} \tag{2.22}$$

The two algebraic equations, in the Fourier transforms $\hat{p}$ and $\hat{\tau}$, allow a non-vanishing solution only if the coefficient determinant vanishes. This gives a complex relationship between ω and k of the form $f(k, \omega) = 0$, which is equivalent to two real relations and gives the real and imaginary parts of k as a function of ω. The dispersion relation which one obtains from (2.22) is also called the Kirchhoff equation:

$$l_q\left[\gamma l_\tau + ic\frac{1}{\omega}\right]k^4 + \left[1 - i\frac{\omega}{c}(l_\tau + \gamma l_q)\right]k^2 - \frac{\omega^2}{c^2} = 0\,. \tag{2.23}$$

This equation can be solved exactly and the real and imaginary parts of k can be plotted in terms of ω. However, what we want to emphasize here is the following result: for high values of the frequency ω the real part of k^2 becomes negligible with respect to the imaginary part which increases with increasing frequency ω. In fact, from (2.23) we obtain the following behaviour of k^2 at high frequencies:

$$k^2 = \left[a(l_q, l_\tau) + O\left(\frac{1}{\omega^2}\right)\right] + i\left[b(l_q, l_\tau)\frac{\omega}{c} + O\left(\frac{1}{\omega}\right)\right], \tag{2.24}$$

where a and b are constants connected to the characteristic lengths by the equations

$$(2.25)\qquad \begin{cases} a = \left[\dfrac{(1-\gamma)l_\nu + \gamma l_q \pm (l_\nu - \gamma l_q)}{\gamma l_\nu} \pm \dfrac{2l_q - \gamma l_q - l_\nu}{l_\nu - \gamma l_q}\right] \cdot \dfrac{1}{2\gamma l_\nu l_q}, \\ b = [l_\nu + \gamma l_q \pm (l_\nu - \gamma l_q)] \cdot \dfrac{1}{2\gamma l_\nu l_q}. \end{cases}$$

From (2.24) we see that the imaginary part prevails over the real part independently of the choice of the sign in (2.25); in the limit of high frequencies we obtain

$$(2.26)\qquad \mathrm{Re}\{k\} = \mathrm{Im}\{k\} \simeq \frac{1}{\sqrt{2}} \left[\frac{l_\nu + \gamma l_q \pm (l_\nu - \gamma l_q)}{2\gamma l_\nu l_q}\right]^{\frac{1}{2}} \cdot \omega^{\frac{1}{2}},$$

Consequently the phase velocity of the corresponding elementary wave is

$$(2.27)\qquad v_p = \frac{\omega}{\mathrm{Re}\{k\}} \simeq 2 \left[\frac{\gamma l_\nu l_q}{l_\nu + \gamma l_q \pm (l_\nu - \gamma l_q)}\right]^{\frac{1}{2}} \cdot \omega^{\frac{1}{2}},$$

and therefore increases with the frequency. This difficulty can be removed by introducing the terms added in the phenomenological equations (1.2′) and (1.6′). Finally we note that the sign can generally be chosen in order to make the real and imaginary parts positive; however from the asymptotic expression of k^2 (2.24) we can deduce only that the phase velocity in both cases increases according to $\omega^{\frac{1}{2}}$.

3. – The modified Navier-Stokes equation and the new dispersion relation.

Let us now look for linearized hydrodynamic equations assuming that the phenomenological laws which give the heat flux and the viscous pressure tensor are not those of FOURIER and NEWTON (2.5), but the following ones:

$$(3.1)\qquad J_q^\alpha = -k\partial^\alpha T - \tau_q \partial_t J_q^\alpha,$$

$$(3.2)\qquad \pi^{\alpha\beta} = -2\mu U^{\alpha\beta} - (\eta - \tfrac{2}{3}\mu)(\partial_\gamma v^\gamma) g^{\alpha\beta} - \tau_\nu \partial_t \pi_{\alpha\beta}.$$

We note, first of all, that there is no difficulty in introducing (3.2) into the equation of motion (2.2): we merely need to differentiate eq. (2.2) with respect to time and then we get the time derivative of $\pi^{\alpha\beta}$. Taking into account the continuity equation we can write

$$\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = \partial_t(\varrho v^\alpha) + \partial_\beta(\varrho v^\alpha v^\beta),$$

and so we get, directly from (2.2),

$$\partial_t \partial_\beta \pi^{\alpha\beta} = -\partial_t [\partial_t(\varrho v^\alpha) + \partial_\beta(\varrho v^\alpha v^\beta) + \partial^\alpha P] . \tag{3.3}$$

Substituting (3.2) in the right-hand side of the equation of motion and using (3.3), we obtain

$$\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} = -\partial^\alpha P + \mu \Delta v^\alpha + \left(\eta + \frac{1}{3}\mu\right) \partial^\alpha(\partial_\gamma v^\gamma) - \tau_\nu \partial_t [\partial_t(\varrho v^\alpha) + \partial_\beta(\varrho v^\alpha v^\beta) + \partial^\alpha P]. \tag{3.4}$$

Equation (3.4) represents the modified Navier-Stokes equation with only one time constant τ_ν; this is not surprising since we used only eq. (3.2) for the inertia of momentum flow. The procedure for obtaining the modifications in the energy equation (2.3) making use of (3.1) and (3.2) is more cumbersome than the previous one. We must take into account, indeed, the consequences due to the new equation of motion (3.4) on the total energy conservation theorem. Moreover, the modified energy equation can be formally written

$$\begin{aligned} \varrho \frac{\mathrm{d}u}{\mathrm{d}t} = &- \partial^\alpha P + k \Delta T - \\ &- \tau_q \partial_t [\partial_t(\varrho u) + \partial_\beta(\varrho u v^\beta) + P(\partial_\gamma v^\gamma)] - \pi^{\alpha\beta} U_{\alpha\beta} - \tau_q \partial_t (\pi^{\alpha\beta} U_{\alpha\beta}) , \end{aligned} \tag{3.5}$$

where, of course, we have yet to substitute the viscous tensor. Nevertheless, since we are interested in the linearized theory, we remark that only the first three terms in the right-hand side of (3.5) give a contribution, because the other ones are a quadratic expression of velocity gradients by virtue of (3.2). The linearized equations can now be easily obtained. The last term of (3.4) can be written

$$\tau_\nu \partial_t \left[\varrho \frac{\mathrm{d}v^\alpha}{\mathrm{d}t} + \partial^\alpha P \right] ,$$

and, therefore, according to the hypothesis of small perturbation used in the preceding Section, we obtain

$$\tau_\nu \partial_t [\partial_t(\varrho v^\alpha) + \partial_\beta(\varrho v^\alpha v^\beta) + \partial^\alpha P] \simeq \tau_\nu \partial_t [\varrho_0 \partial_t v^\alpha + \partial^\alpha p] . \tag{3.4'}$$

Analogously, the third term of the right-hand side of (3.5) can be written

$$\tau_q \partial_t \left[\varrho \frac{\mathrm{d}u}{\mathrm{d}t} + P(\partial_\beta v^\beta) \right] = \tau_q \partial_t \left[\varrho \frac{\partial h}{\mathrm{d}t} - \frac{\mathrm{d}P}{\mathrm{d}t} \right] ,$$

where we have also made use of the definition of specific enthalpy. Therefore, on account of (2.10) and of the thermodynamic relation already used in Sect. **2**,

we obtain

$$(3.5') \qquad \tau_q \partial_t[\partial_t(\varrho u) + \partial_\beta(\varrho u v^\beta) + P(\partial_\beta v^\beta)] \simeq \tau_q \varrho_0 C_v \partial_t \left[\partial_t \tau - \frac{\gamma - 1}{\alpha\gamma} \partial_t p\right].$$

From (3.4) and (3.5), inserting (3.4') and (3.5'), we obtain the linearized system of equations corresponding to the system: (2.11)-(2.14)

$$(3.6) \qquad \partial_t \delta + \varrho_0 \partial_\beta v^\beta = 0\,,$$

$$(3.7) \qquad \varrho_0 \partial_t v^\alpha = -\partial^\alpha p + \mu \Delta v^\alpha + \left(\eta + \frac{1}{3}\mu\right)\partial^\alpha(\partial_\beta v^\beta) - \tau_\nu \partial_t[\varrho_0 \partial_t v^\alpha + \partial^\alpha p],$$

$$(3.8) \qquad \varrho_0 C_v \partial_t \left[\tau - \frac{\gamma - 1}{\alpha\gamma} p\right] = k \Delta \tau - \tau_q \varrho_0 C_v \partial_{tt} \left[\tau - \frac{\gamma - 1}{\alpha\gamma} p\right],$$

$$(3.9) \qquad \delta = \frac{\gamma}{c^2}(p - \alpha\tau)\,.$$

Here all physical quantities are denoted by the same symbols used in the previous Section. Here again we can confine ourselves to consider the one-dimensional case and substituting (3.9) in (3.6) we obtain

$$(3.10) \qquad (1 + \tau_\nu \partial_t)\partial_{xx} p = \frac{\gamma}{c^2} \partial_t[(1 + \tau_\nu \partial_t)\partial_t(p - \alpha\tau) - l_\nu c \partial_{xx}(p - \alpha\tau)],$$

$$(3.11) \qquad l_q c \partial_{xx} \tau = (1 + \tau_q \partial_t)\partial_t \left[\tau - \frac{\gamma - 1}{\alpha\gamma} p\right],$$

$$(3.12) \qquad \varrho_0 (1 + \tau_\nu \partial_t)\partial_t v = -\partial_x \left[(1 + \tau_\nu \partial_q) p + \frac{\gamma l_\nu}{c} \partial_t (p - \alpha\tau)\right].$$

The system (3.10)-(3.12) reduces to the system (2.15)-(2.17) when $\tau_q = \tau_\nu = 0$ as expected. Again, following the conventional method for studying dispersion and absorption of sound, we can analyse the damped partial waves whose superposition gives us the solution of (3.10) and (3.11). Therefore, on account of (2.21), from (3.10) and (3.11) we get

$$(3.13) \qquad \begin{cases} (1 - i\omega\tau_\nu) k^2 \hat{p} = \dfrac{\gamma}{c^2}[\omega^2(1 - i\omega\tau_\nu)(\hat{p} - \alpha\hat{\tau}) + i\omega l_\nu c k^2 (\hat{p} - \alpha\hat{\tau})]\,, \\ l_q c k^2 \hat{\tau} = i\omega(1 - i\omega\tau_q)\left(\hat{\tau} - \dfrac{\gamma - 1}{\alpha\gamma} \hat{p}\right). \end{cases}$$

From the system (3.13), taking into account that the determinant of the coef-

ficients of $\hat{p}$ and $\hat{\tau}$ must vanish, we obtain the generalized dispersion equation

$$(3.14)\qquad l_q\left[\gamma l_\nu + \frac{i}{\omega}c(1-i\omega\tau_\nu)\right]k^4 + \\ + \left\{(1-i\omega\tau_q)(1-i\omega\tau_\nu) - i\frac{\omega}{c}[(1-i\omega\tau_q)l_\nu + (1-i\omega\tau_\nu)\gamma l_q]\right\}k^2 + \\ - \frac{\omega^2}{c^2}(1-i\omega\tau_q)(1-i\omega\tau_\nu) = 0\,.$$

We can multiply eq. (3.14) by the product $(1+i\omega\tau_q)(1+i\omega\tau_\nu)$ so that the last term is a real one. Furthermore it is useful to introduce the characteristic times

$$(3.15)\qquad t_\nu = \frac{l_\nu}{c} = \frac{\eta + \frac{4}{3}\mu}{\varrho_0 c^2}\,, \qquad t_q = \frac{l_q}{c} = \frac{k}{\varrho_0 c^2 C_p}\,.$$

As we will see by comparing these quantities with those given by the kinetic theory of gases, they have the order of magnitude of the mean time between two collisions (or the inverse of collision frequency). Thus eq. (3.14) can be written

$$(3.16)\qquad Ak^4 + Bk^2 - C = 0\,,$$

where

$$A = A_1 + iA_2 = (a_1 + a_1'\omega^2) + i\frac{1}{\omega}(a_2 + a_2'\omega^2)\,,$$

$$B = B_1 - iB_2 = (1 + b_1'\omega^2 + b_1''\omega^4) - i\omega(b_2 + b_2'\omega^2)\,,$$

$$C = C_1 = c_1'\omega^2 + c_1''\omega^4 + c_1'''\omega^6\,.$$

The coefficients a_i, b_i and c_i are given in terms of the times t_q, t_ν, τ_q and τ_ν by the relations

$$(3.17a)\qquad \begin{cases} a_1 = c^2 t_q(\gamma t_\nu - \tau_q)\,, \\ a_1' = -c^2 t_q \tau_\nu \tau_q(\gamma t_\nu + \tau_\nu)\,, \end{cases} \qquad \begin{cases} a_2 = c^2 t_q\,, \\ a_2' = c^2 t_q[\tau_\nu^2 + \gamma t_\nu(\tau_q + \tau_\nu)]\,, \end{cases}$$

$$(3.17b)\qquad \begin{cases} b_1' = \tau_q^2 + \tau_\nu^2 + t_\nu\tau_\nu + \gamma t_q\tau_q\,, \\ b_2'' = \tau_q\tau_\nu(\tau_q\tau_\nu + t_\nu\tau_q + \gamma t_q\tau_\nu)\,, \end{cases} \qquad \begin{cases} b_2 = t_\nu + \gamma t_q\,, \\ b_2' = t_\nu\tau_q^2 + \gamma t_q\tau_\nu^2\,, \end{cases}$$

$$(3.17c)\qquad c_1' = \frac{1}{c^2}\,, \qquad c_1'' = \frac{\tau_\nu^2 + \tau_q^2}{c^2}\,, \qquad c_1''' = \frac{\tau_\nu^2\tau_q^2}{c^2}\,.$$

We are now able to show that the dispersion equation (3.16) does not lead to the difficulty mentioned at the end of the previous Section. We know that

such a difficulty arises from sound waves for high values of ω. Therefore, if we confine ourselves to consider the values of k^2 for $\omega t_q \gg 1$ (t_ν and t_q have the same order of magnitude), we obtain from (3.16)

$$(3.18)\qquad k^2 = \frac{\omega^2}{2(a_1')^2}\left\{a_1'[-b_1'' \pm (b_1''^2 + 4a_1'c_1''')^{\frac{1}{2}}] + \right.$$
$$\left. + i\,\frac{(a_1'b_2' + a_2'b_1'')[(b_1''^2 + 4a_1'c_1''')^{\frac{1}{2}} \mp b_1''] \mp 2a_1'a_2''c_1'''}{(b_1''^2 + 4a_1'c_1''')^{\frac{1}{2}}}\cdot\frac{1}{\omega}\right\}.$$

Relation (3.18) shows that for high frequencies the imaginary part of k^2 is negligible with respect to the real part. This implies that the real part of k becomes

$$(3.19)\qquad \mathrm{Re}\{k\} \simeq \left[\frac{-b_1'' \pm (b_1''^2 + 4a_1'c_1''')^{\frac{1}{2}}}{2a_1'}\right]^{\frac{1}{2}}\cdot\omega\,.$$

Hence the phase velocity $v_p = \omega/\mathrm{Re}\{k\}$ approaches a constant value, independent of ω, for high frequencies. In Appendix B, according to the remark already made in the Introduction, we deduce the propagation velocity of the wave front directly from the system (3.10)-(3.12) showing that it is equal to the phase velocity obtained from (3.19). We remark moreover that for damped progressive waves we must have

$$(3.20)\qquad \mathrm{Re}\{k\} > 0\,, \qquad \mathrm{Im}\{k\} > 0\,,$$

and, as a consequence,

$$(3.21)\qquad \mathrm{Im}\{k^2\} > 0\,.$$

As we previously noted, the condition (3.21) generally leads to the choice of the sign. This is also evident in the asymptotic expression of k^2 (3.18) where the real part is always positive on account of (3.17). However the choice of the sign depends on the values of τ_q and τ_ν included in the coefficients a_i, b_i and c_i. Of course, if we put $\tau_q = \tau_\nu = 0$ in (3.18) such a relation becomes meaningless because the coefficients a_1', a_2', b_1'', b_2' and c_2''' vanish; in this case we obtain again the relation (2.24) since (3.16) reduces to (2.23). However it is worth-while to remark that if we put alternatively either $\tau_q = 0$ or $\tau_\nu = 0$ the behaviour of k in terms of ω, which follows from (3.16), depends on the choice of the sign. By the way, we observe that this cannot be seen from (3.18), and then we must analyse (3.16) at a different order in ω. From our point of view, such a circumstance is without physical meaning because τ_q and τ_ν, connected to the same origin of the delay phenomenon, cannot be separately vanishing. Nevertheless, if we assume that only one τ vanishes, it can be shown from (3.16) that the phase velocity can be either finite or infinite, for

$\omega \to \infty$, according to the choice of the sign; one thus meets the same difficulties pointed out by FRENKEL ([8]), who considers the time constant τ related only to viscosity.

4. – Comparison with experimental measurements.

In the previous Section we were concerned only with the dispersion equation (3.14) and the resulting properties of the phase velocity on account of phenomenological equations (3.1) and (3.2). If we regard these last two equations as macroscopical laws, the results obtained can be applied equally well to the propagation of sound waves either in a dense medium (homogeneous and isotropic) or in a rarefied gas. However, if we want to compare theoretical results with experimental data, we have to choose the system and therefore to fix experimental parameters C_p, k, μ and c (in addition to ϱ_0 and T_0). In this Section we shall be concerned with experimental results in monoatomic rarefied gases. The most extensive experiments on sound propagation in diluted monoatomic gases are still those of GREENSPAN ([9]) and of MEYER and SESSLER ([10]) in argon; they are of the source problem type and are often referred to in the literature as bases for theoretical discussion ([11,12]). In those experiments sound is generated by a piezoelectric transmitter and detected by a movable piezoelectric receiver. As the distance between transmitter and receiver is changed, the phase and amplitude of the signal from the receiver change, and from this one can infer the phase velocity and absorption length. Of course, the main interest of a comparison with experiments in monoatomic gases is strictly linked to the possibility of deriving eqs. (3.1) and (3.2) from the kinetic theory of gases. By a statistical procedure CATTANEO ([1]) obtained the following expression for

$$\tau_q = \frac{\overline{l_v^2}}{\overline{v l_v}}, \tag{4.1}$$

where $\overline{l_v^2}$ is the square mean value of the free path and $\overline{v l_v}$ is the mean value of the product between velocity and free path of the molecules with velocity v. It is easy to prove, at least referring to the elementary arguments on transport properties which lead to (4.1), that $\tau_v = \tau_q$. Therefore, on the basis of the kinetic theory of gases, we can deduce that the two constants τ_v and τ_q must

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([8]) J. FRENKEL: *Kinetic Theory of Liquids*, Chap. 4 (Oxford, 1947).
([9]) M. GREENSPAN: *Journ. Acoust. Soc. Am.*, **28**, 644 (1956); **31**, 155 (1959).
([10]) E. MEYER ad G. SESSLER: *Zeits. Phys.*, **149**, 15 (1957).
([11]) See, *e.g.*, J. D. FOCH and G. W. FORD: *The dispersion of sound in monoatomic gases*, in *Studies in Statistical Mechanics*, Vol. **5** (Amsterdam, 1970).
([12]) L. SIROVICH and J. K. THURBER: *Journ. Acoust. Soc. Am.*, **37**, 329 (1965).

be equal and of the order of magnitude of $\Lambda/\bar{v}$, as one infers from (4.1) (Λ = mean free path). We will assume at once $\tau_\gamma = \tau_q = \tau$ and we will have at our disposal, in the dispersion equation (3.14), only one parameter, t_γ and t_q being fixed by the nature of the gas. Therefore, the comparison with the experimental data should provide, first of all, an explicit value of τ consistent with the order of magnitude expected from (4.1). However the main point of interest in such an analysis is the behaviour of the phase velocity in terms of the frequency ω. The experiments of GREENSPAN and of MEYER and SESSLER give

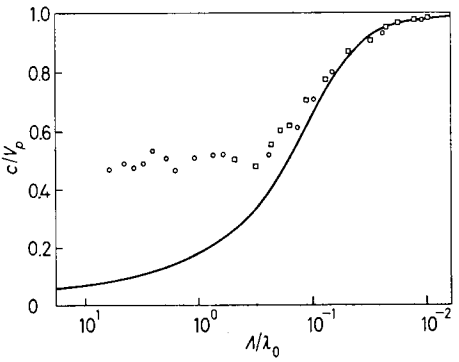

Fig. 1. – Dispersion in argon; experimental data from GREENSPAN ([9]) and MEYER and SESSLER ([10]): o experimental values of MEYER and SESSLER, □ experimental values of GREENSPAN, ——— KIRCHHOFF.

the dispersion of sound waves and show (see Fig. 1) that the ratio c/v_p tends to an almost constant value when ω increases (in Fig. 1 the ratio Λ/λ_0, known as Knudsen number, is proportional to ω), in contrast to the previsions of the Navier-Stokes equation according to which this ratio should vanish. In order to compare these results with those expected from eq. (3.14), we briefly recall that the state equation for a rarefied gas is

$$\frac{P}{\varrho} = \frac{KT}{m}, \tag{4.2}$$

where K is the Boltzmann constant and m the mass of the molecules. From (4.2) it follows that

$$K_s = \frac{1}{\varrho}\left(\frac{\partial P}{\partial \varrho}\right)_s = \frac{1}{\gamma P},$$

and, as a consequence, the propagation velocity c of the sound is

$$c = \frac{1}{(\varrho_0 K_s)^{\frac{1}{2}}} = \left(\frac{\gamma KT}{m}\right)^{\frac{1}{2}}, \tag{4.3}$$

where $\gamma = C_p/C_V = \frac{5}{3}$ for monoatomic gases. Taking this into account and also the relation between the coefficient of viscosity μ and the mean free path of molecules we can identify (*) the mean free path with the quantity

$$\Lambda \equiv \frac{\mu}{\varrho_0 c}. \tag{4.4}$$

Furthermore, recalling that the coefficient of bulk viscosity η is vanishing and that the dimensionless ratio

$$f = \frac{k}{\mu C_V} \tag{4.5}$$

is constant both theoretically and experimentally ([13]) for all monoatomic gases, we can write the Kirchhoff dispersion equation using dimensionless coefficients and variables

$$\xi = \frac{\omega\Lambda}{c}, \qquad \varkappa = \Lambda k \tag{4.6}$$

and the Kirchhoff dispersion equation (2.23) becomes

$$\left[\frac{4}{3}f + i\frac{3}{5}f\frac{1}{\xi}\right]\varkappa^4 + \left[1 - i\left(f + \frac{4}{3}\right)\xi\right]\varkappa^2 - \xi^2 = 0, \tag{4.7}$$

where we have used (4.4) and the definitions (2.18) to get

$$\frac{l_V}{\Lambda} = \frac{4}{3}, \qquad \frac{l_q}{\Lambda} = \frac{f}{\gamma} = \frac{3}{5}f.$$

Analogously the dispersion equation (3.14) can be written

$$\left[\frac{4}{3}f + i\frac{3}{5}f\frac{1}{\xi}(1 - i\chi\xi)\right]\varkappa^4 + \left[(1 - i\xi\chi)^2 - i\left(f + \frac{4}{3}\right)\xi(1 - i\xi\chi)\right]\varkappa^2 - \xi^2(1 - i\xi\chi)^2 = 0, \tag{4.8}$$

where $\chi = c\tau/\Lambda$, Using (4.8) one can calculate the ratio $\varkappa/\xi$ in terms of ξ to get a direct comparison with the experimental data plotted in Fig. 1. In

(*) See J. D. FOCH and G. W. FORD: ref. ([11]).

([13]) S. CHAPMAN and T. G. COWLING: *The Mathematical Theory of Nonuniform Gases* (Cambridge, 1958), p. 241.

fact, from the definitions (4.6) it follows that

$$\frac{c}{v_p} = \frac{\mathrm{Re}\{k\}}{\omega/c} = \mathrm{Re}\left\{\frac{\varkappa}{\xi}\right\},$$

and that ξ is proportional to the ratio Λ/λ_0, where the sound wavelength is indicated by $\lambda_0 = 2\pi c/\omega$:

$$\xi = \omega\frac{\Lambda}{c} = 2\pi\frac{\Lambda}{\lambda_0}.$$

One can see immediately from the graph of Fig. 1 that at low frequencies the behaviour of c/v_p corresponding to the Kirchhoff equation follows the experimental data, whereas there are marked discrepancies in the Knudsen regime where the wavelength is short compared with the mean free path. Therefore, by comparing equations (4.8) and (4.7) one sees that, for $\Lambda/\lambda_0 \lesssim 10^{-1}$, the value of the constant τ must be such that the product $\xi\chi = \omega\tau$ is negligible with respect to unity, while, for $\Lambda/\lambda_0 \simeq 1$, it can be of the order of unity. An estimate of τ can be obtained by imposing the condition $\omega_0\tau = 1$, where ω_0 is the frequency corresponding to $\Lambda/\lambda_0 = 1$. Taking into account that $\Lambda/\lambda_0 = \Lambda\omega/2\pi c$ we have

$$\omega_0 = 2\pi\frac{c}{\Lambda} = \frac{4}{3}2\pi\frac{1}{t_V}, \tag{4.9}$$

where we have used the relation $t_V = 4\mu/3\varrho_0 c^2 = 4\Lambda/3c$.

From the condition $\omega_0\tau = 1$ we obtain

$$\tau = \frac{1}{\omega_0} = \frac{1}{2\pi}\frac{\Lambda}{c}, \tag{4.10}$$

so that τ is of the order of magnitude of the reciprocal of the collision frequency Λ/c according to (4.1). The quantity $\mathrm{Re}\{\varkappa/\xi\} = c/v_p$ for different values of the frequency, or Λ/λ_0, has been calculated with the value of τ from (4.10). From numerical calculations we were able to see that the value of τ which best fits the experimental data on dispersion is slightly lower than $\tau = (1/\pi)\Lambda/c$, thus confirming the considerations about its order of magnitude. In the following Table we have reported some values of c/v_p for $\tau = \Lambda/\pi c$,

TABLE I. – *Numerical values of the ratio c/v_p derived from eq. (4.7) or Kirchhoff equation, from eq. (4.8) and from the experimental data; the range of values of the latter ones corresponds to an evaluation of the uncertainty due to the interpolation procedure.*

Λ/λ_0	0.25	0.5	1	2	4	7
$(c/v_p)_{\mathrm{Kirch}}$	0.40	0.26	0.19	0.13	0.10	0.07
$(c/v_p)_{(4.8)}$	0.52	0.43	0.44	0.47	0.48	0.49
$(c/v_p)_{\mathrm{exp}}$	0.51 ÷ 0.55	0.46 ÷ 0.50	0.50 ÷ 0.52	0.46 ÷ 0.50	0.46 ÷ 0.49	0.46 ÷ 0.49

the corresponding experimental data and the values calculated from the Kirchhoff equation.

These results are plotted in Fig. 2.

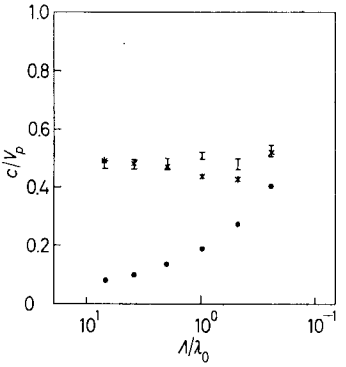

Fig. 2. – Dispersion in argon for some values of Λ/λ_0; the range of values of experimental data corresponds to an evaluation of the uncertainty due to the interpolation procedure: I experimental values, • Kirchhoff equation, * eq. (4.8).

As can be seen, the behaviour of c/v_p according to the Kirchhoff equation departs from the experimental data more and more as Λ/λ_0 increases, while the behaviour of c/v_p according to eq. (4.8) remains more or less constant and reproduces fairly well the experimental data in Fig. 1. It is clear that a more detailed numerical analysis is necessary for an accurate determination of τ from the experimental data. This analysis is also useful for examining the contribution of inertial terms to the sound absorption. Although preliminary calculations show that the behaviour of Im $\{\varkappa/\xi\}$ as a function of Λ/λ_0 is in better agreement with experiments than that derived from the Kirchhoff equation, we shall postpone a discussion of this quantity which is very sensitive to the value of τ chosen to fit dispersion data.

5. – Conclusions.

The addition in the phenomenological equations (1.2′) and (1.6′) of terms which describe the inertial property of energy flux as well as momentum flux has led to the two following main results: on the one hand, we have overcome the difficulty related to the impossibility of wave propagation in a viscous medium, and, on the other hand, we have shown that the order of magnitude of the time constant τ inferred from experiments in monoatomic gases coincides with the order of magnitude expected on the basis of statistical considerations. This last circumstance, along with the fact that *both* the inertial terms added allow the overcoming of the paradox concerned with the wave propagation

in a viscous fluid, suggests that a closer inspection of eqs. (1.2′) and (1.6′) from the point of view of the kinetic theory of gases is probably useful taking account also of the developments that this theory has achieved in recent years. In any case, eqs. (1.2′) and (1.6′) can be considered as phenomenological or macroscopical laws. One could indeed assert, following the Maxwellian point of view on relaxation phenomena supported by VERNOTTE, that the establishment of a flux of a thermomechanic quantity, produced by temperature or velocity gradients, is always joined to a reactive «force» which contrasts, in the initial phase, with the establishement of the flux itself. Following this point of view we have shown in this paper that, although the transitory phases are difficult to measure directly by experiments because of the very small value of τ, the consequences of the inertial character of $\pi^{\alpha\beta}$ and $\boldsymbol{J}_q$ becomes relevant when the transitory phase reveals a recurrence in time with a period of the order of (or smaller than) τ, just as in the case of sound waves in rarefied gases.

Finally, we remark that hydrodynamics, treating energy fluxes and momentum fluxes on the same foot, appears to be the natural framework for any relativistic theory of heat transport. Thus, we believe that in a hydrodynamic context the problems arising from relativistic heat equations, as the ones given by KRANYS ([14]) and VAN KAMPEN ([15]), could be better clarified.

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APPENDIX A

It is well known that the notion of wave propagation related to any kind of physical phenomena, governed by a system of differential equations, can be associated with the definition of a characteristic surface or a wave front. We shall briefly recall, following LEVI-CIVITA ([4]), the definition of a wave front. Consider the phenomena described by the system of differential equations

$$(A.1)\qquad E_\mu = \sum_{\nu=1}^{m} \sum_{(\sum j_i = s)} E_{\mu\nu}^{j_0 j_1 \dots j_n} \frac{\partial^s \varphi_\nu}{\partial x_0^{j_0} \partial x_1^{j_1} \dots \partial x_n^{j_n}} + \phi_\mu(x, \varphi, \chi) = 0 \qquad (\mu = 1, 2, \dots, m),$$

([14]) M. KRANYS: *Nuovo Cimento*, **42** B, 51 (1966).
([15]) N. G. VAN KAMPEN: *Physica*, **46**, 315 (1970).

where s is the maximum order of differentiation of m unknown functions φ_ν (here we assume that the maximum order s is equal for all φ_ν) of $n+1$ independent variables x_j; χ is a symbol of the partial derivatives of φ with respect to the variables x_j but of an order lower than s. The hypersurfaces $z(x_0, x_1, \ldots, x_n) = \text{const}$ are called wave fronts if they are the space regions across which the quantities $\partial^s \varphi_\nu / \partial x_0^{j_0} \partial x_1^{j_1} \ldots \partial x_n^{j_n}$ have a discontinuity, whereas the functions φ_ν and their derivatives up to the order $s-1$ are continuous functions. These surfaces are also called characteristic manifolds because the Cauchy-Kowalewski theorem is not valid for the points of those surfaces. This fact has the important consequence that the characteristic manifolds must satisfy a differential equation. One can show that, by introducing the $n+1$ independent variables $z, z_1, z_2, \ldots, z_n$ instead of the variables $x_0, x_1, x_2, \ldots, x_n$, the characteristic manifold must satisfy the following differential equation:

$$\left\| \sum_{(\sum j_i = s)} E_{\mu\nu}^{j_0 j_1 \ldots j_n} p_0^{j_0} p_1^{j_1} \ldots p_n^{j_n} \right\| = 0 \qquad (\mu, \nu = 1, 2, \ldots m), \tag{A.2}$$

where $p_i = \partial z / \partial x_i$. Without solving eq. (A.2), one can find the propagation velocity of the wave front

$$a = \frac{|p_0|}{\left(\sum_{i=1}^{n} p_i^2 \right)^{\frac{1}{2}}}, \tag{A.3}$$

if the p_i are not identically vanishing.

The above-mentioned results can be put in a different way if one considers a homogeneous linear system with constant coefficients instead of the general system (A.1). Under these circumstances the solutions φ_ν can be written as a Fourier integral

$$\varphi_\nu(\boldsymbol{x}, t) = \int \ldots \int \hat{\varphi}(\boldsymbol{k}, \omega) \exp [i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)] \, \mathrm{d}\boldsymbol{k} \, \mathrm{d}\omega , \tag{A.4}$$

where $\boldsymbol{k}$ and $\boldsymbol{x}$ are the n-tuple $(k_1, k_2, \ldots, k_n)$ and $(x_1, x_2, \ldots, x_n)$ respectively. Equation (A.1) can then be written

$$E_\mu = \sum_{\nu=1}^{m} \sum_{(\sum j_i = s)} E_{\mu\nu}^{j_0 j_1 \ldots j_n} k_0^{j_0} k_1^{j_1} \ldots k_n^{j_n} (i)^s \hat{\varphi}_\nu + \hat{\psi}_\mu = 0 , \tag{A.5}$$

where $\hat{\psi}$ is the transform of $\psi(\varphi, \chi)$ and ψ is now a homogeneous linear function of φ and χ and, moreover, in compliance with the preceding notation, we put $x_0 = t$, $k_0 = -\omega$.

From (A.5) an equation follows (a generalization of the Kirchhoff equation) relating k_j and ω. Now we suppose that, whatever the relation between k_j and ω is, the phase velocity $v_p = \omega / k$, where $k = (k_1^2 + k_2^2 + \ldots + k_n^2)^{\frac{1}{2}}$, is finite when ω increases for all values of the propagation vector $\boldsymbol{u} = \boldsymbol{k}/k$. Denoting the limited value of the phase velocity by $v_{p\infty}$, we will show that, if such a limit exists, it is equal to the propagation velocity defined in (A.3). In fact, if $v_{p\infty}$ is limited for any values of the propagation vector $\boldsymbol{u}$, it follows from the definition of phase velocity itself that k_j must be proportional to ω

($k_j \simeq \beta\omega$) for high values of ω. Therefore, the system (A.5) becomes

$$(A.6) \qquad (i)^s \sum_{\nu=1}^{m} \sum_{(\Sigma j_i = s)} E_{\mu\nu}^{j_0 j_1 \dots j_n} k_0^{j_0} k_1^{j_1} \dots k_n^{j_n} \hat{\varphi}_\nu \left[1 + O\left(\frac{1}{\omega}\right)\right] = 0 .$$

Disregarding all terms of the order $1/\omega$, a nontrivial solution of system (A.6) for $\hat{\varphi}_\nu$ exists, for high values of ω, provided the determinant of the coefficients vanishes:

$$(A.7) \qquad \left\| \sum_{(\Sigma j_i = s)} E_{\mu\nu}^{j_0 j_1 \dots j_n} k_0^{j_0} k_1^{j_1} \dots k_n^{j_n} \right\| = 0 .$$

Equation (A.7) is equal to eq. (A.2) and therefore it follows that the quantity defined by the ratio $|k_0| \Big/ \left(\sum_{i=1}^{n} k_i^2\right)^{\frac{1}{2}}$ is identical to the propagation velocity of the wave front, if such a front exists.

APPENDIX B

As mentioned in the Introduction the Navier-Stokes equation, the continuity equation and the state equation $\varrho = \varrho(P)$ lead to the strange result that wave propagation is impossible in a viscous medium ([4,5]). It can be easily seen that if T is an unknown function and the energy equation (2.16) is added to the system, the previous result is unchanged. On the other hand, the addition of inertial terms corresponding to energy and momentum fluxes leads to a limited value of the propagation velocity. For the sake of simplicity we will now show this in a one-dimensional case. Using (A.2) and (A.3) with relation to the system of equations (3.10)-(3.12) we obtain

$$(B.1) \qquad p_0^4 \begin{vmatrix} \varrho_0 \tau_V p_0 & (\tau_V + \gamma t_V) p_1 & -\alpha\gamma t_V p_1 \\ 0 & \dfrac{\gamma - 1}{\alpha\gamma} \tau_q p_0^2 & t_q c^2 p_1^2 - \tau_q p_0^2 \\ 0 & (\tau_V + \gamma t_V) p_1^2 - \dfrac{\gamma \tau_V}{c^2} p_0^2 & \alpha\gamma \left(\dfrac{\tau_V}{c^2} p_0^2 - t_V p_1^2 \right) \end{vmatrix} = 0 ,$$

from which it follows

$$(B.2) \qquad \varrho \tau_V p_0^5 \left[(\tau_V + \gamma t_V) t_q c^2 p_1^4 - \frac{\tau_q \tau_V}{c^2} p_0^4 - (\tau_q \tau_V + \tau_q t_V + \gamma \tau_V t_q) p_1^2 p_0^2 \right] = 0 .$$

Using the relation (3.17), if $p_0 \neq 0$, we have

$$(B.3) \qquad a_1' p_1^4 + b_1'' p_1^2 p_0^2 + c_1''' p_0^4 = 0 ,$$

and therefore

$$\text{(B.4)} \qquad \frac{1}{a} = \left|\frac{p_1}{p_0}\right| = \left[\frac{-b_1'' \pm (b_1''^2 - 4a_1' c_1''')^{\frac{1}{2}}}{2a_1'}\right]^{\frac{1}{2}} = \frac{1}{v_{p\infty}};$$

this completes the proof.

RIASSUNTO

Riprendendo alcune considerazioni svolte in passato dal Cattaneo, si propone una modifica dell'equazione di Navier-Stokes che tiene conto dell'inerzialità del flusso di impulso. Il sistema linearizzato delle equazioni dell'idrodinamica, ove si consideri l'inerzialità del flusso di calore unitamente a quello di impulso, conduce ad una nuova equazione di dispersione che non presenta gli inconvenienti dell'equazione di Kirchhoff. La teoria della dispersione del suono così modificata è coerente con i risultati sperimentali relativi ai gas monoatomici rarefatti.

Модифицированное уравнение Навье-Стокса и его следствия для дисперсии звука.

Резюме (*). — Принимая во внимание некоторые рассмотрения, развитые ранее Каттанео, мы предлагаем модифицированное уравнение Навье-Стокса, которое включает инерциальное свойство потока импульса. Линеаризованная система гидродинамических уравнений, в которых учитываются инерциальные свойства и потока импульса и потока тепла, приводит к новому дисперсионному уравнению, которое не обнаруживает теоретических неудобств уравнения Кирхгофа. Также заметно улучшается согласие с экспериментальными результатами для дисперсии звука в разреженных одноатомных газах.

(*) *Переведено редакцией.*