Quantum Field Theory off Null Planes (*).

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(ricevuto il 2 Luglio 1970)

Summary. — The initial-value problem for hyperbolic differential equations with the initial data given on a null plane in Minkowski space is considered in detail for the Klein-Gordon and Dirac equations. Existence and uniqueness theorems are given. The quantum field-theoretic analogue involves the commutation relations on the null plane and the null translation operator off that plane. The formal theory of interacting fields is stated briefly in the Feynman-Dyson spirit. It is pointed out that an interaction that involves the null co-ordinate derivative of the field in the direction off the initial plane leads to additional complications.

1. – Introduction.

Quantum field theory off null planes was first used by one of us (1) about two years ago. It arose in a natural manner in studying quantum electrodynamics in a laser beam. If such a beam is pictured as a coherent wave train of finite length but infinite width it will fill a null slab, *i.e.* the four-volume between two parallel three-dimensional null hyperplanes in Minkowski space. Null co-ordinates are therefore the natural co-ordinate system.

Quite independent of this physical problem the study of the infinite-momentum limit in current algebras also leads to quantum field theory off null planes (2). So there is now a double purpose in studying this new formulation of quantum field theory.

(*) Work supported in part by a grant from the National Science Foundation.
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(1) R. A. Neville: Ph. D. thesis, Syracuse University (August 1968).

(2) H. Leutwyler: *Acta Phys Austriaca Suppl.*, **5** (1968) (*VII Schladming Meeting*); K. Bardakci and M. B. Halpern: *Phys. Rev.*, **176**, 1686 (1968).

This problem also offers new mathematical questions: the usual formulation of field theory is based on the well-understood Cauchy-Kowalewski theory of hyperbolic equations, which provides theorems for the existence and uniqueness of solutions when the field and its time derivative are known on a spacelike hyperplane. A somewhat different initial-value problem has recently found considerable attention in the theory of gravitational radiation. This is the characteristic initial-value problem where the initial data are specified on the surface of a (future) characteristic half-cone [3].

The present problem asks for initial data on a null plane and the conditions for existence and uniqueness of solutions for that situation. Given a point above such a plane its (past) characteristic cone would cut out of it a hypersurface area which is infinite and has a parabolic boundary. This infinite domain of dependence that is open in one direction thus differs essentially from the bounded domains of dependence involved in the above two initial-value problems.

An important physical distinction between the initial-value problem on null cones in relativity and the one on null planes used here is the elimination of dynamical fields, moving in the null planes, by boundary conditions implied by the postulated existence of null-translation generators.

It will be the first task of the present paper to study this problem for the most common free fields of physics in order to have a better understanding of the existence and uniqueness questions on which one will have to build. This is done in Sect. **3**, culminating in a few theorems. The important result is that under suitable conditions the knowledge of the field alone (*i.e.* half the usual initial data) on the null plane suffices to yield a unique solution.

In the quantized case the knowledge of the commutation relations on the initial surface is essential. The dynamics is brought about by a self-adjoint operator (the Hamiltonian in the usual case) which generates the unitary transformation characterizing the time development. In the present case these time translations will be replaced by translations in null directions generated by null translation operators which take the place of the Hamiltonian. These operators are actually projections of the four-momentum in null directions. The quantum dynamics is then expressed by null translations, and the physical assumption of the existence of the momentum operators already goes a long way to ensure the desired solution. Section **4** is devoted to this question.

Having thus obtained a better understanding of the free fields propagated

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(3) J. HADAMARD: *Lectures on Cauchy's Problem* (New Haven, Conn., 1923; New York, 1952); R. COURANT and D. HILBERT: *Methods of Mathematical Physics*, Vol. **2** (New York, 1962); F. G. FRIEDLANDER: *Proc. Roy. Soc.*, A **269**, 53 (1962); **279**, 386 (1964); R. PENROSE: *Null hypersurface initial data for classical fields of arbitrary spin and for general relativity* (preprint in P. G. Bergmann's report: *Quantization of generally covariant field theories*, ARL63-56, Wright-Patterson AFB, Ohio).

off a null plane by the new dynamics, we turn to the interaction of fields in Sect. **5**. Here the problem is treated entirely formally, without, however, forgetting the mathematically ill-defined nature of the interaction operator. But in the spirit of the usual Feynman-Dyson approach to field theory, a perturbation solution can be given which is very close indeed to the standard techniques. The main new feature seems to be the difficulty brought about by the dependence of some interaction operators (and perhaps most physically interesting ones) on the derivative of the field in the null direction off the null surface. At the present time this difficulty can be overcome at best by treating the interaction operator itself as an infinite series in the expansion parameter (coupling constant). All problems of convergence are, of course, wide open.

Our results on the free scalar field are complemented by a number of rigorous results obtained recently by KLAUDER, LEUTWYLER, and STREIT (4).

After conclusion of this work our attention was also drawn to other recent papers that treat the Feynman-Dyson approach to null planes in further detail (5).

2. – Null co-ordinates.

We consider $(1+3)$-dimensional Minkowski space with metric signature $+2$ and the usual pseudo-Cartesian co-ordinates as the contravariant components of the position vector $x=(x^0, x^1, x^2, x^3)$. The two null vectors

$$m=\frac{1}{\sqrt{2}}(1,0,0,-1) \quad\text{and}\quad n=\frac{1}{\sqrt{2}}(1,0,0,1) \tag{2.1}$$

satisfy $m_\mu n^\mu = m\cdot n = -1$, $m\cdot m = 0 = n\cdot n$. They are orthogonal to the two unit vectors $e_1=(0,1,0,0)$ and $e_2=(0,0,1,0)$. m, n, e_1, e_2 form a basis, so that the vector x can be written

$$x = um + vn + x_1 e_1 + x_2 e_2 \, . \tag{2.2}$$

The null co-ordinates u and v are therefore defined by

$$u \equiv -n\cdot x = \frac{1}{\sqrt{2}}(x^0 - x^3) \quad\text{and}\quad v \equiv -m\cdot x = \frac{1}{\sqrt{2}}(x^0+x^3)\, . \tag{2.3}$$

The co-ordinate u increases along m, and v along n. For any vector A the cor-

(4) J. R. KLAUDER, H. LEUTWYLER and L. STREIT: *Nuovo Cimento*, **66** A, 536 (1970).
(5) S.-J. CHANG and S.-K. MA: *Phys. Rev.*, **180**, 1506 (1969); J. B. KOGUT and D. E. SOPER: *Phys. Rev. D*, March 15 (1970).

responding components along m and n must be defined by

$$(2.4)\qquad A_u \equiv -m\cdot A = \frac{1}{\sqrt{2}}(A^0 + A^3) \quad \text{and} \quad A_v \equiv -n\cdot A = \frac{1}{\sqrt{2}}(A^0 - A^3)\,,$$

so that A_u is the coefficient of the basis vector $n(!)$, and A_v of m:

$$(2.5)\qquad A = A_v m + A_u n + A_1 e_1 + A_2 e_2\,.$$

As a special case of (2.3) the derivative operation is:

$$(2.6)\qquad \partial_u \equiv -m\cdot\partial = -\frac{\partial}{\partial u} \quad \text{and} \quad \partial_v \equiv -n\cdot\partial = -\frac{\partial}{\partial v}\,.$$

It will be convenient to use the notation $\boldsymbol{x}$ and $\boldsymbol{A}$ for the corresponding vectors in the two-dimensional subspace spanned by e_1 and e_2. Similarly, we we shall use $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{A}}$ for the vectors in the subspace spanned by n, e_1, e_2, *i.e.* in the null hyperplane $u = \text{const}$.

From (2.5) follows that two vectors A and B have the inner product

$$(2.7)\qquad A\cdot B = -A_v B_u - A_u B_v + \boldsymbol{A}\cdot\boldsymbol{B}\,.$$

The corresponding metric tensor, labelled by m, n, e_1, e_2, therefore has the form

$$(2.8)\qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In analogy to the usual situation when initial data are given on a space-like plane $x^0 = \text{const}$ and the development in time $t = x^0$ is sought, the null plane $u = \text{const}$ (or $v = \text{const}$) will carry initial data and u (or v) shall play the role of time. For symmetry reasons we need to investigate only $u = \text{const}$.

3. – Existence and uniqueness of the initial-value problem on a null plane.

Consider the Klein-Gordon equation in null co-ordinates

$$(3.1)\qquad 2\partial_u\partial_v\phi(x) = (\boldsymbol{\partial}^2 - m^2)\phi(x)\,,$$

with $\phi(x)$ specified on the null plane $u = u_0$. Under what conditions does there exist a solution to this initial-value problem and when is it unique?

A first integral of (3.1) on $u = u_0$ is

$$\partial_u \phi(u_0, v, \boldsymbol{x}) = -\frac{1}{4}\int_{-\infty}^{\infty} \varepsilon(v - v')(\boldsymbol{\partial}^2 - m^2)\phi(u_0, v', \boldsymbol{x})\,\mathrm{d}v' + F(u_0, \boldsymbol{x})\,, \tag{3.2}$$

provided the integral converges. Since ϕ is given on u_0, this equation will yield a unique function $\partial_u\phi$ on u_0 provided conditions are imposed which ensure that $F(u_0, \boldsymbol{x}) = 0$. This can be done by imposing

$$\lim_{|v|\to\infty} \partial_u \phi = 0 \qquad \text{on } u_0. \tag{3.3}$$

One can now ask what further conditions are necessary such that a step-by-step integration starting with (3.2) will converge and yield $\phi(x)$ at u, a finite distance off the null plane u_0.

While such a method is certainly feasible, the following considerations lead to the desired answer in a very short and elegant way. Note that we do not want to assume that ϕ in (3.1) be infinitely differentiable or, even stronger, that ϕ be analytic. Of course, when treated as a generalized function ϕ is n-fold differentiable if the test functions are.

Furthermore, we note that the procedure indicated by (3.2) shows a breakdown for $m = 0$ if Minkowski space is limited to $1+1$ dimensions. The right side of (3.1) then vanishes. We shall exclude this exceptional case; but otherwise the following considerations are applicable to any number of dimensions $1 + s$ $(s > 0)$ and $m \neq 0$ or $m = 0$.

The Cauchy problem *i.e.* the problem of finding a solution ϕ of (3.1) with given ϕ and $\partial\phi/\partial t$ on a spacelike plane $t = 0$ has a unique solution which has the well-known form

$$\phi(x) = \int \Delta(x - x')\, \frac{\overleftrightarrow{\partial}}{\partial t'}\, \phi(x')\,\mathrm{d}^3x'\,, \tag{3.4}$$

where $\overleftrightarrow{\partial}$ means $\overrightarrow{\partial} - \overleftarrow{\partial}$ and where Δ is the usual invariant function (see Appendix II). The integration extends over a bounded domain since $\Delta(x - x') = 0$ for $x - x'$ spacelike. In fact any domain which contains the «domain of dependence» of the point x (the *closed* hyperdisc, a three-dimensional closed ball, cut by the past light cone with vertex at x from the hyperplane $t = 0$) will surffice.

Now the eq. (3.1) leads in a well-known manner to the divergence-free expression

$$\Delta(x - x')\,\overleftrightarrow{\partial'_\mu}\phi(x')\,.$$

In Fig. 1 the u-v plane of Minkowski space is shown. The point x is denoted by P. The $t=0$ plane intersects the backward cone from P. We consider the four-dimensional volume V whose projection in the u-v plane is indicated

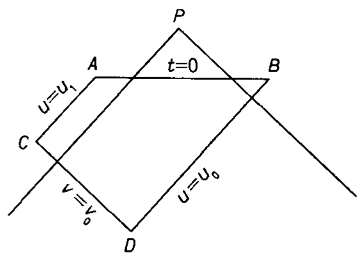

Fig. 1. – Four-volume of integration for the proof of eqs. (3.5) and (3.7).

by $ABDC$. It is infinite in the $\boldsymbol{x}$ direction and bounded by the plane $t=0$, and the null planes $v=v_0$, $u=u_0$, and $u=u_1$. Gauss' theorem yields

$$0=\int_V \partial'^{\mu}[\Delta(x-x')\,\partial'_{\mu}\phi(x')]\,\mathrm{d}^4x' = \left(\int_{AB}+\int_{BD}+\int_{DC}+\int_{CA}\right)\Delta(x-x')\,\nu\cdot\overleftrightarrow{\partial'}\phi(x')\,\mathrm{d}^3\sigma' .$$

The integrations over the hyperplane parts are indicated symbolically. The integrals over the distant ends in the $|\boldsymbol{x}|\to\infty$ directions are omitted, because these timelike areas are spacelike with respect to P so that Δ vanishes. But the same is true for CA. Taking into account that $\nu=-m$ and $\nu=-n$ on CD and DB, respectively, we find with (3.4)

$$\phi(x)=\int_{CD}\Delta(x-x')\,\frac{\overleftrightarrow{\partial}}{\partial u'}\,\phi(x')\,\mathrm{d}^2\boldsymbol{x}'\,\mathrm{d}u' + \int_{DB}\Delta(x-x')\,\frac{\overleftrightarrow{\partial}}{\partial v'}\,\phi(x')\,\mathrm{d}^2\boldsymbol{x}'\,\mathrm{d}v' . \tag{3.5}$$

But the normal to a null plane lies in that plane; therefore the knowledge of ϕ on the null planes implies its normal derivatives ($\partial\phi/\partial u$ on $v=v_0$ and $\partial\phi/\partial v$ on $u=u_0$). Thus we have the following theorem.

Theorem 1. The solution of the Klein-Gordon equation (3.1) for $m\geqslant 0$ is uniquely determined by (3.5) in the convex region bounded by the wedge formed by the null planes $u=u_0$ and $v=v_0$ if ϕ is specified on these planes; the knowledge of ϕ on the wedge but outside the characteristic cone (backward null cone) is not necessary.

In the evaluation of (3.5) the knowledge of Δ and its normal derivative on a null plane is necessary. This auxiliary information is contained in Appendix II.

A special case of great interest is obtained by moving the null plane

$v = v_0$ to the distant past, and by taking $\phi = 0$ on it. This means

$$\lim_{v \to -\infty} \phi = 0 , \qquad \text{for all } \boldsymbol{x},\ u \geqslant u_0. \tag{3.6}$$

One thus obtains

$$\phi(x) = \int_{u'=u_0} \Delta(x - x') \frac{\overleftrightarrow{\partial}}{\partial v'} \phi(x') \,\mathrm{d}^2\boldsymbol{x}' \,\mathrm{d}v' \tag{3.7}$$

and the following theorem.

Theorem 2. Given ϕ on the null plane $u = u_0$ and the asymptotic condition (3.6), then the Klein-Gordon equation (3.1) for any $m \geqslant 0$ has a unique solution given by (3.7) in the half-space $u > u_0$.

As a *corollary* we have that any solution of (3.1) which satisfies (3.6) and which vanishes on a null plane $u = \text{const}$, vanishes everywhere in $u \geqslant u_0$.

It is easily seen that the asymptotic condition (3.6) is not only sufficient to yield uniqueness but is also necessary for $m = 0$. The reason lies simply in the fact that for $m = 0$ any differentiable function of one null co-ordinate only, $\phi = f(u)$ for example, is a solution. Thus there are solutions which can be entirely contained in a region $u > u_0$; these would contradict the corollary if they were not eliminated by (3.6). Physically, (3.6) eliminates waves in a null direction parallel to a $u = \text{const}$ plane.

For $m \neq 0$ this argument breaks down. However, a simple example shows that the condition (3.6) is also here necessary. Let $\varphi(\boldsymbol{x})$ be a solution of $(\boldsymbol{\partial}^2 - m^2)\varphi(\boldsymbol{x}) = 0$; then $\phi(x) = \theta(u - u_0)\varphi(\boldsymbol{x})$ is a solution of (3.1) which vanishes for $u < u_0$. Without (3.6) the solution (3.7) would not be unique.

We note that (3.6) is, of course, weaker than square-integrability, which would exclude the above example.

In eq. (3.5) the integrals extend from the edge of the wedge (at D) to $+\infty$, but those parts of the null planes which are outside the characteristic cone from P do not contribute. Similarly, in eq. (3.7) the v-integration extends from $-\infty$ to $+\infty$, even though the open set $v' \in (v, \infty)$ does not contribute, since Δ vanishes for spacelike arguments. However one cannot terminate the integration on the characteristic cone because of the singularity on it. It is exactly there where the distribution nature of Δ enters in an essential way. The points of the characteristic cone must all be interior points of the domain of integration.

It follows that $\phi(x)$ will be correctly given by these equations for any $u > u_0$ but in the limit as u approaches u_0 a convergence condition for $v \to +\infty$ of the integral in (3.7) will become necessary. With this additional restriction

$$\lim_{v \to \infty} \phi = 0 , \qquad \text{on } u = u_0 ,$$

one can easily verify the consistency of (3.7) by taking P on the u_0 null plane: from (A-II.5) with $u = u_0$

$$\phi(x) = \frac{1}{4}\int_{-\infty}^{\infty} \varepsilon(v-v')\,\mathrm{d}v' \int \delta_2(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\overleftrightarrow{\partial}}{\partial v'}\,\phi(u, v', \boldsymbol{x}')\,\mathrm{d}^2\boldsymbol{x}' =$$

$$= \frac{1}{4}\int_{-\infty}^{\infty} \varepsilon(v-v')\,\mathrm{d}v'\,\frac{\overleftrightarrow{\partial}}{\partial v'}\,\phi(u, v', \boldsymbol{x}) = -\frac{1}{2}\int_{-\infty}^{\infty}\phi(u, v', \boldsymbol{x})\,\frac{\partial}{\partial v'}\,\varepsilon(v-v')\,\mathrm{d}v' = \phi(u, v, \boldsymbol{x})\,.$$

Here we used integration by parts and $\partial\varepsilon(v)/\partial v = 2\delta(v)$. Similarly,

$$\frac{\partial\phi}{\partial v}(x) = \frac{1}{2}\int_{-\infty}^{\infty} \delta(v-v')\,\mathrm{d}v' \int \delta_2(\boldsymbol{x}-\boldsymbol{x}')\,\frac{\overleftrightarrow{\partial}}{\partial v'}\,\phi(u, v', \boldsymbol{x}')\,\mathrm{d}^2\boldsymbol{x}' =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \delta(v-v')\,\mathrm{d}v'\,\frac{\overleftrightarrow{\partial}}{\partial v'}\,\phi(u, v', \boldsymbol{x}) = \int_{-\infty}^{\infty}\delta(v-v')\,\mathrm{d}v'\,\frac{\partial}{\partial v'}\,\phi(u, v', \boldsymbol{x}) = \frac{\partial}{\partial v}\,\phi(u, v, \boldsymbol{x})\,.$$

We conclude this Section by noting that the Dirac equation can be treated in a very similar manner. In null co-ordinates it is

$$\gamma_v\,\partial_u\,\psi = \big((\boldsymbol{\gamma}\cdot\boldsymbol{\partial}) + m - \gamma_u\,\partial_v\big)\,\psi\,, \tag{3.8}$$

where

$$\gamma_u = -\,m\cdot\gamma = \frac{1}{\sqrt{2}}\,(\gamma^0 + \gamma^3) \quad \text{and} \quad \gamma_v = -\,n\cdot\gamma = \frac{1}{\sqrt{2}}\,(\gamma^0 - \gamma^3)\,, \tag{3.9}$$

so that

$$\gamma_u^2 = 0\,, \quad \gamma_v^2 = 0\,, \quad \{\gamma_u, \gamma_v\} = -2\,, \quad [\gamma_u, \gamma_v] = -2\gamma^0\gamma^3\,. \tag{3.10}$$

The initial-value problem with ψ given on a spacelike plane ($t' = 0$, say) is well known. Its unique solution is

$$\psi(x) = \int_{t'=0} S(x-x')\,\gamma_0\,\psi(x')\,\mathrm{d}^3x' \tag{3.11}$$

with $S = (\gamma\cdot\partial - m)\Delta$. Applying Gauss' theorem to the divergence of the divergence-free expression

$$S(x-x')\,\gamma_\mu\,\psi(x')\,,$$

yields, in complete analogy to the Klein-Gordon case,

$$\psi(x) = -\int_{CD} S(x-x')\gamma_u \psi(x')\,\mathrm{d}^2\boldsymbol{x}'\,\mathrm{d}u' - \int_{DB} S(x-x')\gamma_v \psi(x')\,\mathrm{d}^2\boldsymbol{x}'\,\mathrm{d}v' . \tag{3.12}$$

This equation corresponds to (3.5). Theorem 1 is therefore also applicable to the Dirac equation (3.8) with its unique solution (3.12).

The removal of CD to the distant past requires again

$$\lim_{v\to-\infty} \psi = 0 \qquad \text{for all } \boldsymbol{x},\ u > u_0 . \tag{3.13}$$

The solution is then given by the initial data on one null plane only ($u' = u_0$),

$$\psi(x) = -\int_{u'=u_0} S(x-x')\gamma_v \psi(x')\,\mathrm{d}^2\boldsymbol{x}'\,\mathrm{d}v' . \tag{3.14}$$

Theorem 2′. Given ψ on the null plane $u = u_0$ and the asymptotic condition (3.13), then the Dirac equation (3.8) has a unique solution given by (3.14) in the half-space $u > u_0$, valid for $m \geqslant 0$.

4. – Null transitions.

At this point it is not clear whether problems of physical interest with initial data on a null plane would yield unique solutions, because it may not be possible to satisfy the necessary asymptotic conditions on physical grounds.

In order to investigate this problem we recall the very basic condition of translation invariance common to all closed physical systems. Associated with this invariance is the existence of the four infinitesimal generators, *i.e.* the four-vector P. In particular, quantum dynamics is characterized by the unitary operator $\exp[iHt]$, where $H = P^0$ in the generator of time translations. This is the content of Heisenberg's equation of motion. It gives a complete specification of the dynamics provided the commutation relations are known. We shall now apply this idea to translations along null directions.

Translation invariance of the quantum field $\phi(x)$ implies

$$\phi(x) = \exp[-iP\cdot x]\phi(0)\exp[iP\cdot x] , \tag{4.1}$$

from which we can obtain the well-known differential form

$$\partial_\mu \phi(x) = i[\phi(x), P_\mu] . \tag{4.2}$$

Translations in the null direction m are therefore given by the null translation generator

$$P_u = -m\cdot P \tag{4.3}$$

and the equation

$$\phi(u, v, \boldsymbol{x}) = \exp[iP_u u]\phi(0, v, \boldsymbol{x})\exp[-iP_u u]\,. \tag{4.4}$$

The null-translation generator P_u is easily obtained by inspection of the classical limit of the Klein-Gordon field ϕ. One finds (see Appendix I)

$$P_u = \int :\partial\phi^\dagger\cdot\partial\phi + m^2\phi^\dagger\phi: \mathrm{d}^3\bar{x}\,. \tag{4.5}$$

This operator is self-adjoint and positive definite for the free field ϕ.

If we wish to translate off the null surface $u = u_0$ we need to know the commutation relations on that surface. From Appendix II (A-II.5),

$$[\phi(x), \phi(x')]_{u=u'} = 0\,, \qquad [\phi(x), \phi^\dagger(x')]_{u=u'} = -\frac{i}{4}\varepsilon(v-v')\delta_2(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{4.6}$$

This is a surprising result because it implies that the canonical commutator on the null plane is

$$[\phi(x), \pi(x')]_{u=u'} = \frac{i}{2}\delta_3(\bar{x}-\bar{x}')\,, \tag{4.7}$$

since

$$\pi(x) = \partial\phi^\dagger(x)/\partial v\,. \tag{4.8}$$

The unexpected factor $\frac{1}{2}$ in (4.7) is however not an error. It is due to the lack of independence of ϕ and its canonical conjugate, π, on a null plane. A modification of the usual canonical formalism taking this dependence into account is necessary. Such a formalism is not unknown in general relativity (6). Its use leads indeed to the commutation relation (4.7).

Returning to the computation of the u-derivative by means of (4.4) to (4.6) one has

$$\partial_u\phi(x) = i[\phi(x), P_u] = -\frac{1}{4}\int_{-\infty}^{\infty}\varepsilon(v-v')(\partial^2-m^2)\phi(u, v', \boldsymbol{x})\,\mathrm{d}v'. \tag{4.9}$$

(6) P. A. M. DIRAC: *Proc. Roy. Soc.*, A **246**, 326 (1958); P. G. BERGMANN and A. B. KOMAR: in *Recent Developments in General Relativity* (New York, 1962), p. 31. We are indebted to our colleague, J. GOLDBERG, for informative discussions on this point.

Apart from the operator character of ϕ this result is identical to (3.2) with $F(u, \boldsymbol{x}) = 0$ for all u. Thus, the equivalent to the asymptotic condition (3.6) is implied by the existence of the null translation.

The recovery of the Klein-Gordon equation by differentiation of (4.9) is trivial. Thus we are lead to the following result:

Theorem 3. Any representation of the commutation relations (4.6) by operator-valued distributions on Hilbert space for which P_u (4.5) is a self-adjoint operator and for which $[\phi, P_u]$ has a meaning, is a solution of the Klein-Gordon equation, given by (4.4).

We note that the commutation relations do not contain the mass; nor does the Hilbert space of the associated test functions. The latter involves the measure for the inner product in momentum space $\mathrm{d}^3\bar{p} = \mathrm{d}^2\boldsymbol{p}\,\mathrm{d}p_v/|p_v|$ (see *e.g.* (A-II.4)), which is independent of the mass. As was pointed out by KLAUDER, LEUTWYLER, and STREIT [4], this has the important consequence that, in contradiction to the commutation relations on a spacelike plane, the representations of (4.6) which refer to different masses are unitarily equivalent. The mass that appears in the Klein-Gordon equation (3.1) enters only through P_u, (4.5). We also note that the translations P_v and $\boldsymbol{P}$ in the null plane are mass-independent:

$$(4.10)\qquad P_v = 2\int :\partial_v\phi^\dagger\,\partial_v\phi:\mathrm{d}^3\bar{x}\,, \qquad \boldsymbol{P} = \int :\boldsymbol{\partial}\phi^\dagger\,\partial_v\phi + \partial_v\phi^\dagger\,\boldsymbol{\partial}\phi:\mathrm{d}^3\bar{x}\,,$$

as shown in Appendix I.

The quantum field-theoretic analogue of the classical initial-value problem on a null plane governed by Theorem 2 is simply the statement: The self-adjoint operator P_u, (4.5) together with the algebra (4.6) determines $\phi(x)$ uniquely for all x via (4.4) for any $\phi(0, v, \boldsymbol{x})$ of that algebra given on the plane $u = 0$.

This indicates the completeness of the field on the null plane, which is reflected by its irreducibility [4].

Null translations can also be carried out for the Dirac case. Here we find (see Appendix I)

$$(4.11)\qquad P_u = \tfrac{1}{2}\int :\bar{\psi}\gamma_u \overleftrightarrow{\partial_v}\psi:\mathrm{d}^3\bar{x}$$

and the rather complicated anticommutation relations (Appendix II)

$$(4.12)\qquad \{\psi(x),\ \psi(x')\}_{u=u'} = 0\ ,$$

$$(4.13)\qquad \{\psi(x), \bar{\psi}(x')\}_{u=u'} = \frac{i}{2}\gamma_u\delta_3(\bar{x}-\bar{x}') + \frac{i}{4}\varepsilon(v-v')(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}-m)\delta_2(\boldsymbol{x}-\boldsymbol{x}') + \\ + \frac{i}{16}\gamma_v(\boldsymbol{\partial}^2-m^2)\delta_2(\boldsymbol{x}-\boldsymbol{x}')\int\varepsilon(v-v'')\,\mathrm{d}v''\,\varepsilon(v''-v')\,.$$

First we observe the presence of so-called « Schwinger terms » in this relation. But this formal expression must of course be understood as a generalized function. Using test functions f and g in the three-space v, $\boldsymbol{x}$, we can write (4.13) more meaningfully as

$$\{\psi(f), \bar{\psi}(g)\}_{u=u'} = \frac{i}{2}\gamma_u(f, g) - \frac{4}{i}((\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)f, G) - \frac{i}{16}\gamma_v((\boldsymbol{\partial}^2 - m^2)F, G)\,, \tag{4.14}$$

where F is defined in terms of f by

$$F(v, \boldsymbol{x}) \equiv \int_{-\infty}^{v} - \int_{v}^{\infty} f(v', \boldsymbol{x})\,\mathrm{d}v'$$

and analogously G in terms of g.

Actually (4.13) gives more than necessary. Consider the anticommutators

$$\begin{cases} \{\psi(x), \bar{\psi}(x')\}_{u=u'}\gamma_v = \dfrac{i}{2}\gamma_u\gamma_v\delta_3(\bar{x} - \bar{x}') - \dfrac{i}{4}\gamma_v\varepsilon(v - v')(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\,\delta_2(\boldsymbol{x} - \boldsymbol{x}')\,, \\ \gamma_v\{\psi(x), \bar{\psi}(x')\}_{u=u'} = \dfrac{i}{2}\gamma_v\gamma_u\delta_3(\bar{x} - \bar{x}') + \dfrac{i}{4}\gamma_v\varepsilon(v - v')(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} - m)\,\delta_2(\boldsymbol{x} - \boldsymbol{x}')\,. \end{cases} \tag{4.15}$$

Equations (4.15) correspond to the canonical commutation relations since the canonical conjugate of ψ is $\pi = \bar{\psi}\gamma_v$ when the « time » is u and the Lagrangian is (A-I.9). We note again the « noncanonical » appearance of (4.15). The comments made in connection with (4.7) are also valid here.

Even more restrictive than (4.15) are the relations obtained by multiplying the first of these by γ_u on the left, the second by γ_u on the right, leaving only a single term on the right side:

$$\begin{cases} \gamma_u\{\psi(x), \bar{\psi}(x')\}_{u=u'}\gamma_v = -\dfrac{i}{4}\gamma_u\gamma_v\varepsilon(v - v')(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\,\delta_2(\boldsymbol{x} - \boldsymbol{x}')\,, \\ \gamma_v\{\psi(x), \bar{\psi}(x')\}_{u=u'}\gamma_u = -\dfrac{i}{4}\gamma_v\gamma_u\varepsilon(v - v')(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\,\delta_2(\boldsymbol{x} - \boldsymbol{x}')\,. \end{cases} \tag{4.16}$$

As we shall see below, only these anticommutation relations, rather than the more general ones in (4.13) are needed in the evaluation of the derivatives $\partial_u\psi$ and $\partial_u\bar{\psi}$.

But we also notice an explicit dependence of the anticommutator on the mass of the field. Clearly, a unitary equivalence of representations associated with different masses cannot be expected of this algebra.

By means of (4.16) it is easily verified that the null translation operators P_u, (4.11) and P_v,

$$P_v = \tfrac{1}{2}\int :\bar{\psi}\gamma_v \overset{\leftrightarrow}{\partial}_v \psi:\mathrm{d}^3\bar{x} \tag{4.17}$$

lead to the Dirac equation

$$\gamma_v\partial_u\psi = i\gamma_v[\psi(x), P_u] = \frac{i}{2}\int d^3\bar{x}'\gamma_v\{\psi(x), \bar{\psi}(x')\}\gamma_u\overleftrightarrow{\partial}_{v'}\psi(x') =$$

$$= \frac{1}{8}\gamma_v\gamma_u(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\int dv'\varepsilon(v-v')\overleftrightarrow{\partial}_{v'}\psi(u, v', \boldsymbol{x}) =$$

$$= \frac{1}{4}\gamma_v\gamma_u(\boldsymbol{\gamma}\cdot\boldsymbol{\partial} + m)\int\psi(u, v', \boldsymbol{x})\,\partial_v\varepsilon(v-v') = -\frac{1}{2}\gamma_v\gamma_u(\boldsymbol{\gamma}\cdot\boldsymbol{\partial}+ m)\psi(x)\,.$$

Similarly,

$$\gamma_u\partial_v\psi = i\gamma_u[\psi(x), P_v] = -\tfrac{1}{2}\gamma_u\gamma_v((\boldsymbol{\gamma}\cdot\boldsymbol{\partial}) + m)\,\psi(x)\,.$$

Combining these results and observing $\{\gamma_u, \gamma_v\} = -2$, which follows from (3.10), we find the Dirac equation (3.8). Thus we conclude with the following theorem.

Theorem 4. Any representation of the anticommutation relations (4.12) and (4.16) by operator-valued distributions on Hilbert space for which P_u and P_v of (4.11) and (4.17) are self-adjoint operators and for which $[\psi, P_u]$ and $[\psi, P_v]$ have a meaning, is a solution of the Dirac equation (3.8).

Of course, if we use the homogeneous Green function S instead of the null translation generators and represent the Cauchy solution by (3.14) then only

$$\psi_v(x) \equiv \gamma_v\psi(x)$$

is needed. Its anticommutation relations are

$$(4.18)\qquad \{\psi_v(x), \psi_v(x')\}_{u=u'} = 0\,,\qquad \{\psi_v(x), \bar{\psi}_v(x')\}_{u=u'} = i\gamma_v\delta_3(\bar{x}-\bar{x}')\,,$$

i.e. are *mass independent.* The more general commutator (4.13) then follows from (4.18) and (3.14). But then an asymptotic condition like (3.13) must be imposed.

5. – Interacting fields.

The considerations at the beginning of Sect. **3** for the free scalar field can easily be generalized to equations of the form

$$(5.1)\qquad 2\partial_u\partial_v\Phi(x) = L\Phi(x)\,,$$

where L is a linear differential operator which is symmetric and does not contain ∂_u. The latter condition is essential so that the integration over v yields an explicit expression for $\partial_u\Phi$ in terms of an integral over Φ in the null plane.

Despite this severe restriction there are field-theoretic problems of considerable physical interest of this type. With minimal electromagnetic coupling, $D_\mu \equiv \partial_\mu - ieA_\mu$, eq. (3.1) becomes

$$(D_u D_v + D_v D_u)\,\Phi = (\boldsymbol{D}^2 - m^2)\,\Phi\,. \tag{5.2}$$

This equation will be of the form (5.1) with L independent of ∂_u provided

$$A_v \equiv -n\cdot A = 0\,. \tag{5.3}$$

This condition is satisfied for a transverse external field moving in the z-direction so that $A = \boldsymbol{A}$. A laser field in the form of a wave train of finite length can be described this way with $\boldsymbol{A} = \boldsymbol{A}(u)$ and $\boldsymbol{A}(u) = 0$ for $|u| > u_0$. This would be a plane wave with infinite wave fronts in the $\boldsymbol{x}$ direction.

It is therefore not surprising that an exact solution for this system has been known for a long time ([7]), although it was not put in terms of null coordinates, where it takes its simplest and most tractable form ([1]).

This particular system also clarifies the physical significance of condition (3.6) and even suggests the stronger condition of making Φ compact in v. Otherwise the scalar electron described by Φ is not outside the laser beam at any finite times $t < t_0$ and $t > t_1$. Previous treatments of this problem have led to serious ambiguities because the interaction time was not finite. These matters will be discussed in detail in a separate publication devoted to quantum eletrodynamics in a laser beam ([8]).

Returning now to (5.1) and the importance of the requirement that L be independent of ∂_u, we consider the quantized problem corresponding to (5.2), *i.e.* the interaction with a quantized electromagnetic field. The condition (5.3) can be satisfied here only by choosing a special gauge. A gauge transformation $A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda$ can always be chosen so that one component of the potential, *e.g.* A'_v, vanishes. This is the gauge used also by KOGUT and SOPER ([5]). In a forthcoming paper ([8]) we shall show that this gauge, characterized by (5.3), is indeed necessary if one works with initial data on a null plane.

The Lagrangian method of Appendix I can be generalized to minimal electromagnetic interaction in a standard way. One finds for the null translation operator which translates Φ off the initial null plane $u = u_0$

$$P_u = \int \mathrm{d}^3\,\bar{x}[(\boldsymbol{D}\Phi)^\dagger\cdot\boldsymbol{D}\Phi + m^2\Phi^\dagger\Phi - ieA_u(\Phi^\dagger \overleftrightarrow{D}_v\Phi)]\,. \tag{5.4}$$

([7]) D. M. VOLKOV: *Zeits. Phys.*, **94**, 250 (1935).
([8]) R. A. NEVILLE and F. ROHRLICH: *Quantum electrodynamics in a laser beam* (to be published in *Phys. Rev.*).

Here $\overleftrightarrow{D}$ is defined as $\overrightarrow{D}-\overleftarrow{D}^\dagger$. Similarly, the null translation operator in the $u=$ const plane is

$$P_v=\int \mathrm{d}^3\bar{x}[2\partial_v\Phi^\dagger\partial_v\Phi+ieA_v(\Phi^\dagger\overleftrightarrow{\partial}_v\Phi)]\,. \tag{5.5}$$

These operators do not depend on ∂_u even though the gauge (5.3) has not been used in their derivation. However the associated translations (4.4) will not reproduce the field equation (5.2) unless the gauge choice (5.3) is made.

When A_μ is the quantized self-field or a superposition of that field with an external field, these expressions are no longer mathematically well-defined operators in Hilbert space. Interacting fields cannot be normal ordered in general. We shall take them here in the usual formal sense, in the spirit of the conventional (divergent) Lagrangian field theory.

In particular, one can go into the Dirac picture (interaction picture) by means of the usual formally unitary transformation and use the free-field commutation relations (4.6) on the null plane. Or, if one can solve the external field problem at hand exactly, one can go to the Furry picture and use the commutation relations which take into account the external field only. An example of the latter method will be given for the quantum electrodynamics of the laser beam ([8]).

For the Dirac picture one separates P_u

$$P_u=P_u^{(0)}+P_u^{(1)}, \tag{5.6}$$

and one finds for the interaction operator in the Dirac picture

$$P^{(D)}(u)=\int :[-ie\,a\cdot(\phi^+\overleftrightarrow{\partial}\phi)+e^2a^2\phi^+\phi]:\mathrm{d}^3\bar{x}\,. \tag{5.7}$$

Here $\phi(x)$ and $a(x)$ are the particle and electromagnetic fields in the Dirac picture (free fields). The usual Dyson solution for the scattering operator is then given by means of a positive-u–ordered (rather than time-ordered) expression:

$$S=U_+\exp\left[-i\int P^{(D)}(u)\,\mathrm{d}u\right]. \tag{5.8}$$

This is converted into normal-ordered products by Wick's theorem which carries over unchanged. In particular

$$\langle U_+(\phi(x)\phi^\dagger(x'))\rangle_0= \tag{5.9}$$

$$=\theta(u-u')\Delta_+(x-x')+\theta(u'-u)\Delta_+(x'-x)=\Delta_c(x-x')\,,$$

which is the usual causal propagator (in our normalization). The reason for this lies in the fact that for timelike x one has $\theta(\pm u)=\theta(\pm t)$, while for spacelike x the θ-functions in (5.9) combine to 1, since $\Delta_+(x)=\Delta_+(-x)$.

It follows that the Feynman rules can be taken over formally unchanged, but it will of course be convenient to use null co-ordinates in p-space as well as in x-space. The propagator in p-space then follows from Appendix II.

It seems rather fortuitous that the minimal electromagnetic coupling permits an easy elimination of the ∂_u terms of L in (5.1) by a suitable choice of gauge. It is clearly conceivable that there exist interactions in nature where the field equation (5.1) will contain ∂_u terms in L. The question of existence and uniqueness of solutions for such equations with initial data on a null plane is apparently open. But it seems that the corresponding null-translation operator P_u would then also have to contain ∂_u terms. This means that the transformation to the Dirac picture would yield an integral equation for $P_u^{(D)}(u)$ and not an explicit expression. If that integral equation can be solved by treating the ∂_u terms as a perturbation one would be led to

$$P_u^{(D)}(u) = g\sum_{n=0}^{\infty} g^n\, V^{(n)}(u)\,. \tag{5.10}$$

The « interaction Hamiltonian » in the Dyson S-matrix (5.8) would then be an infinite series in the coupling constant of this perturbation.

In any case, it is clear that the presence of ∂_u terms in L would lead to considerable complications.

Appendix I

Derivation of the null-translation operators.

In a classical field theory of a complex field ϕ_A with N components ($A=1, 2, \ldots, N$) the well-known theorem by Noether states that the translation invariance of the Lagrangian density $\mathscr{L}[\phi_A, \partial\phi_A]$ implies the existence of a divergence-free tensor

$$T_{\mu\nu} = \frac{\delta\mathscr{L}}{\delta\partial^\mu\phi_A}\,\partial_\nu\phi_A + \frac{\delta\mathscr{L}}{\delta\partial^\mu\phi_A^*}\,\partial_\nu\phi_A^* - g_{\mu\nu}\mathscr{L}\,. \tag{A-I.1}$$

This, in turn, implies a conserved vector P which generates the translations and which is defined in terms of an integral over a three-dimensional hypersurface with element $d^3\sigma$,

$$P_\mu \equiv \int T_{\alpha\mu}\, d^3\sigma^\alpha\,. \tag{A-I.2}$$

This theorem is quite general and is not restricted to the usual Minkowski diagonal metric. If the surface normal is n so that $d^3\sigma^\alpha = n^\alpha d^3\sigma$, then the projection of P on any vector ν is

$$\nu\cdot P=\int n\cdot T\cdot\nu\,\mathrm{d}^3\sigma\,. \tag{A-I.3}$$

With the metric (2.8) and the notation of that Section we have for the complex scalar field

$$\mathscr{L}=\partial_u\phi^*\,\partial_v\phi+\partial_v\phi^*\,\partial_u\phi-\boldsymbol{\partial}\phi^*\cdot\boldsymbol{\partial}\phi-m^2\phi^*\phi \tag{A-I.4}$$

and the well-known

$$T_{\mu\nu}=-\,\partial_\mu\phi^*\,\partial_\nu\phi-\partial_\nu\phi^*\,\partial_\mu\phi-g_{\mu\nu}\mathscr{L}\,. \tag{A-I.5}$$

The null momenta $P_u=-m\cdot P$ and $P_v=-n\cdot P$ then follow from (A-I.3) as integrals over the null plane $u=\text{const}$ with normal vector n given by (2.1) and $\mathrm{d}^3\sigma=\mathrm{d}^2\boldsymbol{x}\,\mathrm{d}v\equiv\mathrm{d}^3\overline{x}$

$$P_u=\int(\boldsymbol{\partial}\phi^*\cdot\boldsymbol{\partial}\phi+m^2\phi^*\phi)\,\mathrm{d}^3\overline{x}\,, \tag{A-I.6}$$

$$P_v=\int 2\partial_v\phi^*\,\partial_v\phi\,\mathrm{d}^3\overline{x}\,. \tag{A-I.7}$$

The two momenta $\boldsymbol{P}\equiv\hat{\boldsymbol{x}}\cdot P$ in the two space directions orthogonal to m and n are

$$\boldsymbol{P}=\int(\boldsymbol{\partial}\phi^*\,\partial_v\phi+\partial_v\phi^*\,\boldsymbol{\partial}\phi)\,\mathrm{d}^3\overline{x}\,. \tag{A-I.8}$$

For the quantized theory the translation operators of the free field are defined as the normal ordered products of functionals formally identical to the classical case. This leads to (4.5) and (4.10).

For the Dirac field the choice for the Lagrangian is

$$\mathscr{L}=-\,\tfrac{1}{2}\bar\psi(\overleftrightarrow{\boldsymbol{\partial}}-\gamma_u\overleftrightarrow{\partial}_v-\gamma_v\overleftrightarrow{\partial}_u)\psi-m\bar\psi\psi\,, \tag{A-I.9}$$

and the translation generators are in the notation of (3.9) and Sect. **2**,

$$P_u=\tfrac{1}{2}\int\bar\psi\gamma_u\overleftrightarrow{\partial}_v\psi\,\mathrm{d}^3\overline{x}\,, \tag{A-I.10}$$

$$P_v=\tfrac{1}{2}\int\bar\psi\gamma_v\overleftrightarrow{\partial}_v\psi\,\mathrm{d}^3\overline{x}\,, \tag{A-I.11}$$

$$\boldsymbol{P}=\tfrac{1}{2}\int\bar\psi\boldsymbol{\gamma}\overleftrightarrow{\partial}_v\psi\,\mathrm{d}^3\overline{x}\,. \tag{A-I.12}$$

APPENDIX II

The invariant function on a null plane.

The invariant function is defined by its Fourier representation

$$\Delta(x) = \frac{i}{(2\pi)^3}\int \exp[ip\cdot x]\,\varepsilon(p_0)\,\delta(p^2+m^2)\,\mathrm{d}^4p\,. \tag{A-II.1}$$

Null co-ordinates in p-space yield

$$\begin{cases} p_u = -m\cdot p = \dfrac{1}{\sqrt{2}}(p^0+p^3)\,, \\ p_v = -n\cdot p = \dfrac{1}{\sqrt{2}}(p^0-p^3)\,, \end{cases} \tag{A-II.2}$$

so that on the mass shell

$$\frac{p_u}{p_0} = \frac{1}{\sqrt{2}}\left(1+\frac{p^3}{p^0}\right) > 0\,, \qquad \frac{p_v}{p_0} > 0\,. \tag{A-II.3}$$

Therefore $\Delta(x)$ can be written, using the notation of Sect. 2,

$$\Delta(x) = \frac{i}{(2\pi)^3}\int \exp[i\boldsymbol{p}\cdot\boldsymbol{x} - ip_u u - ip_v v]\,\varepsilon(p_v)\,\delta(\boldsymbol{p}^2+m^2-2p_u p_v)\,\mathrm{d}^2\boldsymbol{p}\,\mathrm{d}p_v\,\mathrm{d}p_u\,.$$

For $p_v > 0$ and $p_v < 0$ this expression yields $\Delta_+(x)$ and $\Delta_-(x)$ in null co-ordinates. The p_u integration is easily done, yielding

$$\Delta(x) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{d}^2\boldsymbol{p}\int_0^{\infty}\frac{\mathrm{d}p_v}{|p_v|}\exp[i\boldsymbol{p}\cdot\boldsymbol{x}]\sin\left[p_v v + \frac{\boldsymbol{p}^2+m^2}{|2p_v|}u\right]. \tag{A-II.4}$$

If we put $u = 0$,

$$\Delta(x)|_{u=0} = \tfrac{1}{4}\varepsilon(v)\,\delta_2(\boldsymbol{x})\,. \tag{A-II.5}$$

Because of the symmetry in u and v, an analogous calculation using $\varepsilon(p_0) = \varepsilon(p_u)$ and integrating first over p_v yields

$$\Delta(x)|_{v=0} = \tfrac{1}{4}\varepsilon(u)\,\delta_2(\boldsymbol{x})\,. \tag{A-II.6}$$

We note that $\varepsilon(v)$ in (A-II.5) (and similarly $\varepsilon(u)$ in (A-II.6)) is defined by the integral

(A-II.7) $$\varepsilon(v) = \frac{2}{\pi}\int_0^\pi \frac{\sin p_v v}{p_v}\,\mathrm{d}p_v\,,$$

so that it is not only specified for $v > 0$ and $v < 0$ as $\varepsilon(v) = +1$ and -1, but also for $v = 0$, giving $\varepsilon(0) = 0$.

The scalar free field has the commutation relations

(A-II.8) $$[\phi(x), \phi^\dagger(x')] = -i\Delta(x - x')\,.$$

Restriction of this equation to the $u = u'$ hyperplane now yields the result (4.6) in view of (A-II.5).

For the spinor field we must restrict

(A-II.9) $$\{\psi(x), \bar{\psi}(x')\} = iS(x - x')\,,$$

where $S(x) = (\gamma\cdot\partial - m)\Delta(x)$, to the $u = u'$ plane. Since Δ must satisfy the same Klein-Gordon equation as ϕ in (4.9) we have

(A-II.10) $$\partial_u \Delta(x)|_{u=0} = -\frac{1}{4}\int_{-\infty}^{\infty}\varepsilon(v - v')(\boldsymbol{\partial}^2 - m^2)\Delta(u, v', \boldsymbol{x})|_{u=0}\,\mathrm{d}v' = -\frac{1}{16}\int_{-\infty}^{\infty}\varepsilon(v - v')\,\varepsilon(v')\,\mathrm{d}v'\cdot(\boldsymbol{\partial}^2 - m^2)\,\delta_2(\boldsymbol{x})\,.$$

The restriction of $S(x)$ to $u = 0$ is therefore

(A-II.11) $$S(0, v, \boldsymbol{x}) = (\gamma\cdot\boldsymbol{\partial} - m - \gamma_u\partial_v - \gamma_u\partial_u)\Delta(u, v, \boldsymbol{x})|_{u=0} = \frac{1}{4}(\gamma\cdot\boldsymbol{\partial} - m)\varepsilon(v)\,\delta_2(\boldsymbol{x}) + \frac{1}{2}\gamma_u\delta_3(\overline{x}) + \frac{1}{16}\gamma_v(\boldsymbol{\partial}^2 - m^2)\,\delta_2(\boldsymbol{x})\int_{-\infty}^{\infty}\varepsilon(v - v')\,\varepsilon(v')\,\mathrm{d}v'.$$

The result (4.13) now follows from this and (A-II.9).

● RIASSUNTO (*)

Si considera in dettaglio per le equazioni di Klein-Gordon e Dirac il problema dei valori iniziali per le equazioni differenziali iperboliche con i dati iniziali in un piano nullo nello spazio di Minkowski. Si espongono i teoremi di esistenza e unicità. L'analogo per la

(*) *Traduzione a cura della Redazione.*

teoria dei campi quantici comporta l'uso di relazioni di commutazione sul piano nullo e dell'operatore di traslazione nulla fuori di quel piano. Si enuncia brevemente la teoria formale dei campi interagenti, nello spirito della teoria di Feynman-Dyson. Si mettono in evidenza le complicazioni ulteriori che sorgono dalla derivata della coordinata nulla del campo nella direzione al di fuori del piano iniziale.

Квантовая теория поля вне нулевых плоскостей.

Резюме (*). — Для уравнений Клейна-Гордона и Дирака подробно рассматривается проблема начальных значений для гиперболических дифференциальных уравнений с заданными начальными данными на нулевой плоскости в пространстве Минковского. Приводятся теоремы существования и единственности. Теоретический аналог квантового поля включает коммутационные соотношения на нулевой плоскости и оператор нулевых трансляций вне этой плоскости. В смысле Фейнмана-Дайсона вкратце формулируется формальная теория взаимодействующих полей. Отмечается, что взаимодействие, которое включает нулевую координатную производную поля в направлении от начальной плоскости, приводит к дополнительным усложнениям.

(*) *Переведено редакцией.*