Conformal Group in Minkowsky Space. Unitary Irreducible Representations (*).

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Summary. — The three fundamental series of unitary irreducible representations of the Lie algebra of the conformal group in Minkowsky space are described. We use the homomorphism between conformal group and the group $G_{2,2}$ of linear transformations which leave invariant the quadratic form $-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2$.

1. – Introduction.

The conformal group in Minkowsky space has been studied from various points of view by several authors [1-4]. This group is an extension of the proper inhomogeneous Lorentz group, which includes dilatation (change of scale) and uniform accelerations. It is a simple Lie group of rank three and order 15, and it is the lowest-order semi-simple group containing the inhomogeneous

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(1) E. Cunningham: *Proc. Lond. Math. Soc.*, **8**, 77 (1910).

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(3) J. A. Schouten: *Rev. Mod. Phys.*, **21**, 421 (1949).

(4) Y. Murai: *Prog. Theor. Phys.*, **11**, 441 (1954).

Lorentz group. This property and the fact that some important field equations are invariant under its transformations, are well known arguments giving physical importance to the conformal group.

From a quite general point of view, independently of any field-theoretical description of interacting particles, it may be noted that, in principle, using as starting data only the measurements of angles in space-time in collision phenomena (that is angles between pairs of ingoing or outgoing particles and their β-values) we can determine the mass-ratios of the involved particles and also check the validity of the conservation principles associated with Lorentz-invariance. The physical results obtained only with this type of angular observations (in adimensional form) are clearly invariant under the conformal group. This remark is a further argument showing that the conformal group could have an important role in physics, even if it is a critical question if really all physics can be deduced from observations on asymptotic states only.

MURAI ([5]) has tried the classification of the unitary irreducible representations of the conformal group extending a method used by THOMAS for the De Sitter group. Unfortunately this extension seems to be uncorrect, as we show below.

For the groups of type A_l the complete set of commuting operators ([6]) which can be used for classifying the basic functions of the irreducible representations contains $2l+l(l-1)/2$ operators. l are the invariants of A_l characterizing the irreducible representation; the remaining are conveniently chosen ([7]) as the l infinitesimal operators H_i defining the rank, and the $l(l-1)/2$ invariants of the sugroups in the chain $A_{l-1}\supset A_{l-2}\supset \dots \supset A_2\supset A_1$. For the groups of type D_l, the complete set of commuting operators ([6]) contains $l+l(l-1)$ operators, l of which are the invariants of D_l and the remaining $l(i-1)$ are usually taken ([8]) as the invariants of the subgroups in the chain

$$B_{l-1}\supset D_{l-1}\supset B_{l-2}\supset \dots \supset B_1\supset D_1 .$$

In this chain only the invariant of D_1 is linear.

For the case of the conformal group since it is equivalent to the group of real linear transformations on a six-dimensional space, leaving invariant the quadratic form $-x_1^2-x_2^2+x_3^2+x_4^2+x_5^2+x_6^2$, the corresponding algebra is of type $A_3\equiv D_3$.

The number of independent commuting operators is then 9. In the classification given by MURAI, it is implicitly assumed that 5 operators besides the

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([5]) Y. MURAI: *Prog. Theor. Phys.*, **9**, 147 (1953).

([6]) G. RACAH: *Group Theory and Spectroscopy* (Princeton, 1951), p. 49.

([7]) M. GEL'FAND and M. L. ZETLIN: *Dokl. Akad. Nauk S.S.S.R.*, **71**, 825 (1950).

([8]) M. GEL'FAND and M. L. ZETLIN: *Dokl. Akad. Nauk S.S.S.R.*, **71**, 1017 (1950).

3 invariants, suffice to specify completely the basic functions. The choice of these 5 operators does not correspond to any of the two schemes given above and it is very difficult (if not impossible) to find some other set of 6 independent commuting operators containing these five.

In the following Section, we show the homomorphism between the conformal group and the group $G_{2,2}$ of the series of real forms $G_{p,q}$. It is known [9] that these groups have $q+1$ fundamental series of unitary irreducible representations. The three fundamental series d_0, d_1, d_2 of the conformal group are described in Sect. 3.

2. – Homomorphism between $G_{2,2}$ and the conformal group in Minkowsky space.

We shall start by considering the well-known homomorphism between the proper six-dimensional rotation group R_6 which leaves invariant the real quadratic form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \tag{2.1}$$

and the four-dimensional unitary unimodular group $SU(4)$ which leaves invariant the quadratic form

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = z^\dagger z \,.$$

Let $\mathscr{A}$ be the set of 4×4 complex matrices such that

$$A_{ij} = - A_{ji} = \tfrac{1}{2}\varepsilon_{ijkl}\bar{A}_{kl} \tag{2.2}$$

the matrices $A\in\mathscr{A}$ depend clearly on 6 real parameters:

$$\left\{\begin{aligned} &A_{12} = x_1 + ix_2\,, && A_{34} = \bar{A}_{12} = x_1 - ix_2\,,\\ &A_{13} = -\bar{A}_{24} = x_3 + ix_4\,, && A_{14} = \bar{A}_{23} = x_5 + ix_6\,. \end{aligned}\right. \tag{2.3}$$

The transformation

$$A' = UA\tilde{U} \tag{2.4}$$

is such that $A'\in\mathscr{A}$ if $A\in\mathscr{A}$ and $U\in SU(4)$; in fact, A' is clearly antisymmetric and in order to show that it satisfies (2.2) it will suffice to restrict our-

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[9] M. L. GRAEV: *Dokl. Akad. Nauk S.S.S.R.*, **98**, 517 (1954).

selves to the case of $U=1+B$ with B infinitesimal ($\tilde{B}=-\bar{B}$); we have

$$\tfrac{1}{2}\varepsilon_{ijkl}\overline{A'_{kl}}=\tfrac{1}{2}\varepsilon_{ijkl}(\overline{A}_{kl}+\overline{B}_{kr}\overline{A}_{rl}+\overline{A}_{ks}\overline{\tilde{B}}_{sl})=$$
$$=A_{ik}+\tfrac{1}{4}\varepsilon_{ijkl}\varepsilon_{pqrl}\overline{B}_{kr}A_{pq}=A_{ij}+(A\tilde{B})_{ij}+(BA)_{ij}=A'_{ij}\,.$$

The tranformation (2.4) induces a linear real transformation R_u on the six parameters x_i, and since

$$\operatorname{Tr}(A'^{\dagger}A')=\operatorname{Tr}(A^{\dagger}A)=4(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)\,, \tag{2.5}$$

R_u conserves $\sum x_i^2$; furthermore it is a proper transformation since it can be transformed into the identity by a continuous change in U, so $R_u\in R_6$.

The property

$$A''=VA'\tilde{V}=(VU)A(VU)^{\sim}$$

shows that the application $U\to R_u$ is an homomorphism. It is not difficult to find the unitary matrices corresponding to the 15 elementary rotations, for instance

$$\begin{Vmatrix} e^{i\alpha/2} & 0 & 0 & 0\\ 0 & e^{-i\alpha/2} & 0 & 0\\ 0 & 0 & e^{-i\alpha/2} & 0\\ 0 & 0 & 0 & e^{i\alpha/2}\end{Vmatrix}\to\begin{Vmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&\cos\alpha&-\sin\alpha\\0&0&0&0&\sin\alpha&\cos\alpha\end{Vmatrix},$$

$$\begin{Vmatrix}\cos\alpha/2 & -\sin\alpha/2 & 0 & 0\\ \sin\alpha/2 & \cos\alpha/2 & 0 & 0\\ 0 & 0 & \cos\alpha/2 & -\sin\alpha/2\\ 0 & 0 & \sin\alpha/2 & \cos\alpha/2\end{Vmatrix}\to\begin{Vmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&\cos\alpha&0&-\sin\alpha\\0&0&0&0&1&0\\0&0&0&\sin\alpha&0&\cos\alpha\end{Vmatrix},$$

etc.; clearly $SU(4)$ is the covering group of R_6. Let us consider now the group $G_{2,2}$ of 4×4 unimodular matrices which leave invariant the quadratic form

$$-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2=z^\dagger s z$$

with

(2.6)
$$s=\begin{Vmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{Vmatrix}.$$

If $M\in G_{2,2}$, $z'^\dagger s z'=z^\dagger M^\dagger s M z=z^\dagger s z$, so that

(2.7)
$$M^\dagger s M = s \quad \text{or} \quad s M^\dagger s M = 1\,, \qquad \det M = 1\,.$$

In complete analogy with the preceding case, the set of antisymmetric matrices with the condition

(2.8)
$$A_{ij}=-A_{ji}=\tfrac{1}{2}\varepsilon_{ijkl}(\overline{s\bar{A}s})_{kl}\,,$$

transforms in itself under the transformation

$$A'=MA\tilde{M}$$

if $M\in G_{2,2}$.

To prove it we consider again $M=1+B$ with B infinitesimal:

$$s\tilde{B}s=-\bar{B}\,,$$

then

$$\tfrac{1}{2}\varepsilon_{ijkl}(sA's)_{kl}=\tfrac{1}{2}\varepsilon_{ijkl}\left[(\overline{s\bar{A}s})_{kl}+(s\bar{B}s)_{kr}(s\bar{A}s)_{rl}+(s\bar{A}s)_{ks}(\overline{s\tilde{B}s})_{sl}\right]=$$
$$=A_{ij}+\tfrac{1}{4}\varepsilon_{ijkl}\varepsilon_{rlpq}(s\bar{B}s)_{kr}A_{pq}=A_{ij}+(A\tilde{B})_{ij}+(BA)_{ij}=A'_{ij}\,.$$

We have now, defining A_{12}, A_{13}, A_{14} as in (2.3):

(2.9)
$$\mathrm{Tr}\,(sA'^\dagger sA')=\mathrm{Tr}\,(sA^\dagger sA)=-4(-x_1^2-x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)\,,$$

which proves the homomorphism between the conformal group and $G_{2,2}$.

3. – Unitary irreducible representations of $G_{2,2}$.

Graev [9] has given the general form of the unitary irreducible representation of the groups $G_{p,q}$, defined as groups of unimodular complex matrices of order $p+q$ $(p \geq q)$ leaving invariant the nondegenerated hermitean form

$$-|z_1|^2-|z_2|^2-\ldots-|z_p|^2+|z_{p+1}|^2+\ldots+|z_{p+q}|^2.$$

In his work, GRAEV found $q+1$ types d_0, d_1, ..., d_q of fundamental series of irreducible representations.

In the following we explicitate the formulas for the fundamental series d_0, d_1, d_2 of the group $G_{2,2}$.

a) The series d_0 is associated with the realization of the group obtained taking as invariant hermitean form the expression

$$z^\dagger s z = |\bar{z}_1 \bar{z}_2 \bar{z}_3 \bar{z}_4| \cdot \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{vmatrix} . \tag{3.1}$$

If g is an arbitrary matrix of the group, we write

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} ,$$

where g_{ij} is a 2×2 matrix.

The invariance of $z^\dagger s z$ implies $g^\dagger s g = s$, that is

$$\begin{cases} g_{11}^\dagger g_{11} - g_{21}^\dagger g_{21} = 1 , \\ g_{12}^\dagger g_{12} - g_{22}^\dagger g_{22} = -1 , \\ g_{11}^\dagger g_{12} - g_{21}^\dagger g_{22} = 0 . \end{cases} \qquad (\mathrm{Det}\, g = 1). \tag{3.2}$$

The representations are given in the Hilbert space of the functions $f(x, y, t)$ of the three complex 2×2 matrices x, y, t satisfying the following conditions:

α) For y, f fixed, j is a homogeneous polynomial in x_{11}, x_{12} of a given degree m_1; and for x, t fixed, f is a homogeneous polynomial in y_{11}, y_{12} of a given degree m_2.

β) f is an analytical function in the four elements of t in the region R where $1-tt^\dagger$ is definite positive. Another part of d_0 is obtained taking f as antianalytical function in t.

γ) For a given integer r, the integral

$$\|f\|^2 = \int |fu[(1-t^\dagger t)^{-\frac{1}{2}}, v(1-tt^\dagger))^{-\frac{1}{2}}, t]|^2 \cdot (\det(1-tt^\dagger)^{r-4} \cdot \mathrm{d}\mu(u)\mathrm{d}\mu(v)\mathrm{d}\mu(t), \tag{3.3}$$

converges, where u and v are 2×2 unitary unimodular matrices; $\mathrm{d}\mu(u)$, $\mathrm{d}\mu(v)$ the corresponding invariant measures, $\mathrm{d}\mu(t)$ the product of the differentials of the real and imaginary parts of the elements of t. The integration extends on the compact spaces of the matrices u and v, and on the region R defined in β).

The conditions α), β), γ) fix the three parameters m_1, m_2, r characterizing the different irreducible representations.

The operator of the representation is given by

$$T_g f(x, y, t) = f[x(\bar{g}_{11} + \tilde{t}\bar{g}_{21}), y(tg_{12} + g_{22}), (tg_{12} + g_{22})^{-1}(tg_{11} + g_{21})] \cdot \\ \cdot (\det(tg_{12} + g_{22}))^{-r}. \tag{3.4}$$

In order to give the explicit form of the infinitesimal generators we put

$$g = 1 + \varepsilon\alpha, \quad \varepsilon \text{ real} \ll 1 \qquad \text{and} \qquad g_{ij} = 1 + \varepsilon\alpha_{ij}.$$

From (3.2):

$$\left|\begin{array}{l} \alpha_{11}^\dagger + \alpha_{11} = 0, \\ \alpha_{12} - \alpha_{21}^\dagger = 0, \\ \alpha_{22}^\dagger + \alpha_{22} = 0, \end{array}\right. \qquad \operatorname{Tr}\alpha_{11} + \operatorname{Tr}\alpha_{22} = 0. \tag{3.5}$$

The transformation of f by $g = 1 + \varepsilon\alpha$ becomes now

$$\begin{aligned} T_g f(x, y, t) &= f\big[x(1 + \varepsilon\bar{\alpha}_{11}) + \tilde{t}\varepsilon\bar{\alpha}_{21}), y(t\varepsilon\alpha_{12} + 1 + \varepsilon\alpha_{22}), (t\varepsilon\alpha_{12} + 1 + \varepsilon\alpha_{22})^{-1} \cdot \\ &\quad \cdot \big(t(1 + \varepsilon\alpha_{11}) + \varepsilon\alpha_{21}\big)\big]\left[\det(t\varepsilon\alpha_{12} + 1 + \varepsilon\alpha_{22})\right]^{-r} = \\ &= f[x + \varepsilon(x\bar{\alpha}_{11} + x\tilde{t}\bar{\alpha}_{21}), y + \varepsilon(y\alpha_{22} + yt\alpha_{12}), t + \varepsilon(t\alpha_{11} + \alpha_{21} - t\alpha_{12}t - \alpha_{22}t)] \cdot \\ &\quad \cdot [1 - \varepsilon r \cdot \operatorname{Tr}(\alpha_{22} + t\alpha_{12})]. \end{aligned} \tag{3.6}$$

If T_α is the operator of the representation corresponding to the infinitesimal generator α, we have

$$\begin{aligned} T_\alpha &= x(\bar{\alpha}_{11} + \tilde{t}\bar{\alpha}_{21}) \cdot \frac{\partial}{\partial x} + y(\alpha_{22} + t\alpha_{12}) \cdot \frac{\partial}{\partial y} + \\ &\quad + (t\alpha_{11} + \alpha_{21} - t\alpha_{12}t - \alpha_{22}t) \cdot \frac{\partial}{\partial t} - r\operatorname{Tr}(\alpha_{22} + t\alpha_{12}), \end{aligned} \tag{3.7}$$

where

$$A \cdot \frac{\partial}{\partial B} = \Sigma_{ij} A_{ij} \frac{\partial}{\partial B_{ij}} .$$

We can conveniently take the following realization of the 15 independent infinitesimal operators, in agreement with (3.5):

$$(3.8) \quad \begin{cases} A_i = \dfrac{1}{2}\begin{vmatrix} e_i & 0 \\ 0 & e_i \end{vmatrix}, & B_\mu = -\dfrac{1}{2}\begin{vmatrix} e_\mu & 0 \\ 0 & -e_\mu \end{vmatrix}, \\ C_\mu = -\dfrac{i}{2}\begin{vmatrix} 0 & e_\mu \\ e_\mu & 0 \end{vmatrix}, & D_\mu = \dfrac{1}{2}\begin{vmatrix} 0 & e_\mu \\ -e_\mu & 0 \end{vmatrix}, \end{cases}$$

with

$$e_\mu^2 = -1 , \quad e_i e_j = e_k , \quad e_0 = i \qquad (\mu = 0, 1, 2, 3;\ i, j, k = 1, 2, 3).$$

From the commutation relations deduced by (3.8) we can obtain the following correspondence with the physical co-ordinate transformations in Minkowsky space:

$$A_i \to -\left(x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}\right),$$

$$C_i \to -\left(x_i \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_i}\right),$$

$$C_0 \to x_\mu \frac{\partial}{\partial x_\mu} \qquad \text{(dilatation)},$$

$$\frac{B_\mu + D_\mu}{\sqrt{2}} \to \frac{\partial}{\partial x_\mu},$$

$$\frac{B_i - D_i}{\sqrt{2}} \to \frac{1}{2} x_\mu x^\mu \frac{\partial}{\partial x_i} - x_i x_\mu \frac{\partial}{\partial x_\mu} \qquad \text{(spatial acceleration)},$$

$$\frac{B_0 - D_0}{\sqrt{2}} \to -\frac{1}{2} x_\mu x^\mu \frac{\partial}{\partial x_0} - x_0 x_\mu \frac{\partial}{\partial x_\mu} \qquad \text{(temporal acceleration)}.$$

The representation of (3.8) using the eq. (3.7) becomes

$$(3.9) \quad \begin{cases} T_{A_i} = \dfrac{1}{2}\left[x\bar{e}_i \cdot \dfrac{\partial}{\partial x} + y e_i \cdot \dfrac{\partial}{\partial y} + (te_i - e_i t) \cdot \dfrac{\partial}{\partial t}\right], \\ T_{B\mu} = -\dfrac{1}{2}\left[x\bar{e}_\mu \cdot \dfrac{\partial}{\partial x} - y e_\mu \cdot \dfrac{\partial}{\partial y} + (te_\mu + e_\mu t) \cdot \dfrac{\partial}{\partial t} + r \operatorname{Tr}(te_\mu)\right], \\ T_{C\mu} = \dfrac{i}{2}\left[x\tilde{t}\bar{e}_\mu \cdot \dfrac{\partial}{\partial x} - y t e_\mu \cdot \dfrac{\partial}{\partial y} - (e_\mu - te_\mu t) \cdot \dfrac{\partial}{\partial t} + r \operatorname{Tr}(te_\mu)\right], \\ T_{D\mu} = \dfrac{1}{2}\left[x\tilde{t}\bar{e}_\mu \cdot \dfrac{\partial}{\partial x} + y t e_\mu \cdot \dfrac{\partial}{\partial y} - (e_\mu + te_\mu t) \cdot \dfrac{\partial}{\partial t} - r \operatorname{Tr}(te_\mu)\right]. \end{cases}$$

The three commuting operators corresponding to the three diagonal matrices in (3.8) are $\left(\text{choosing } e_3 = \begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}\right)$

$$T_{A_3} = \frac{i}{2}\left[-x_{11}\frac{\partial}{\partial x_{11}} + x_{12}\frac{\partial}{\partial x_{12}} + y_{11}\frac{\partial}{\partial y_{11}} - y_{12}\frac{\partial}{\partial y_{12}} + 2\left(t_{21}\frac{\partial}{\partial t_{12}} - t_{12}\frac{\partial}{\partial t_{12}}\right)\right],$$

$$T_{B_3} = \frac{i}{2}\left[\quad x_{11}\frac{\partial}{\partial x_{11}} - x_{12}\frac{\partial}{\partial x_{12}} + y_{11}\frac{\partial}{\partial y_{11}} - y_{12}\frac{\partial}{\partial y_{12}} - 2\left(t_{11}\frac{\partial}{\partial t_{11}} - t_{22}\frac{\partial}{\partial t_{22}}\right)\right],$$

$$T_{B_0} = \frac{i}{2}\left[\quad x_{11}\frac{\partial}{\partial x_{11}} + x_{12}\frac{\partial}{\partial x_{12}} + y_{11}\frac{\partial}{\partial y_{11}} + y_{12}\frac{\partial}{\partial y_{12}} - \right.$$
$$\left. - 2\left(t_{11}\frac{\partial}{\partial t_{11}} + t_{12}\frac{\partial}{\partial t_{12}} + t_{21}\frac{\partial}{\partial t_{21}} + t_{22}\frac{\partial}{\partial t_{22}}\right) - 2r\right].$$

Being

$$x_{11}\frac{\partial}{\partial x_{11}} + x_{12}\frac{\partial}{\partial x_{12}} + y_{11}\frac{\partial}{\partial y_{11}} + y_{12}\frac{\partial}{\partial y_{12}} = m_1 + m_2,$$

for the functions of the space of the representation, we have

$$T_{B_0} = \frac{i}{2}[m_1 + m_2 - 2r] - i\left(t_{11}\frac{\partial}{\partial t_{11}} + t_{12}\frac{\partial}{\partial t_{12}} + t_{21}\frac{\partial}{\partial t_{21}} + t_{22}\frac{\partial}{\partial t_{22}}\right). \tag{3.10}$$

Choosing the basis so that T_{B_0} is diagonal, (3.10) shows that the functions of the basis are homogeneous polynomials in t_{11}, t_{12}, t_{21}, t_{22}. In a similar way can be treated the part of the series d_0 corresponding to the case where f is antianalytical in t.

b) The series d_1 is obtained starting from the realization of $G_{2,2}$ which leaves invariant

$$z^\dagger s' z = z^\dagger \cdot \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \cdot z = z^\dagger \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & -\sigma_3 & 0 \\ 1 & 0 & 0 \end{vmatrix} \cdot z. \tag{3.11}$$

The matrices g leaving invariant (3.11) are related by a similarity transformation M with the matrices which leave invariant (3.1); M is the matrix for which $Ms'M^{-1} = s$.

We write every $g \in G_{2,2}$, according to the structure of the matrix s', as

$$g = \begin{Vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{Vmatrix} .$$

(where for instance g_{21} is a 2×1 rectangular matrix).

Invariance of (3.11) implies $g^\dagger s' g = s'$; for $g = 1 + \varepsilon\alpha$, ε real $\ll 1$, this gives

$$\alpha_{31}^\dagger + \alpha_{31} = 0 ,$$

$$\alpha_{32}^\dagger - \sigma_3 \alpha_{21} = 0 , \qquad \alpha_{22}^\dagger \sigma_3 + \sigma_3 \alpha_{22} = 0 ,$$

$$\alpha_{33}^\dagger + \alpha_{11} = 0 , \qquad \alpha_{23}^\dagger \sigma_3 - \alpha_{12} = 0 , \qquad \alpha_{13}^\dagger + \alpha_{13} = 0 .$$

The representation is given in a Hilbert space of functions $f(t, z)$ where t is a complex number, and z is a triangular matrix of the group $G_{2,2}$, having the form

$$z = \begin{Vmatrix} 1 & 0 & 0 \\ z_{21} & 1 & 0 \\ z_{31} & z_{32} & 1 \end{Vmatrix} .$$

The function f must be analytical (or antianalitical) in t in the region $|t| < 1$, and the integral

$$\|f\|^2 = \int |f(t, z)|^2 \cdot (1 - t\bar{t})^{r-2} \, d\mu(t) \, d\mu(z) ,$$

must converge for a given integer r.

The integration extends on the region $|t| < 1$, and on the whole range of the parameters of the matrix z.

In order to describe the operator T_g of the representation, we note that the product zg can be univocally written as $zg = k\hat{z}$, where $\hat{z}$ is a matrix of $G_{2,2}$ having the same structure of z, and k is a triangular matrix of $G_{2,2}$ of the form:

$$k = \begin{Vmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & k_{33} \end{Vmatrix} .$$

The operator of the representation is given by

$$(3.12)\qquad T_g f(t, z) = T_{k22} f(t, \hat{z})\cdot k_{33}^{-r'}|k_{33}|^{r'+i\varrho-2} =$$
$$= f\left[\left(t(k_{22})_{12} + (k_{22})_{22}\right)^{-1}\cdot\left(t(k_{22})_{11} + (k_{22})_{21}\right), \hat{z}\right]\cdot\left(t(k_{22})_{12} + (k_{22})_{22}\right)^{-r}\cdot k_{33}^{-r'}\cdot|k_{33}|^{r'+i\varrho-2}.$$

The explicit expression of $\hat{z}$, k_{22}, k_{33} are:

$$(3.13)\qquad \begin{cases} k_{33} = z_{31}g_{13} + z_{32}g_{23} + g_{33}\,, \\ \hat{z}_{32} = k_{33}^{-1}(z_{31}g_{12} + z_{32}g_{22} + g_{32})\,, \\ k_{22} = z_{21}g_{12} + g_{22} - (z_{21}g_{13} + g_{23})\cdot\hat{z}_{32}\,, \\ \hat{z}_{31} = k_{33}^{-1}(z_{31}g_{11} + z_{32}g_{21} + g_{31})\,, \\ \hat{z}_{21} = k_{22}^{-1}\left(z_{21}g_{11} + g_{21} - (z_{21}g_{13} + g_{23})\hat{z}_{31}\right)\,. \end{cases}$$

The parameters characterizing the unitary irreducible representation, are r, r', ϱ; (r, r' integer, ϱ real).

We limit ourselves to give the expression of the three commuting operators corresponding to the diagonal forms of α (being $g = 1 + \varepsilon\alpha$):

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \qquad B = \tfrac{1}{2}\begin{vmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{vmatrix}, \qquad C = \begin{vmatrix} 0 & 0 & 0 \\ 0 & i\sigma_3 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

Being z an element of $G_{2,2}$, the condition (3.11) implies that it depends only on five real parameters.

We can take for instance

$$(3.14)\qquad z = \begin{vmatrix} 1 & 0 & 0 & 0 \\ (z_{21})_1 & 1 & 0 & 0 \\ (z_{21})_2 & 0 & 1 & 0 \\ z_{31} & (z_{32})_1 & (z_{32})_2 & 1 \end{vmatrix} \equiv$$
$$\equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ x_2 + ix_3 & 1 & 0 & 0 \\ x_4 + ix_5 & 0 & 1 & 0 \\ \frac{1}{2}(x_2^2 + x_3^2 - x_4^2 - x_5^2) + ix_1 & x_2 - ix_3 & -x_4 + ix_5 & 1 \end{vmatrix}.$$

The expressions for T_A, T_B, T_C deduced from (3.12), (3.13), (3.14) are:

$$(3.15)\quad \begin{cases} T_A f(t; x_1, x_2, x_3, x_4, x_5) = \\ \qquad = \left[2x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2} + x_3 \dfrac{\partial}{\partial x_3} + x_4 \dfrac{\partial}{\partial x_4} + x_5 \dfrac{\partial}{\partial x_5} - (i\varrho - 2)\right]\cdot f\,, \\ T_B f(t; x_1, x_2, x_3, x_4, x_5) = \\ \qquad = i\left[x_2 \dfrac{\partial}{\partial x_2} + x_3 \dfrac{\partial}{\partial x_3} + x_4 \dfrac{\partial}{\partial x_4} + x_5 \dfrac{\partial}{\partial x_5} + \dfrac{r - r'}{2}\right]\cdot f\,, \\ T_C f(t; x_1, x_2, x_3, x_4, x_5) = \\ \qquad = i\left[2t \dfrac{\partial}{\partial t} - x_2 \dfrac{\partial}{\partial x_2} - x_3 \dfrac{\partial}{\partial x_3} + x_4 \dfrac{\partial}{\partial x_4} + x_5 \dfrac{\partial}{\partial x_5} + r\right]\cdot f\,. \end{cases}$$

c) The series d_2 corresponds to the realization of $G_{2,2}$ with the matrix of the quadratic form

$$s'' = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} \equiv \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

The elements $g \in G_{2,2}$ can be written in the form

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} \qquad \text{with } g_{ij} \quad 2\times 2 \text{ matrices.}$$

The representations are defined in the space of the functions $f(z, t)$, where $z = x + iy$ is a complex number, t is an antihermitean 2×2 matrix

$$t = \begin{vmatrix} it_1 & t_3 + it_4 \\ -t_3 + it_4 & it_2 \end{vmatrix} \qquad \text{and} \qquad \int |f|^2\, dx\, dy\, dt_1\, dt_2\, dt_3\, dt_4 < \infty\,.$$

In this space the operators of the representations are given by

$$(3.16)\qquad T_g f(z, t) = f(\hat{z}, \hat{t})\cdot \alpha(z, k_{22})\cdot |\det k_{22}|^{i\varrho}\,,$$

where

$$k_{22} = tg_{12} + g_{22}\,, \qquad \hat{t} = k_{22}^{-1}(tg_{11} + g_{21})\,, \qquad \hat{z} = \frac{(k_{22})_{11}z + (k_{22})_{21}}{(k_{22})_{12}z + (k_{22})_{22}}\,,$$

$$\alpha(z, k_{22}) = |(k_{22})_{12}z + (k_{22})_{22}|^{-m+\varrho'-2}\big((k_{22})_{12}z + (k_{22})_{22}\big)^m\,.$$

Each representation is characterized by the three numbers ϱ, ϱ', m being m integer and ϱ, ϱ' real.

As for the series d_1, we give the generators corresponding to the three diagonal matrices of the Lie algebra

$$A=\frac{1}{2}\begin{vmatrix} e_3 & 0 \\ 0 & e_3 \end{vmatrix}, \quad B=\frac{i}{2}\begin{vmatrix} e_3 & 0 \\ 0 & -e_3 \end{vmatrix}, \quad C=\frac{1}{2}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}, \quad e_3=\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix},$$

$$T_A=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+t_4\frac{\partial}{\partial t_3}-t_3\frac{\partial}{\partial t_4}+\frac{im}{2},$$

$$T_B=\quad x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-t_1\frac{\partial}{\partial t_1}+t_2\frac{\partial}{\partial t_2}-\frac{\varrho'-2}{2},$$

$$T_C=\quad t_1\frac{\partial}{\partial t_1}+t_2\frac{\partial}{\partial t_2}+t_3\frac{\partial}{\partial t_3}+t_4\frac{\partial}{\partial t_4}-\frac{\varrho'-2}{2}-i\varrho .$$

The calculation of the other generators and of the invariants for the three series is straightforward, but more involved. A more interesting point will be to look for possible physical meanings of these irreducible representation; the variety of the mathematical structure of the three series d_0, d_1, d_2 seems indeed very promising.

RIASSUNTO

Sono descritte le tre serie fondamentali di rappresentazioni irriducibili dell'algebra di Lie del gruppo conforme nello spazio di Minkowsky. Si fa uso dell'omomorfismo tra il gruppo conforme e il gruppo $G_{2,2}$ delle trasformazioni lineari che lasciano invariante la forma quadratica $-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2$.