IL NUOVO CIMENTO VOL. XXV, N. 3 1° Agosto 1962

Covariant Quantization of the Gravitational Field.

JOHN R. KLAUDER

Department of Physics University - Bern
Bell Telephone Laboratories Murray Hill - N. J.

(ricevuto il 28 Marzo 1962)

Summary. — In any quantum theory, in which the metric tensor of Einstein's gravitational theory is also quantized, it becomes meaningless to ask for an initial space-like surface on which to specify the conventional field commutators. The covariant quantum formalism, in which all fields either commute or fail to do so only when the field's points coincide, is proposed as being suitable to quantize gravity. The extension of the covariant quantum formalism to general boson fields that interact in an intrisically nonlinear way with external fields is analysed in some detail. This formalism is applied to the case of the free gravitational field. In a functional representation, the measure on metrics is found to be that proposed by Misner. A basic state of the quantized gravitational theory is proposed, which involves a summation over all permissible metrics in the entire space-time manifold.

1. – Introduction.

Interest in the quantization of the Einstein gravitational field stems from a variety of reasons. These range i) from the modest desire for completeness in the quantization of known fields and a curiosity about intrinsically nonlinear fields, ii) to a belief that divergence difficulties in electrodynamics, etc., may be alleviated if the space-time metric is quantized, and thus the divergence-bearing sharp light cone is smeared, iii) to the speculation that quantized space-time is itself a sufficiently rich structure to represent nature. The quantization of the gravitational field is being studied with a variety of approaches (1). BERGMANN and co-workers emphasize strongly the importance

(1) See, for example: *Proc. of the Conference on the Role of Gravitation in Physics*, C. and B. DEWITT Editors (University of North Carolina, 1957). For more recent work see the forthcoming volume *The Theory of Gravitation*, L. WITTEN Editor, (New York).

of isolating the true observables of the theory in analogy with accepted quantum mechanical practice ([2]). ARNOWITT, DESER and MISNER in a series of papers have re-expressed general relativity in terms of independently-specifiable, co-ordinate-transformation-invariant canonical variables, secured by a repeated elimination of the constraint equations ([3]). The latter formulation is closely related to that of Dirac based on a recent extension ([4]) of his well-known Hamiltonian formalism with constraints ([5]). In many such treatments general covariance is forsaken in order to exhibit the unique role assigned to the time co-ordinate that is demanded by Hamiltonian formalisms. Even in cases where general covariance is explicitly maintained the basic quantum-mechanical postulates still remain logically equivalent tot hose of the conventional Hamiltonian formalism ([6]).

In so far as these formalisms are transcriptions of techniques successful in a flat Lorentz space-time, they ignore a unique problem peculiar to general relativity. Conventional field theories deal, in particular, with commutation rules, which, when employed for the fields separated by a space-like interval, have an especially simple form. Whether two nearby points are or are not space-like is a *metric*-question that can be asked (and in principle answered) not only in a flat space but also in any space with a preassigned curved metric as well. However as soon as the space-time metric $g_{\mu\nu}(x)$ becomes a dynamical variable—as in Einstein's theory—then an initial space-like surface on which to specify commutators of any two fields becomes a meaningless concept. This impossibility to define an initial space-like surface for commutators arises, in particular, in the quantum theory of the free gravitational field, where the only field is then the metric tensor. We devote our attention in this paper to this simplest example of a generally covariant quantum theory, that of the free gravitational field.

In an effort to circumvent the problem introduced by the absence of space-like surfaces, we seek an alternative quantum formulation, especially one with different commutation rules. An appropriate formalism, suitable for our purposes, has been discussed by several authors ([7-11]) with respect to its appli-

([2]) P. G. BERGMANN: *Rev. Mod. Phys.*, **33**, 510 (1961).

([3]) R. ARNOWITT, S. DESER and C. W. MISNER: *Phys. Rev.*, **116**, 1322 (1959), and following papers.

([4]) P. A. M. DIRAC: *Proc. Roy. Soc. London*, A **246**, 333 (1958); *Phys. Rev.*, **114**, 924 (1959).

([5]) P. A. M. DIRAC: *Can. Journ. Math*, **2**, 129 (1950).

([6]) B. S. DEWITT: *Journ. Math. Phys.*, **2**, 151 (1961), and preprint.

([7]) J. SCHWINGER: *Proc. Nat. Acad. Sci.*, **37**, 452 (1951).

([8]) Y. NAMBU: *Progr. Theor. Phys. (Japan)*, **4**, 331, 399 (1949).

([9]) F. COESTER: *Phys. Rev.*, **95**, 1318 (1954).

([10]) J. M. JAUCH: *Helv. Phys. Acta*, **29**, 287 (1954).

([11]) J. G. VALATIN: *Proc. Roy. Soc. London*, A **229**, 221 (1955).

cations to Lorentz-invariant theories. However, as we shall see, it is a simple task to apply such methods successfully to generally covariant theories as well [12].

The essence of this form of quantum theory that we shall adopt can be briefly stated as follows [11,13]:

A) All operator fields either commute everywhere or fail to commute only at the same space-time point, *i.e.*, if the two points of field evaluation are coincident.

B) Dynamics is added to this space by requiring that all physically acceptable « dynamical » vectors belong to an invariant subspace annihilated by an appropriate Hermitian combination of field operators.

As a simple example of this covariant formalism let us briefly consider its application to an Hermitian scalar field $\varphi(x)$ in a Lorentz space. For all points x and y this field satisfies

$$[\varphi(x), \varphi(y)] = 0\,, \tag{1.1}$$

consistent with postulate *A*).

Besides φ we introduce an Hermitian field $\pi(x)$ which commutes everywhere with itself, but together with φ satisfies [14]

$$[\varphi(x), \pi(y)] = -i\,\delta(x-y)\,, \tag{1.2}$$

with $\delta(x)$ a four-dimensional δ-function.

The form for the Hermitian dynamical operator of postulate *B*) follows from the conventional action principle in an « external field ». As a general example, consider the Hermitian action sum

$$I\{\varphi, \pi\} = I\{\varphi\} + \int \varphi\cdot\pi\,\mathrm{d}x\,, \tag{1.3}$$

where the dot signifies an Hermitian symmetrizing operation:

$$\varphi\cdot\pi = \tfrac{1}{2}(\varphi\pi + \pi\varphi)\,.$$

The term $I\{\varphi\}$ represents the unperturbed action, *e.g.*, that appropriate to a free particle field of rest mass m, perhaps involving in addition a self-inter-

(12) For a preliminary account of this work see: J. R. KLAUDER: *Nuovo Cimento*, **19**, 1059 (1961).

(13) J. V. NOVOZILOV and A. V. TULUB: *Forts. d. Phys.*, **6**, 50 (1958).

(14) We choose units such that $\hbar = c = 16\pi G = 1$.

action term proportional to φ^4, etc. The dynamical Hermitian operator of interest for postulate B) is defined by

$$i\,[\pi(x),\, I\{\varphi, \pi\}]\,. \tag{1.4}$$

Assume for the moment that φ and π were classical c-number fields and, apart from a factor i, the brackets in (1.2) and (1.4) were « Poisson brackets ». Here, we have made a generalization of the ordinary Poisson brackets for fields such that the bracket between a field and its « conjugate » field is proportional to a *four*-dimensional δ-function. In this classical analogue, then, a *constraint* requiring eq. (1.4) to vanish generates the classical equations of motion. Finally, when these fields are quantized the constraint requiring (1.4) to vanish becomes a subsidiary condition imposed on acceptable dynamical vectors, in accord with standard procedures ([15]). (This heuristic argument serves only to make plausible the postulated dynamical constraint equation. It can, of course, be derived as a consequence of the conventional quantum-mechanical formalism for the Lorentz-invariant theories.)

All of the dynamical statements are contained in the constraint

$$i[\pi(x),\, I\{\varphi, \pi\}]\,|\Omega) = 0\,, \tag{1.5}$$

which all dynamical vectors $|\Omega)$ are required to satisfy. Solutions to (1.5) may be written in the form

$$|\Omega) = \exp\left[i\,I\{\varphi\}\right]|\omega_0)\,, \tag{1.6}$$

where $|\omega_0)$ is the eigenvector of π with eigenvalue zero: $\pi(x)\,|\omega_0) = 0$. Inasmuch as (1.5) involves a differential operator, various solutions for $|\Omega)$ in (1.6) arise for various supplementary boundary conditions. Of these solutions it is convenient to select one—commonly the vacuum-vacuum transition element for the present example—for further analysis. Call this choice $|\Omega_0)$; all physical information is contained therein. In particular, fo rour example, the inner product $(\omega_\pi|\Omega_0)$ of $|\Omega_0)$ with $|\omega_\pi)$, an eigenvector of the operator $\pi(x)$, is equivalent to the Schwinger T-product generating functional ([7]). Interpretations then may proceed along standard lines ([16]).

Equations (1.2) and (1.5) can be taken as a postulational basis for a quantum theory ([10,11]), which is therefore quite symmetric in its treatment of space and time. Equation (1.2), with which (1.5) is evaluated, is a commutation rule not valid simply on a space-like surface but valid everywhere; it « asks »

([15]) E. Fermi: *Rev. Mod. Phys.*, **4**, 125 (1932).

([16]) H. Lehmann, K. Symanzik and W. Zimmermann: *Nuovo Cimento*, **1**, 214 (1954).

simply whether the points x and y coincide or not, which is not a metric question. The extension of this covariant postulational basis to the gravitational field is the subject of the present paper. Our work is somewhat related to that of KLEIN ([17]) who proposed commutation rules such as (1.2) for various fields directly from the point of view of general covariance. Following up the work of KLEIN, LAURENT ([18]) suggested that eq. (1.2), applied to symmetric tensor fields, should have a bearing on the quantum theory of gravitation. However both KLEIN and LAURENT consider nonhermitian subsidiary conditions, analogous to (1.5), involving principally annihilation-like operators. This non-Hermitian choice is of course suggested by the Gupta-Bleuler approach to the electromagnetic field. While this difference is in part a matter of choice ([19]) we feel that the applicability of Hermitian subsidiary constraints in the covariant formalism under discussion is well substantiated by the correspondence of the above and other Lorentz-covariant examples to conventional quantum mechanics ([13]). At the present stage of development it seems preferable to ask for a functional representation directly in terms of the field of interest (Schrödinger representation), rather than a representation in terms of an increasing number of bare quanta, *i.e.*, gravitons (Fock representation). KLEIN and LAURENT do not consider any particular realizations of the operators and vectors they discuss.

The necessary extension of the covariant quantization formalism involves one new feature not heretofore treated. It is a conventional assumption that the field under study (for example, φ above) enters linearly into the interaction Lagrangian (see (1.3)). If we choose $g^{\mu\nu}$ as basic gravitational variables, then the very nature of the gravitational field dictates the interaction term

$$\int v_{\mu\nu}\cdot g^{\mu\nu}\sqrt{-g}\,\mathrm{d}x\,, \tag{1.7}$$

where v_{μ} is dynamically independent of $g^{\mu\nu}$. In (1.7) it is seen that $g^{\mu\nu}$ does not enter linearly. An alternate possibility is to adopt $g_{\mu\nu}$ as basic; then the

([17]) O. KLEIN: in *Niels Bohr and the Development of Physics* (London, 1955), p. 96.

([18]) B. E. LAURENT: *Ark. for Fys.* (*Sweden*), **16**, 237 (1959).

([19]) The operator $-\lambda\,\partial/\partial x$ is Hermitian in the Schrödinger representation, but it acts as a shift operator on a state

$$\Psi=\sum_{n=0}^{\infty}a_n x^n\,,$$

which, for example, represents a generating function for amplitudes in an harmonic oscillator basis. The interpretation of x is of course quite different in these cases, as is the representation of the inner product in Hilbert space.

interaction term

$$\int w^{\mu\nu}\cdot g_{\mu\nu}\sqrt{-g}\,\mathrm{d}x\,, \tag{1.8}$$

is appropriate where $w^{\mu\nu}$ is taken dynamically independent of the metric tensor. Again the metric does not enter linearly. There is naturally a close relationship between these two descriptions. In order to be able to discuss and compare these different descriptions of the gravitational field, we extend the conventional covariant quantization formalism in Section **2** to examples wherein the field of interest does not enter the interaction term linearly.

The extension developed in Section **2** is particularly interesting in regard to a functional realization of the Hilbert space in question. In this representation either the metric tensor or its conjugate ($v_{\mu\nu}$ or $w^{\mu\nu}$) are taken as diagonal, *i.e.*, acting as multiplication on functionals. The conjugate to the diagonal variable is then represented by functional differentiation. Non-linear interaction terms such as (1.7) or (1.8) will have an influence on both the form taken by the functional differentiation and on the formal resolution of unity in terms of the eigenvectors of the diagonal operator. Furthermore, the form taken by the resolution of unity has a direct bearing on the question of the « measure on metrics » in a Feynman sum-over-histories formulation of quantized gravity. The measure we find (eq. (3.16)) is identical to that found by MISNER [20] from invariance arguments, and by LAURENT [21] from a transformation Jacobian. We emphasize that in the present analysis the form assumed by the measure on metrics is a consequence of the form of the interaction term, say eq. (1.7). The realization of the Hilbert space in terms of functionals is discussed in Section **2** in a general way, and is applied to the gravitational field in Section **3**.

In Section **4** an important four-dimensional physical state vector for the gravitational field is discussed that treats all metrics equivalently. Finally a direct operator approach to covariant quantization is suggested by means of a distribution analysis of the basic four-dimensional commutation relations and of the dynamical constraint.

2. – General properties of the covariant quantum formalism.

We shall discuss the quantization of a general set of boson fields, $f_A(x)$, $A=1, 2, ..., N$, which we distinguish by a subscript A, or any capital Latin letter, taking the values 1 to N. For conventional tensor fields A stands for

---

([20]) C. W. MISNER: *Rev. Mod. Phys.*, **29**, 497 (1957).

([21]) B. E. LAURENT: *Ark. for Fys. (Sweden)*, **16**, 279 (1959).

the collective space-time indices of the tensor. If we adopt the summation convention for these indices as well, then a general action functional in the presence of « external fields » denoted by X^A, is simply

$$I\{f, X\} = I\{f\} + \int F_A[f(y)]\cdot X^A(y)\,\mathrm{d}y\,. \tag{2.1}$$

$I\{f\}$ represents the action functional in the absence of external fields, and F_A signifies a set of invertable functions of the N fields f_A at a point. As in (1.3), the dot denotes an hermitization operation.

The generalized four-dimensional commutation rules are

$$\big[F_A[f(x)], X^B(y)\big] = -i\delta_A^B\,\delta(x-y) \tag{2.2}$$

and

$$\big[f_A(x), f_B(y)\big] = \big[X^A(x), X^B(y)\big] = 0\,.$$

We adopt an Hermitian operator form of the conventional equations of motion and constrain the physical state vectors to be their null eigenvectors:

$$\left\{\frac{\delta I\{f\}}{\delta f_A(x)} + \frac{\partial F_B[f(x)]}{\partial f_A(x)}\cdot X^B(x)\right\}|\Omega) = 0\,. \tag{2.3}$$

The basic field variables $X^B(x)$ define a formal set of simultaneous eigenvectors $|\omega_{X'})$, such that

$$X^B(x)|\omega_{X'}) = X'^B(x)|\omega_{X'})\,, \tag{2.4}$$

where X' is the c-number eigenfield.
The matrix element of direct physical interest is

$$(\omega_{X'}|\Omega)\,, \tag{2.5}$$

which we subject to the normalization condition $(\omega_0|\Omega)=1$ when $X'=0$. We now proceed to analyse eqs. (2.2)–(2.4) so as to study (2.5) further.

The field X^B is dynamically independent of F_A and of any function thereof. In particular X^B is independent of the basic fields f_A themselves. It follows that the commutator of X^B with any function of f_A can be at most a function of f_A. Therefore (2.2) always implies

$$(\partial F_A/\partial f_C)\,[f_C(x), X^B(y)] = -i\,\delta_A^B\,\delta(x-y)\,. \tag{2.6}$$

Since F_A is assumed invertable the matrix $\partial F_A/\partial f_C$ has an inverse, say $\partial f_A/\partial F_C$.

Hence

$$[f_C(x),\, X^B(y)] = -\, i(\partial f_C/\partial F_B)\,\delta(x-y)\,,$$

or equivalently

$$[f_C(x),\, X^B(y)]\,(\partial F_B/\partial f_A)(y) = -\, i\,\delta^A_C\,\delta(x-y)\,.$$

Bringing the transformation matrix within the commutator we find

$$[f_C(x),\, \chi^A(y)] = -\, i\,\delta^A_C\,\delta(x-y)\,, \tag{2.7}$$

where the Hermitian operator

$$\chi^A \equiv (\partial F_B/\partial f_A)\cdot X^B\,. \tag{2.8}$$

According to (2.2), we may say that F_A is conjugate to X^A; we now see as a consequence that f^A is conjugate to the Hermitian operator χ^A. The pair of conjugate variables F_A and X^B are unitarily related to the conjugate pair f_A and χ^B.

The question arises as to the connection between eq. (2.3) and a possible alternate choice, which is also at first sight seemingly valid. Consider the constraint

$$\left\{\frac{\delta I\{f\}}{\delta F_C(x)} + X^C(x)\right\}|\tilde{\Omega}) = 0\,. \tag{2.9}$$

This equation demands also that some vector, $|\tilde{\Omega})$, be the eigenvector of an Hermitian operator. It is just the quantum transcription of the classical equations of motion assuming F_C itself to be the « basic » field. Equation (2.9) is not simply unitarily equivalent to (2.3) because for this purpose it would be necessary to change F into f and X into χ everywhere they appear. In order to show the relation between (2.3) and (2.9) we derive an equation of the form of (2.9) directly from (2.3). This derivation will be very useful in evaluating (2.5) as well.

From (2.3) it follows that

$$\left\{\frac{\delta I\{f\}}{\delta F_C(x)} + \frac{\partial f_A}{\partial F_C}\,\chi^A\right\}|\Omega) = 0\,, \tag{2.10}$$

which differs from (2.9) in that the last operator is not Hermitian. The desired Hermitian operator X^C is just

$$X^C = (\partial f_A/\partial F_C)\cdot\chi^A \tag{2.11}$$

which is a consequence of the general rule

$$(X^A \cdot B) \cdot C = X^A \cdot (BC)$$

valid for any B and C which are functions of f alone. With (2.11), eq. (2.10) becomes

$$\left\{\frac{\delta I\{f\}}{\delta F_C(x)} + X^C(x) + \frac{1}{2}\left[\frac{\partial f_A}{\partial F_C}, \chi^A\right]\right\}|\Omega) = 0 . \tag{2.12}$$

The non-hermitian part is now displayed in the commutator. This term is proportional to $\delta(x-x) = \delta(0)$, a formal, infinite factor, but its functional form is of more interest. Thus

$$\left[\frac{\partial f_A}{\partial F_C}, \chi^A\right] = \frac{\partial F_D}{\partial f_A}\left[\frac{\partial f_A}{\partial F_C}, X^D\right] = -i\delta(0)\frac{\partial F_D}{\partial f_A}\frac{\partial^2 f_A}{\partial F_D \partial F_C},$$

and on interchanging the D and C derivatives,

$$\left[\frac{\partial f_A}{\partial F_C}, \chi^A\right] = -i\delta(0)\left(\frac{\partial F_D}{\partial f_A}\frac{\partial}{\partial F_C}\left(\frac{\partial f_A}{\partial F_D}\right)\right) = \tag{2.13}$$

$$= -i\delta(0)\frac{\partial}{\partial F_C}\ln|\partial f/\partial F| = [\ln|\partial f/\partial F|, X^C],$$

where

$$|\partial f/\partial F| \equiv \det[\partial f_A(x)/\partial F_B(x)] .$$

Therefore the additional term in (2.12) is a gradient (or a commutator with respect to X^C). This suggests defining

$$|\Omega) = H|\tilde{\Omega}) , \tag{2.14}$$

where $H = \Pi_x H(x)$ is a functional of the operator f_A alone. The factor H fails to commute only with X^C in (2.12), and therefore

$$H\left\{\frac{\delta I}{\delta F_C} + X^C\right\}|\tilde{\Omega}) + \{[X^C, H] + \tfrac{1}{2}H[\ln|\partial f/\partial F|, X^C]\}|\tilde{\Omega}) = 0 . \tag{2.15}$$

If we let

$$H(x) = |\partial f/\partial F|^{\frac{1}{2}} ,$$

then the last bracket in (2.15) vanishes.

Since

$$H = \Pi_x |\partial f/\partial F|^{\frac{1}{2}} \equiv |Df/DF|^{\frac{1}{2}} \tag{2.16}$$

is nonsingular, eq. (2.15) leads to (2.9). Thus we have deduced (2.9) from (2.3) and at the same time related the state $|\Omega)$ to the state $|\tilde{\Omega})$. Now H, although adjusted to be unity in the special case $F_A = f_A$, is certainly not a unitary transformation. Rather H arises from the transformation to new variables. Such a factor has an analogue in many elementary problems, one of which we now illustrate.

The inner product of states in an elementary one-particle system has the form

$$\langle\psi_1|\psi_2\rangle = \int \psi_1^*(r)\, r^2\, \mathrm{d}r\, \psi_2(r)\,, \tag{2.17}$$

when expressed in spherical co-ordinates (radial part only). The representation of the radial momentum operator

$$p_r = -\, i r^{-1}(\partial/\partial r) r \tag{2.18}$$

is Hermitian in this form of inner product. It is often very convenient to define « wave functions » $u(r) \equiv r\psi(r)$, in which case (2.17) becomes simply

$$\int u_1^*(r)\, \mathrm{d}r\, u_2(r)\,. \tag{2.19}$$

In this inner product $p_r = -\, i\, \partial/\partial r$. One might be tempted then to introduce in the abstract one-particle Hilbert space a « state »

$$|u\rangle \equiv r\,|\psi\rangle\,, \tag{2.20}$$

where here r is an operator. However an attempt to realize the inner product of two such states would, from (2.17), give

$$\langle u_1|u_2\rangle = \int u_1^*(r) r^2\, \mathrm{d}r\, u_2(r)\,,$$

in contradiction with (2.19). Instead an equation like (2.20) *signifies a change in the weight function* taking place in that realization of the Hilbert space by functions in which r is diagonal, *i.e.*, where r acts simply as multiplication. It can be argued that neither realization is strictly correct, but that one form (here (2.17)) is more useful to study (rotational) invariance properties, while the other form (here (2.19)) is perhaps more useful for computations.

We now identify relation (2.14) as an analogue of (2.20), namely that both states $|\Omega)$ and $|\tilde{\Omega})$ have the same normalization, it is only their representations which differ, a difference readily displayed in a representation which diagonalizes $f_A(x)$.

Let us, therefore, introduce formal states $|f')$ in which the operators f_A and F_A are diagonal:

$$(2.21)\qquad \begin{cases} f_A(x)\,|f') = f'_A(x)\,|f') \\ F_A(x)\,|f') = F'_A(x)\,|f') \equiv F_A[f'(x)]\,|f') \,. \end{cases}$$

We further adopt the state $|\Omega)$ as « more fundamental » than $|\tilde{\Omega})$ (analogous to $|\psi\rangle$ in preference to $|u\rangle$). Therefore we are interested in a realization of the Hilbert space for the states $|\Omega)$, which we define as

$$(2.22)\qquad (\Omega_1|\Omega_2) = \int \Omega_1^*(f)\,H^{-1}\,\mathscr{D}f\,H\,\Omega_2(f)\,.$$

Here H simply stands for a c-number functional of f_A like that given by (2.16), and $\mathscr{D}f$ signifies a measure on histories yet to be determined. It is clear according to (2.14) that the identification $\tilde{\Omega}(f) = \Omega(f)/H(f)$ leads to the alternate form (analogous to (2.19))

$$(2.23)\qquad (\Omega_1|\Omega_2) = \int \tilde{\Omega}_1^*(f)\,H\,\mathscr{D}f\,\tilde{\Omega}_2(f)\,.$$

We study the measure $\mathscr{D}f$ on f-histories by using the eigenstates $|\omega_X)$, defined in eq. (2.4), which are appropriate for the operator X^A. These vectors also provide a realization of the Hilbert space, which we take in the form

$$(2.24)\qquad (\Omega_1|\Omega_2) = \int \Omega_1^*(X)\,DX\,\Omega_2(X)\,.$$

where DX is translationally invariant:

$$(2.25)\qquad DX = D(X + X') \propto \Pi_{x,A}\,\mathrm{d}X^A(x)\,.$$

(The symbol D as part of a measure will always be used along with a translationally invariant measure.) The uniform weighting in eqs. (2.24) and (2.25) may be justified in several ways. For example, such a weighting is appropriate in conventional theories with a simple interaction term $F_A(f) \equiv f_A$. Since the nonlinearity enters in the field f_A and has nothing to do with the test field X^A, the appropriate measure (2.25) should remain unchanged. Alternately we can observe that our interacting classical field X^A must ultimately be produced by some external system. Justification for a nonquantum treatment of this system, as with all test apparatus, is that its inertial aspects are enormous; it only disturbs the quantum system and is not disturbed by it. When the inertial aspects become increasing larger, only smaller and smaller deviations from equilibrium are important. Thus the limiting action functional for a test system is simply quadratic in the probe field, an adequate representation for small deviations. As is well known, a translationally invariant history measure is appropriate for actions quadratic in the fields ([22]).

Equation (2.24) has a direct bearing on the measure on f-histories. Adopting a convenient shorthand notation

$$FX = \int F_A(x)\, X^A(x)\, \mathrm{d}x \;,$$

etc., and putting aside questions of normalization, we introduce a Fourier transformation over histories

$$\Omega(F) = \int \exp\,[iFX]\,\Omega(X)\,DX \;,$$

and obtain for (2.24)

$$(\Omega_1\,|\,\Omega_2) = \int \Omega_1^*(F)\,DF\,\Omega_2(F)\;. \tag{2.26}$$

The transformation from F_A to f_A introduces a Jacobian,

$$DF = |DF/Df|\,Df = H^{-2}\,Df\;,$$

and comparison with (2.22) shows that

$$\mathscr{D}f = H^{-1}Df \propto \Pi_x\,|\partial F/\partial f|^{\frac{1}{2}}\Pi_A\,\mathrm{d}f_A(x)\;, \tag{2.27}$$

which apart from a normalization factor expresses the form of the needed measure. We now study the physically important matrix element $(\omega_{X'}|\,\Omega)$, and discover as well the reason for our interest in the unusual measure $\mathscr{D}f$.

The solution to eq. (2.3) can be written in the form

$$|\,\Omega) = \exp\,[\,iI\{f\}]\,|\,\tilde{\omega}_0) \tag{2.28}$$

where the necessary constraint is now

$$\chi^A(x)\,|\,\tilde{\omega}_0) = 0\;,$$

i.e., $|\tilde{\omega}_0)$ is an eigenvector of χ^A with eigenvalue zero, whence the subscript.

([22]) For the particular external field variables discussed in Section **3**.1 the validity of eqs. (2.24) and (2.25) can be explicitly verified by more conventional procedures.

A solution of similar form may be written for eq. (2.9), namely

$$|\tilde{\Omega}) = \exp[iI\{f\}]|\omega_0) \,, \tag{2.29}$$

where the vector $|\omega_0)$ satisfies simply

$$X^c(x)|\omega_0) = 0 \,,$$

and is thus one of the vectors defined in (2.4). The relation (2.14) between $|\Omega)$ and $|\tilde{\Omega})$ together with (2.28) and (2.29) imply that

$$|\tilde{\omega}_0) = H|\omega_0) \,, \tag{2.30}$$

and thus, that the eigenstate $|\tilde{\omega}_0)$ is related through the operator H to the state $|\omega_0)$ whose representation properties we know.

Combination of (2.28) and (2.30) determines our matrix element of interest to be

$$(\omega_X|\Omega) = (\omega_X|\exp[iI\{f\}]H|\omega_0) \,. \tag{2.31}$$

The proper representation of the states in this matrix element is clearly given by (2.24), and thus by (2.22).

Because

$$(\omega_X|f) = \omega_X^*(f) \propto \exp[iFX]$$

it then follows that

$$(\omega_X|\Omega) = N^{-1}\int \exp[iFX + iI\{f\}]\mathscr{D}f \,, \tag{2.32}$$

N representing a cumulative normalization factor determined by the requirement $(\omega_0|\Omega) = 1$. Note that the factor H in (2.31) cancelles the H^{-1} in (2.22) leaving just the measure $\mathscr{D}f$ defined by (2.27).

Equation (2.32) represents the desired resolution of the physical matrix element in terms of a realization of the Hilbert space by f-history functionals. The form of (2.32) is not unlike that of the Feynman sum-over-histories. However important differences in interpretation should be observed. The integrals in (2.32) extend over *all* space-time; no variable boundary values are preserved to characterize « the propagator ». Instead the « label » in (2.32) is provided by the test function X^A with which the system interacts. Equation (2.32) therefore represents the interaction of the quantized field f_A with the entire external system. The form taken by the quantization—notably the form of $\mathscr{D}f$—is dictated by the interaction term FX. It is thus dictated by the way in which the « conjugate variables » enter into the action functional.

This is of course in agreement with the conventional time development form of the sum-over-histories.

It remains to discuss (2.32) in the light of the resolutions in (2.22) and (2.23). Why did a result arise which seems to be intermediate to both of these choices? This arose simply because, according to (2.29) it is the vector $|\tilde{\Omega})$ which through $|\omega_0)$, is directly related to the ket of interest, $|\omega_X)$. The alternate matrix element

$$(\omega_X|\tilde{\Omega})$$

would be expressed in a more conventional form like (2.32) with an additional factor H^{-1}. The choice of this latter matrix element, however, fails to recognize that the basic variable is f_A and not F_A; it fails to recognize that the interaction term is really nonlinear in f_A, and only formally linear in the variables F_A. This argument is admittedly not compelling, but unfortunately the correct choice can not be discovered by a study of our elementary analogue in (2.17); this example is merely a reparametrization of essentially linear interactions. Our two possibilities coincide, however, in the linear interaction case $F_A = f_A$. Perhaps (2.32) could be looked at in the elementary framework as the inner product of two vectors, one of which is best interpreted as a «$|\psi\rangle$» vector while the other is best interpreted as a «$|u\rangle$» vector. In the four-dimensional form of quantum mechanics, such inner products are of basic importance, in contrast to the conventional formalism. The basic weight function for the measure on histories (in the functional representation of the matrix element of physical interest) is thus altered and, as we shall illustrate in the case of the gravitational field, the group of invariance transformations is altered as a consequence.

3. – Application to the gravitational field.

The application of the preceding general formalism to the Einstein gravitational field is straightforward. We adopt the « free » conventional action functional

$$I\{g^{\mu\nu}\} = \int R_{\mu\nu} g^{\mu\nu} \sqrt{-g}\, \mathrm{d}x \,, \tag{3.1}$$

expressed in natural units ([14]). As « basic » gravitational variables we adopt first the contravariant form of the metric tensor $g^{\mu\nu}(x)$. These variables correspond to the variables f_A of Section **2**. From these variables we can form other quantities, such as $g_{\mu\nu}$ or $g = \det g_{\mu\nu}$. The Ricci tensor $R_{\mu\nu}$ is constructed from $g^{\alpha\beta}$, $g_{\sigma\tau}$ and their derivatives.

A) *Contravariant metric tensor as basic.* – For the interaction term for the gravitational field with an external source we adopt the action

$$\int v_{\mu\nu} \cdot g^{\mu\nu} \sqrt{-g}\, \mathrm{d}x \,. \tag{3.2}$$

The symmetric field $v_{\mu\nu}(x) = v_{\mu\nu}(x)$ represents the external source (*i.e.*, X^A) and is, by assumption, functionally independent of $g^{\mu\nu}$ (and thus of $g_{\mu\nu}$). The non-linear appearance of $g^{\mu\nu}$ in (3.2) is clear, and it follows that the relevant function F_A is determined by

$$F_A \to \mathfrak{g}^{\mu\nu}(g^{\alpha\beta}) \equiv g^{\mu\nu} \sqrt{-g} \,. \tag{3.3}$$

The fundamental commutation relations then take the form

$$[\mathfrak{g}^{\mu\nu}(x),\, v_{\alpha\beta}(y)] = -\tfrac{1}{2} i [\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha]\, \delta(x-y) \,, \tag{3.4}$$

$$[g^{\mu\nu}(x),\, g^{\alpha\beta}(y)] = [v_{\mu\nu}(x),\, v_{\alpha\beta}(y)] = 0 \,. \tag{3.5}$$

A straightforward calculation at a point shows that

$$\frac{\partial \mathfrak{g}^{\mu\nu}}{\partial g^{\alpha\beta}} = \frac{1}{2} [\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha - g^{\mu\nu} g_{\alpha\beta}] \sqrt{-g} \,. \tag{3.6}$$

In analogy with eq. (2.8) we define a new Hermitian quantity

$$\mathfrak{V}_{\alpha\beta} = v_{\mu\nu} \cdot \frac{\partial \mathfrak{g}^{\mu\nu}}{\partial g^{\alpha\beta}} \,,$$

$$\mathfrak{V}_{\alpha\beta} = v_{\alpha\beta} \cdot \sqrt{-g} - \tfrac{1}{2} v_{\mu\nu} \cdot g^{\mu\nu} g_{\alpha\beta} \sqrt{-g} \,; \tag{3.7}$$

thus $\mathfrak{V}_{\alpha\beta}$ corresponds to the operator χ^A. This new variable obeys the commutation relation

$$[g^{\mu\nu}(x),\, \mathfrak{V}_{\alpha\beta}(y)] = -\tfrac{1}{2} i [\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha]\, \delta(x-y) \,. \tag{3.8}$$

Additional commutation relations may be found as a direct consequence of the above relations. We list several of these without proof, although, as an example we discuss the first of these in the Appendix. Some additional commutation relations are:

$$\left[\sqrt{-g(x)},\, v_{\alpha\beta}(y)\right] = -\tfrac{1}{2} i g_{\alpha\beta}\, \delta(x-y) \,, \tag{3.9}$$

$$\left[\sqrt{-g(x)},\, \mathfrak{V}_{\alpha\beta}(y)\right] = \tfrac{1}{2} i \mathfrak{g}_{\alpha\beta}\, \delta(x-y) \,,$$

$$[(-g)^{\frac{5}{2}} g^{\mu\nu}(x),\, \mathfrak{V}_{\nu\sigma}(y)] = 0 \,. \tag{3.10}$$

Still further relations may be found by raising or lowering various indicies. For example,

$$[g_{\mu\nu}(x), \mathfrak{V}_{\alpha\beta}(y)] = - g_{\mu\lambda}(x)\,[g^{\lambda\tau}(x), \mathfrak{V}_{\alpha\beta}(y)]\, g_{\tau\nu}(x)\,,$$

which follows as a simple identity. Let us define various contravariant Hermitian v-operators by

$$v_{\mu}{}^{\nu} = v_{\mu\sigma}\cdot g^{\sigma\nu}\,; \qquad v^{\mu\nu} = v_{\tau\sigma}\cdot g^{\mu\tau} g^{\nu\sigma}\,, \tag{3.11}$$

and similarly for $\mathfrak{V}_{\alpha\beta}$. Then, for example,

$$[O(x), v^{\mu\nu}(y)] = g^{\mu\tau}(y)\,[O(x), v_{\tau\sigma}(y)]\, g^{\nu\sigma}(y)\,,$$

etc., where O is an arbitrary g-dependent operator. Note that

$$[v^{\mu\nu}(x), v_{\tau\beta}(y)] \neq 0$$

since $v^{\mu\nu}$ is not functionally independent of the metric.

The dynamical constraint equation, analogous to (2.3), is given by

$$\{\mathfrak{G}_{\mu\nu}(x) + \mathfrak{V}_{\mu\nu}(x)\}\,|\Omega) = 0\,, \tag{3.12}$$

where we employ the conventional abbreviation

$$\mathfrak{G}_{\mu\nu} \equiv (R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R)\sqrt{-g}\,.$$

The corresponding alternate dynamical equation like (2.9) is simply

$$\{R_{\mu\nu}(x) + v_{\mu\nu}(x)\}\,H^{-1}|\Omega) = 0\,. \tag{3.13}$$

A derivation of (3.13) from (3.12) shows on the basis of the general study in Section **2** that

$$H(x) = \det{}^{\frac{1}{2}}[\partial g^{\beta\alpha}/\partial \mathfrak{g}^{\mu\nu}] = [-g(x)]^{-\frac{5}{4}}, \tag{3.14}$$

a calculation considerably simplified by the observation that apart from the factor $(-g)^{\frac{1}{2}}$ the matrix on the right-hand side of (3.6) equals its own inverse. Thus only the $(-g)^{-\frac{1}{2}}$ part of $\partial g^{\alpha\beta}/\partial\mathfrak{g}^{\mu\nu}$ effectively survives the determinant operation ([23]).

The matrix element of physical interest is $(\omega_v|\Omega)$, where $|\omega_v)$ is a simultaneous eigenvector of the operator $v_{\alpha\beta}(x)$. In a functional representation of

([23]) See also the remarks made in connection with eq. (3.28).

this Hilbert space in terms of functionals of metric, eq. (2.32) becomes

$$(\omega_v\,|\,\Omega) = N^{-1}\int \exp\left|i\int(v_{\mu\nu}+R_{\mu\nu})\mathfrak{g}^{\mu\nu}\,\mathrm{d}x\right|\mathscr{D}g\,, \tag{3.15}$$

where, from (2.27) and (3.14),

$$\mathscr{D}g \propto \Pi_x[-g(x)]^{\frac{5}{2}}\,\Pi_{\mu\leq\nu}\,\mathrm{d}g^{\mu\nu}(x)\,. \tag{3.16}$$

This measure on metrics is just the one proposed originally by MISNER ([20]) based on invariance arguments in his study of a Feynman quantization of general relativity. Here we observe that this measure follows as a consequence of the form taken by the interaction with an external source.

Before a further discussion of (3.15) is made we point out that a similar equation can also be derived under differing hypotheses.

B) Covariant metric tensor as basic. – Suppose instead of the contravariant tensor $g^{\alpha\beta}$ we wished to call the covariant tensor $g_{\alpha\beta}$ « basic ». Then the interaction term

$$\int w^{\alpha\beta}\cdot g_{\alpha\beta}\sqrt{-g}\,\mathrm{d}x, \tag{3.17}$$

would involve yet a new operator $w^{\alpha\beta}$ which is functionally independent of $g_{\alpha\beta}$. Clearly $v^{\mu\nu}$ (in 3.11) is not a direct candidate for $w^{\mu\nu}$. The new commutation relation replacing (3.4) is

$$[\mathfrak{g}_{\alpha\beta}(x),\,w^{\sigma\tau}(y)] = -\tfrac{1}{2}i[\delta^\sigma_\alpha\delta^\tau_\beta+\delta^\sigma_\beta\delta^\tau_\alpha]\,\delta(x-y)\,. \tag{3.18}$$

It follows from the relation

$$\frac{\partial\mathfrak{g}_{\alpha\beta}}{\partial g_{\sigma\tau}} = \frac{1}{2}[\delta^\sigma_\alpha\delta^\tau_\beta+\delta^\sigma_\beta\delta^\tau_\alpha+g_{\alpha\beta}g^{\sigma\tau}]\sqrt{-g}\,, \tag{3.19}$$

that the Hermitian operator

$$\mathfrak{W}^{\sigma\tau}(x) \equiv w^{\alpha\beta}(x)\cdot(\partial\mathfrak{g}_{\alpha\beta}/\partial g_{\sigma\tau}) = w^{\sigma\tau}\cdot\sqrt{-g}+\tfrac{1}{2}\,w^{\alpha\beta}\cdot g_{\alpha\beta}g^{\sigma\tau}\sqrt{-g}\,, \tag{3.20}$$

satisfies

$$[g_{\alpha\beta}(x),\,\mathfrak{W}^{\sigma\tau}(y)] = -\tfrac{1}{2}i[\delta^\sigma_\alpha\delta^\tau_\beta+\delta^\sigma_\beta\delta^\tau_\alpha]\,\delta(x-y)\,. \tag{3.21}$$

According to the discussion in *A*) we may lower the indices of $\mathfrak{W}^{\sigma\tau}$ and raise those of $g_{\alpha\beta}$ with only a net change of sign. Thus eq. (3.21) implies

$$[g^{\alpha\beta}(x), \mathfrak{W}_{\sigma\tau}(y)] = \tfrac{1}{2} i[\delta^{\alpha}_{\sigma}\delta^{\beta}_{\tau} + \delta^{\alpha}_{\tau}\delta^{\beta}_{\sigma}]\,\delta(x-y)\,,$$

where $\mathfrak{W}_{\sigma\tau} = \mathfrak{W}^{\mu\nu}\cdot g_{\nu\tau}g_{\mu\sigma}$.

A comparison of this result with (3.8) indicates that

$$\mathfrak{W}_{\sigma\tau} = -\mathfrak{V}_{\sigma\tau}\,; \tag{3.22}$$

any possible difference in the form of a function of the metric is ruled out by the nondependence of $v_{\alpha\beta}$ and $w^{\alpha\beta}$ on $g_{\alpha\beta}$.

An appropriate constraint equation in the present case is

$$\{-\mathfrak{G}^{\alpha\beta} + \mathfrak{W}^{\alpha\beta}\}\,|\Lambda) = 0\,, \tag{3.23}$$

$|\Lambda)$ being a new state vector. The new matrix element of physical interest is now

$$(\lambda_{w'}|\Lambda)\,, \tag{3.24}$$

where we define

$$w^{\alpha\beta}(x)\,|\lambda_{w'}) = w'^{\alpha\beta}(x)\,|\lambda_{w'})\,. \tag{3.25}$$

The formal solution to (3.23) is

$$|\Lambda) = \exp\,[i\,I\{f\}]\,|\tilde{\lambda}_0)\,, \tag{3.26}$$

where

$$\mathfrak{W}^{\alpha\beta}(x)\,|\tilde{\lambda}_0) = 0\,.$$

This solution can be related to the solution of the alternate constraint

$$\{-R^{\mu\nu} + w^{\mu\nu}\}\,|\tilde{\Lambda}) = 0\,.$$

Here the solution is

$$|\tilde{\Lambda}) = \exp\,[i\,I\{f\}]\,|\lambda_0)\,,$$

and based on the general analysis of Section **2**, it follows that

$$|\tilde{\lambda}_0) = H'\,|\lambda_0)\,, \tag{3.27}$$

$$H' = |\,\partial g_{\alpha\beta}/\partial \mathfrak{g}_{\mu\nu}|^{\frac{1}{2}}\,. \tag{3.28}$$

In order to evaluate (3.28) we use the following simple argument, for which we introduce the 10×10 matrices $M(A)$ defined by

$$\tfrac{1}{2}[\delta^\sigma_\alpha \delta^\tau_\beta + \delta^\sigma_\beta \delta^\tau_\alpha + A g_{\alpha\beta} g^{\sigma\tau}] .$$

It follows directly that

$$M(A)\,M(B) = M(A + B + 2AB) , \tag{3.29}$$

an algebraic relation obeyed also by their determinants $\mathrm{d}(A) \equiv \det M(A)$. The appropriate solution to (3.29) for the determinants is

$$\mathrm{d}(A) = (1 + 2A)^2 ,$$

and is independent of $g_{\alpha\beta}$ for all A. Application of this result to (3.19) then shows that

$$H' = \Pi_x \tfrac{1}{3}\,[-g(x)]^{-\frac{5}{2}} = \mathscr{M} H , \tag{3.30}$$

namely, that apart from a formal normalization factor $\mathscr{M}$, H' and H are identical. We now derive (3.23) and an expression for (3.24) from the relations established in part A) above.

If we raise both indicies of (3.12), then

$$\left\{\mathfrak{G}^{\alpha\beta}(x) + g^{\alpha\mu} g^{\beta\nu} \mathfrak{P}_{\mu\nu}(x)\right\} |\Omega) = 0 .$$

In order to hermitize the last term, we multiply by $\Pi_x\,[-g(x)]^{\frac{5}{2}}$, and employ eq. (3.10). Thus, with the definition (3.14) for H, we find

$$\left\{\mathfrak{G}^{\alpha\beta}(x) + g^{\alpha\mu} \mathfrak{P}_{\mu\nu} g^{\nu\beta}(x)\right\} H^{-1} |\Omega) = 0 .$$

The identification (3.22) then leads to

$$\left\{\mathfrak{G}^{\alpha\beta}(x) - \mathfrak{W}^{\alpha\beta}(x)\right\} H^{-1} |\Omega) = 0 ,$$

and comparison of this equation with (3.23) indicates that

$$|A) = H^{-1} |\Omega) . \tag{3.31}$$

From the general arguments of Section **2** we must interpret (3.31) as a change taking place in the weight function of the functional realization of the Hilbert space.

To determine what weight change is to be associated with $(\lambda_w|$ in (3.24) we argue as follows for the particular case $(\lambda_0|$. If we ignore the numerical weight factor $\mathscr{M}$, then we find

$$H\,|\omega_0) = |\tilde{\omega}_0) = H\,|\tilde{\lambda}_0) = H^2\,|\lambda_0)$$

from eqs. (2.30), (3.26), (3.27) and (3.31). Thus generalizing to nonzero eigenvalues, we obtain

$$|\lambda_w) = H^{-1}\,|\omega_v)\,. \tag{3.32}$$

The content of (3.31) and (3.32) is that $|\Lambda)$ and $|\lambda_w)$ are not to evaluated in the conventional form of inner product eq. (2.22), but rather in the alternate form (2.23). Still ignoring the factor $\mathscr{M}$, we find

$$(\lambda_w|\Lambda) = (\lambda_w|\exp[iI\{f\}]H\,|\lambda_0) = \int \lambda_w^*(f)\,HDf\,\exp[iI\{f\}]\,\lambda_0(f)\,. \tag{3.33}$$

In the present application of (2.23) the variable f_A are the covariant metric $g_{\mu\nu}$ and

$$\lambda_w^*(f) \propto \exp\left|i\int w^{\alpha\beta}\mathfrak{g}_{\alpha\beta}\,\mathrm{d}x\right|\,.$$

Therefore (3.33) becomes

$$(\lambda_w|\Lambda) = N^{-1}\int \exp\left[i\int (w^{\alpha\beta}+R^{\alpha\beta})\mathfrak{g}_{\alpha\beta}\,\mathrm{d}x\right]\mathscr{D}g\,, \tag{3.34}$$

where

$$\mathscr{D}g \propto \Pi_x\,[-g(x)]^{-\frac{5}{2}}\,\Pi_{\mu\le\nu}\,\mathrm{d}g_{\mu\nu}(x)\,,$$

which is just an alternate and equivalent form for (3.16). Therefore the same measure on metrics arises in the covariant metric formulation as arose in the contravariant metric formulation.

The similarity in the form of the functional representation of (3.15) and (3.34) is noteworthy. The expression (3.15) is discussed somewhat further in the next section.

4. – Conclusion.

We have generalized the conventional covariant quantization procedure to nonlinear interaction terms and have applied this formalism to the gravitational field. The functional representation (such as (3.15), (3.16)) shows a striking formal similarity to the results of Misner for a Feynman quantization of general relativity.

Since the arbitrary field $v_{\mu\nu}$ is at our disposal it is suggestive to choose very simple and symmetric boundary conditions on the integrals in (3.15), namely, to sum over *all permissible metric histories in the entire manifold.* Such a choice singles out no metric in particular as it treats them all alike. Indeed, if the manifold were closed in the time direction this would have to be recognized in summing over all permissible metrics. This important physical state vector we call $|\Omega_0)$; it may well play the role of a « vacuum » state. If we adopt the physical state $|\Omega_0)$ then no space-like surfaces ever enter the discussion; space and time are treated everywhere on an equal footing.

Functional derivatives with respect to the test field $v_{\mu\nu}$ will generate matrix elements of the metric, if we give to (3.15) a sum-over-histories interpretation. It is of course possible, therefore, to ask surface-dependent questions of the final expression $(\omega_v|\Omega_0)$. However the advantage of the present analysis is that it saves all such questions to the very last step.

Unfortunately further calculation of the functional integrals involved in (3.15) seems not possible at present since the techniques for continuous integrations are as yet insufficiently advanced. One may have to be content with exploiting the invariance of $\mathscr{D}g$, as was done by MISNER by ROSEN [24]. The resulting measure on metrics is invariant under the transformation

$$g^{\mu\nu} \to g^{\mu\nu(\pi)} = \pi_\alpha{}^\mu g^{\alpha\beta} \pi_\beta{}^\nu$$

at each point, *i.e.*,

$$\mathscr{D}g^{(\pi)} = \mathscr{D}g$$

which then reflects itself in the structure of (3.15).

Finally it should be remarked that the operator form of the relevant equations (2.2) and (2.3), may be directly approachable with distribution theory. In the case of the gravitational field the commutation relation (3.4) would, after multiplication with $\sqrt{-g}$, become

$$[g(\xi), v(\eta)] = -i(\xi, \eta)\,, \tag{4.1}$$

where

$$g(\xi) = \int \xi_{\alpha\beta} g^{\alpha\beta} \sqrt{-g}\, dx\,,$$

$$v(\eta) = \int \eta^{\alpha\beta} v_{\alpha\beta} \cdot \sqrt{-g}\, dx\,,$$

$$(\xi, \eta) = \int \xi_{\alpha\beta} \eta^{\alpha\beta} \sqrt{-g}\, dx\,.$$

([24]) G. ROSEN: *Thesis* (Princeton, 1959).

Here $\xi_{\alpha\beta}$ and $\eta^{\alpha\beta}$ are two « test » functions in the distribution sense. The dynamical constraint eq. (3.12) becomes

$$\{G(\eta) + V(\eta)\}\,|\,\Omega) = 0\,, \tag{4.2}$$

where

$$G(\eta) = \int \eta^{\alpha\beta}\,\mathfrak{G}_{\alpha\beta}\,\mathrm{d}x\,,$$

$$V(\eta) = \int \eta^{\alpha\beta}\,\mathfrak{V}_{\alpha\beta}\,\mathrm{d}x\,.$$

The remarkable property of the dynamical constraint (4.2) in the case of the gravitational field is that the operators therein, $G(\eta)$ and $V(\eta)$, depend on no higher powers of the local fields than do the basic distributions which appear in (4.1).

It is possible that a rigorous approach to the covariant quantization of the gravitational field could be based on (4.1) and (4.2).

Appendix

Herein we derive the typical commutation relation

$$\left[\sqrt{-g}(x),\, v_{\sigma\tau}(y)\right] = -\tfrac{1}{2}\, i g_{\sigma\tau}\,\delta(x-y)\,, \tag{A.1}$$

and, in particular, show its dependence on the dimension of space-time, which in this section we initially keep arbitrary. Observe that the 1, 1-element of the commutation relation eq. (3.4),

$$\left[\sqrt{-g}\,g^{\mu\nu}(x),\, v_{11}(y)\right] = -\,i\delta^{\mu}_{1}\delta^{\nu}_{1}\,\delta(x-y)\,, \tag{A.2}$$

vanishes unless $\mu=\nu=1$. Consider, then, the commutator

$$\left[\det\left\{\sqrt{-g}\,g^{\mu\nu}(x)\right\},\, v_{11}(y)\right] = C\,. \tag{A.3}$$

In the expansion of the determinant in (A.3) only one term is nonvanishing so that (suppressing x and y)

$$C = \mathrm{Minor}_{11}\left\{\sqrt{-g}\,g^{\mu\nu}\right\}\left[\sqrt{-g}\,g^{11},\, v_{11}\right] = -\,i\,\mathrm{Minor}_{11}\left\{\sqrt{-g}\,g^{\mu\nu}\right\}\delta(x-y)\,. \tag{A.4}$$

By definition

$$\mathrm{Minor}_{11}\left\{\sqrt{-g}\,g^{\mu\nu}\right\} = \det\left\{\sqrt{-g}\,g^{\mu\nu}\right\}\left(\sqrt{-g}\,g^{\mu\nu}\right)^{-1}_{11}\,. \tag{A.5}$$

If $n(\neq 2)$ denotes the dimension of space-time, then

$$\det\left\{\sqrt{-g}\,g^{\mu\nu}\right\} = -(-g)^{(n-2)/2}\,, \tag{A.6}$$

$$(\sqrt{-g}\,g^{\mu\nu})^{-1}_{11} = g_{11}/\sqrt{-g}\,. \tag{A.7}$$

Using eqs (A.6) and (A.7) we find

$$-[(-g)^{(n-2)/2}, v_{11}] = i(-g)^{(n-3)/2}g_{11}\delta(x-y)\,, \tag{A.8}$$

and, if we assume that the commutator of $v_{\alpha\beta}$ with any function of $g^{\alpha\beta}$ depends only on $g^{\alpha\beta}$, then

$$(n-2)(-g)^{(n-3)/2}\left[\sqrt{-g}, v_{11}\right] = -i(-g)^{(n-3)/2}g_{11}\delta(x-y)\,.$$

Finally, generalizing to each of the elements of $v_{\sigma\tau}$, it follows that

$$\left[\sqrt{-g(x)}, v_{\sigma\tau}(y)\right] = -\frac{i}{n-2}\,g_{\sigma\tau}\delta(x-y)\,. \tag{A.9}$$

which, as desired, reduces to (A.1) for $n=4$.

RIASSUNTO (*)

In ogni teoria quantistica, nella quale sia anche quantizzato il tensore metrico della teoria gravitazionale di Einstein, diventa privo di senso richiedere una superficie spaziale iniziale su cui specificare i commutatori di campo convenzionali. Si designa il formalismo quantistico covariante, in cui tutti i campi o commutano o non commutano solo quando i punti dei campi coincidono, come adatto a quantizzare la gravità. Si analizza un po' dettagliatamente l'estensione del formalismo quantistico covariante a campi bosonici generali che interagiscono in modo intrinsecamente non lineare. Si applica questo formalismo al caso del campo libero gravitazionale. Si trova che, in una rappresentazione funzionale, la misura sulle metriche è quella proposta da Misner. Si propone uno stato fondamentale della teoria gravitazionale quantizzata, che comporta una sommatoria estesa a tutte le metriche possibili in tutto il complesso spazio-tempo.

(*) *Traduzione a cura della Redazione.*