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# The Lorentz-Covariant Approximation Method in General Relativity. - I.

R. P. KERR (*)

*Trinity College - Cambridge*

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**Summary.** — The Lorentz-covariant approximation method for the field outside a set of localized particles has been analysed. It is found that as well as the usual equations of motion and energy derived by Eistein, Infeld and Hoffman for the quasi-static approximation, there are three further equations, the equations of spin, which must be satisfied by the structural parameters of each particle. These equations also appear as surface integral conditions in the quasi-static approximation. Furthermore, it is shown that it is not necessary to expand the mass, dipole, or spin parameters. those introduced into the lowest approximation being the physical particle parameters. It is only the differential equations satisfied by these that change in the higher orders.

## 1. – Introduction.

In this paper we shall analyse the covariant approximation method for calculating the equations of motion of localized particles in general relativity. By covariant we mean that the approximation equations are Lorents covariant, not covariant under the full co-ordinate transformation group. Previously, it has been usual to consider the derivatives with respect to the non-covariant time coordinate to be small compared to the spatial derivatives, giving

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the usual quasi-static approximation. EINSTEIN, INFELD and HOFFMAN ([1-3]) analysed this problem and showed that each particle must satisfy three equations of motion and an equation of mass. Their method was based on the dipole procedure wherein a dipole field is introduced into each approximation, so that the field equations are integrable in the next approximation. However, their proof assumed the vanishing of the dipole forces and, as has been shown by MOFFAT and KERR ([4]), this is not so. In the lowest approximation one obtains the usual classical dipole forces on a non-symmetric particle in an external field.

Furthermore, as I shall show in a subsequent paper, the four surface integral conditions of Einstein, Infeld and Hoffman are not sufficient for the integrability of the approximation equations. There are also three equations, corresponding to the classical equations of angular momentum, which must be satisfied in every approximation. If a dipole field is introduced into each approximation so that the E.I.H. surface integrals are zero, then the three spin equations will be inconsistent. It is also necessary to introduce a spin field into each approximation, before the field equations can be satisfied. Physically, it is found that a dipole particle will start to rotate under an applied couple.

We shall not consider the quasi-static approximation further in this paper but instead shall analyse the corresponding problem in the Lorentz-covariant approximation. The reason for this is that, although it is much more difficult to integrate the field equations in a simple closed form, we shall have a clearer picture of the structure of the approximation method. In particular, we shall see that the equations of energy, motion, and spin are the only physical equations to be satisfied by the particle constants. The form of these equations is completely identical to that in the quasi-static approximation, except that the time derivative is treated in the same way as the spatial derivatives.

In Sections **2** and **3** we shall show that the field equations may be integrated in each approximation, provided that we expand the mass, dipole and spin parameters for each particle. It will be shown that each of these parameters must satisfy a first or second order differential equation, as in the quasi-static approximation. However, it can be shown that the total field is a function of the mass, etc., rather than of the individual $n$-th approximation order components. This being so, we would expect that the physical equations of mass, motion, and spin would also be functions of these total mass, dipole and spin constants, and that the differential equations satisfied by the $n$-th order para-

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([1]) A. EINSTEIN, L. INFELD and *H.* HOFFMAN: *Ann. Math.*, **39**, 66 (1938).
([2]) A. EINSTEIN and L. INFELD: *Ann. Math.*, **41**, 797 (1940).
([3]) A. EINSTEIN and L. INFELD: *Can. Journ. Math.*, **1**, 109 (1949).
([4]) J. MOFFAT and R. KERR: paper. Copies available.

meters do not affect the physical field. This was the motivating idea behind the E.I.H. dipole procedure. However, they attempted to show that only the total dipole constants would appear in the equations of motion by proving that the dipole forces were zero in every approximation, which is not true as has been shown in [4].

As we shall see in Sect. **4**, exactly the same problem arises in the Lorentz-covariant approximation. It could be proved that it is only the total particle parameters that appear in the physical equations but we shall not do so. Instead, we notice that if the individual $n$-th order parameters do not appear in the physical field or the equations of motion, etc., it is pointless to expand these parameters. Of course, if we do not expand them then we cannot satisfy the approximation equations exactly. However, we show that if the equations of motion, energy and spin are satisfied to the $n$-th approximation then the Einstein field equations are also satisfied *to the same approximation*. The original $n$-th approximation field equations of E.I.H. are not satisfied even in an approximate sense but, if we add together the $n$-th approximation fields up to the $N$-th order to give the physical field to the $N$-th approximation, then the field equations of Einstein will be satisfied to the $N$-th approximation, provided that the seven differential equations of energy, motion and spin are satisfied *to the same approximation*. This is all that can be expected of any approximation method.

## 2. – Lorentz-invariant $\lambda$-expansions.

In this Section we shall derive an approximation method for solving the field equations outside a set of weakly interacting, fast moving, bodies. Obviously, if we wished to consider fundamental particles we should have to take into account the Bremstrahlung emission which plays a large part in the theory. This is also the case with classical theories, such as quantum theory or electrodynamics, where the spontaneous emission of radiation plays an important role in the motion of the elementary particles.

For a macroscopic body, the rate of emission will be governed by the equations of statistical mechanics, and consequently may be considered to be a continuous function of the time for a stable body in a weak gravitational field. Thus, it should be possible to derive meaningful equations of motion for such bodies.

In the following, we shall consider the field around a set of particles, or « singularities », which we shall distinguish by Latin superfixes over the appropriate functions. As it is not possible to expand the field at the singularities, and as we do not wish to introduce an energy-momentum tensor, we shall enclose the particles by arbitrary three dimensional surfaces with the topo-

logical properties of a time-like cylinder. To specify the position of the $p$-th particle, we shall take a representation world line, $\overset{p}{z}_\mu(\overset{p}{s})$, inside the $p$-th surface, $\overset{p}{S}$. Since the actual shape of the surfaces is not important, we shall define them by the following conditions.

Firstly, we define a one dimensional parameter, $\overset{p}{s}$, along each world line by means of the Galilean metric tensor (*). We then extend the domain of definition of this parameter to a four dimensional region, enclosing the particle, by means of the equations,

$$x^\mu = \overset{p}{z}_\mu(\overset{p}{s}) + \gamma^\mu \,, \tag{2.1}$$

$$\overset{p}{v}_\mu \gamma_\mu = \overset{p}{v}_\mu (x^\mu - \overset{p}{z}_\mu(\overset{p}{s})) = 0 \,, \tag{2.2}$$

where throughout this paper we shall raise and lower the particle tensors with the Galilean tensor, rather than the unknown Einstein tensor. These equations correspond to taking the planes of constant $\overset{p}{s}$ perpendicular to the world line, and they will have a unique solution for $\overset{p}{s}$ as a function of $x^\lambda$ inside $S$ provided that the derivative, $\partial s/\partial x^\mu$, is not infinite. If we differentiate (2.2) with respect to the parameter $x^\alpha$, we have the equation

$$\frac{\partial s}{\partial x^\alpha} \dot{v}_\beta \gamma^\beta + v_\beta \left( \delta^\beta_\alpha - v^\beta \cdot \frac{\partial s}{\partial x^\beta} \right) = 0 \,, \tag{2.3}$$

and so

$$\frac{\partial s}{\partial x^\alpha} = \frac{v_\alpha}{1 - (\dot{v}_\nu \gamma^\nu)} \,, \tag{2.4}$$

so that $s$ will be a single valued function of position provided that $\dot{v}_\nu \cdot \gamma^\nu < 1$ inside the surface. We define the surface, $\overset{p}{S}$, by the condition that $(\gamma_\lambda \cdot \gamma^\lambda)$ is a constant everywhere on the surface; *i.e.* in the rest system of the world line the surface has constant Galilean radius. The condition for single valuedness means that the acceleration of the world line should be small compared to the dimensions of the system. We do not want to take a world line oscillating rapidly across the particle, as we should then have to disentangle the violent motion of this form the slow acceleration of the physical body. This means that we must find some way of fixing the position of the world line inside the particle. We shall leave this to a later section.

We shall impose the De Donder co-ordinate conditions on the fundamentatensor

$$g^{\mu\nu}{}_{,\nu} = 0 \,, \tag{2.5}$$

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(*) $\eta_{\lambda\beta}$ is the Galilean metric tensor, $(+1, -1, -1, -1)$; $(ds)^2 = \eta_{\lambda\beta} dx^\lambda dx^\beta$, and $v^\alpha = dx^\lambda/ds$. We shall denote differentiation with respect to $s$ by a dot, e.g. $\dot{v}^\lambda = dv^\lambda/ds$. Where there is no ambiguity, we shall omit the particle index, $p$.

where

$$\mathfrak{g}^{\mu\nu} = (-g)^{\frac{1}{2}} \cdot g^{\mu\nu},$$

and $g = \det(g_{\alpha\beta})$. Later, we shall consider whether this introduces any non-physical equations into the theory. We are not concerned here with whether these conditions have any physical significance of their own, since, whether this is so or not, they are the most convenient to use in any reiteration procedure. From the explicit expression for the curvature tensor in Appendix A, we see that the approximation field equations reduce to D'Alembert equations whenever these co-ordinate conditions are satisfied.

It is fundamental in the solution of any non-linear equation by successive approximation that the field equations should first be expressed in terms of a set of independent variables. We shall take

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}, \tag{2.6}$$

and express the field equations in terms of the $h^{\lambda\beta}$. FOCK has shown that, in a co-ordinate system satisfying (2.5), the field equations reduce to

$$\Box h^{\mu\nu} = 2\Lambda^{\mu\nu}, \tag{2.7}$$

where $\Lambda^{\mu\nu}$ is a non-linear function of the $h^{\lambda\beta}$ and their derivatives. The complete expression for the Einstein tensor, as well as its reduced form using (2.5), is given in Appendix A. It may be shown that the $\Lambda^{\mu\nu}$ are absolutely convergent power series in the $h^{\alpha\beta}$, provided that the moduli of the characteristic roots of $\eta_{\mu\lambda} h^{\lambda\nu}$ are all less than unity. A simpler, though not so sensitive, condition is that

$$\text{mod}\,(h^{\lambda\beta}) < \tfrac{1}{4}. \tag{2.8}$$

The « obvious » way to solve (2.7) is to use *successive reiteration*,

$$\Box \underset{(n)}{h}{}^{\mu\nu} = 2 \underset{(n)}{\overset{*}{\Lambda}}{}^{\mu\nu}(\underset{(n-1)}{h}{}^{\lambda\beta}), \tag{2.9}$$

where $\underset{(n)}{\overset{*}{\Lambda}}{}^{\mu\nu}$ consists of all terms in the expansion of the field equations which do not contain more than $n(h^{\lambda\beta})$. However, there are two things wrong with this intuitive approach. The first is that there would be too much unnecessary labour involved if we intended to stop at the $N$-th reiteration, since it would be sufficiently accurate to use $(\underset{(1)}{h}{}^{\cdot\cdot})^N$ in the last reiteration rather than $(\underset{(N-1)}{h}{}^{\cdot\cdot})^N$. The difference is negligible in the $N$-th reiteration. This would not matter if we intended to carry the process on to infinity (whatever that may

mean in a logical sense!), but in practise one has to cut the process off at some finite reiteration.

The second thing wrong with this approach is that we have to insure that the limiting field, $\underset{(\infty)}{h}^{\mu\nu}$, will satisfy the co-ordinate conditions. On the surface, the simplest way to do this seems to be to make each successive approximation satisfy it exactly, *i.e.*

$$\underset{(n)}{h}^{\mu\nu}{}_{,\nu} = 0 \,, \tag{2.10}$$

but, as we shall see in the next section, this is not so. Nevertheless, for the present we shall assume that (2.10) is to be satisfied. If we take the divergence of (2.9) we see that $\overset{*}{\underset{(n)}{\Lambda}}^{\mu\nu}{}_{,\nu}$ must be zero, and this will not be so, in general, even if the co-ordinate conditions are satisfied exactly in every previous reiteration.

Fortunately, the classical solution of the first problem is also the solution to the second. In the standard work on non-linear equations by Bogoliubov, we see that the usual method used to solve such differential equations is to expand the field in powers of an *indeterminate parameter*, $\lambda$,

$$h_{\mu\nu}(\lambda) = \sum_r \lambda^r \underset{r}{h}^{\mu\nu} \,, \tag{2.11}$$

which corresponds to the expansion method introduced by EIHSTEIN, INFELD and HOFFMAN in 1938 for the quasi-static approximation. It must be stressed that this « parameter » has no physical significance whatsoever. The $\lambda$-series in (2.11) is not the physical field — that is obtained by replacing $\lambda$ by unity. The significance of $\lambda$ is that it should act as a weighting factor to define the order of magnitude of different terms in the approximation procedure, thereby reducing the labour involved in the reiteration methods. Some times, when there is an obvious constant present in the differential equations, it is convenient to expand in terms of this rather than an indeterminate constant, the physical constant playing the role of a weighting factor. However, there is no obvious parameter to chose in the covariant approximation.

If we expand the field in powers of this parameter, and then equate to zero the coefficient of $\lambda^n$ in the field equations, we obtain a new set of approximation equations,

$$\Box \underset{n}{h}^{\mu\nu} = 2\underset{n}{\Lambda}^{\mu\nu} \,. \tag{2.12}$$

Hence, the *reiteration* field must satisfy the equation,

$$\tfrac{1}{2}\Box \underset{(n)}{h}^{\mu\nu} = \underset{(n)}{\Lambda}^{\mu\nu}(\underset{(r)}{h}^{\lambda\beta};\, r<n) = \sum_r^n \underset{r}{\Lambda}^{\mu\nu} \,, \tag{2.13}$$

where

$$\underset{(n)}{h}{}^{\mu\nu} = \sum_{1}^{n} \underset{r}{h}{}^{\mu\nu}, \tag{2.14}$$

and $\underset{n}{\Lambda}{}^{\mu\nu}$, and so $\underset{(n)}{\Lambda}{}^{\mu\nu}$, will be a finite polynomial in the previous approximations.

We shall now state an important theorem that was first proved by EINSTEIN and INFELD, in 1949, for the quasi-static approximation.

> *Theorem 1.* If the gravitational field equations of Fock are expanded as a power series in an indeterminate parameter, and the coefficients of the successive powers equated to zero, (2.12), then the divergences, $\underset{n}{\Lambda}{}^{\mu\nu}{}_{,\nu}$ and $\underset{(n)}{\Lambda}{}^{\mu\nu}{}_{,\nu}$, will both be zero, provided that the field equations and the co-ordinate conditions are satisfied in all previous approximations.

This follows from the Bianchi identities and is proved in a more general form in Sect. **4**. It follows immediately that, provided that (2.12) and (2.10) are satisfied in every previous approximation, any solution of the field equations in the $n$-th approximation must satisfy

$$\Box \underset{n}{h}{}^{\mu\nu}{}_{,\nu} = 0\,, \tag{2.15}$$

as may be proved by taking the divergence of (2.12). This does not imply that the co-ordinate conditions are satisfied by any solution of the field equations. In the next section, however, we shall see that the De Donder conditions may be satisfied exactly in the $n$-th approximation by the addition of a solution of the homogeneous wave equation onto any particular solution of (2.16). This will give the equations of mass, motion, and spin.

## 3. – The solution of the field equations.

Firstly, we must find a particular solution of the field equations outside the surfaces, $\overset{p}{S}$. The final results will, however, be independant of the surface chosen, as we can always use analytic continuation to extend the approximation field inside the surface. Of course, the solution obtained in this manner would be singular at the world lines, and so physically meaningless there. Even for the Schwarzschild metric, when we take a finite number of terms in this asymptotic expansion, the successive approximation fields bear very little relationship with the exact solution at and near the singularity.

Let $D(x')$ be an invariant Green's function satisfying the wave equation everywhere except the origin. We shall not specify it at present, as the choice is governed by the boundary conditions at infinity or by causal considerations.

We shall write our particular solution of (2.12) in the form,

$$\overset{*}{h}{}^{\mu\nu}_{n} = (2\pi)^{-1} \cdot \int_{V} \mathcal{A}^{\mu\nu}_{n} \cdot D(r - r')\, \mathrm{d}^4 r' \,, \tag{3.1}$$

where $V$ is the 4-volume outside the surfaces. To obtain the general solution of the field equations, we must add an arbitrary solution of the wave equation onto $\overset{*}{h}{}^{\mu\nu}_{n}$,

$$h^{\mu\nu}_{n} = \overset{*}{h}{}^{\mu\nu}_{n} + U^{\mu\nu}_{n} \,, \tag{3.2}$$

where,

$$\Box U^{\mu\nu}_{n} = 0 \,. \tag{3.3}$$

From (3.1) we see that

$$\overset{*}{h}{}^{\mu\nu}_{n\,,\nu} = (2\pi)^{-1} \int_{V} \mathcal{A}^{\mu\nu}_{n}(r') \cdot \frac{\partial}{\partial x^{\nu}} \cdot D(r - r') \cdot \mathrm{d}^4 r' \,. \tag{3.4}$$

Since the Green's function is a function of $(r - r')^2$, we have

$$\frac{\partial}{\partial x^{\nu}} \cdot D(r - r') = -\frac{\partial}{\partial x'^{\nu}} D(r - r') = D(r - r')_{\,,\nu} \,, \tag{3.5}$$

by definition, and so we may reduce (3,3), by integration by parts, to

$$\begin{aligned} \overset{*}{h}{}^{\mu\nu}_{n\,,\nu} &= -(2\pi)^{-1} \int_{V} \mathcal{A}^{\mu\nu}_{n}(r') \cdot \frac{\partial}{\partial x'^{\nu}} \cdot D(r - r') \cdot \mathrm{d}^4 r' = \\ &= (2\pi)^{-1} \int_{V} \mathcal{A}^{\mu\nu}_{n\,,\nu}(r') \cdot D(r - r') \cdot \mathrm{d}^4 r' + (2\pi)^{-1} \sum \int_{p}^{p} \mathcal{A}^{\mu\nu}_{n} \cdot D \cdot \mathrm{d}S'_{\nu} \,, \end{aligned} \tag{3.6}$$

where the $p$ on the integral sign denotes the surface $S^p$, and $\mathrm{d}S'_{\nu}$ is an element of the surface $S^p$ at the point $r'$, pointing along the same direction as the vector $\gamma_{\nu}$. Therefore, provided that the field equations and co-ordinate conditions are satisfied in all previous approximations,

$$h^{\mu\nu}_{n\,,\nu} = (2\pi)^{-1} \sum \int_{p}^{p} \mathcal{A}^{\mu\nu}_{n}(r') \cdot D(r - r') \cdot \mathrm{d}S'_{\nu} + U^{\mu\nu}_{n\,,\nu} \,. \tag{3.7}$$

We shall use (2.1-3) to reduce this surface integral to a line integral along the world lines. We can expand $D(r - r')$, at the point $(s, \gamma^{\alpha})$, as a Taylor's series,

$$D(r - r') = \sum_{t=0}^{\infty} \frac{(-)^t}{t!} \cdot D(r - z(s))_{,(\alpha)_t} \cdot \gamma^{(\alpha)_t} \,, \tag{3.8}$$

where we have used the following notation,

$$F(x^\varrho)_{,(\alpha)_t}\cdot\gamma^{(\alpha)_t} = F(x^\varrho)_{,\alpha_1\alpha_2\ldots\alpha_t}\cdot\gamma^{\alpha_1}\cdot\gamma^{\alpha_2}\ldots\gamma^{\alpha_t}\,. \tag{3.9}$$

Since the surfaces have constant radius and since each $S^p$ is orthogonal to the planes of constant $\overset{p}{s}$, we may take as the surface element $\mathrm{d}^2S\cdot\mathrm{d}u\cdot n_\nu$. $\mathrm{d}^2S$ is a two dimensional element on the surface, $(\gamma)^2 = \text{const.}$, $s(r) = \text{const.}$; $\mathrm{d}u$ is the distance between neighbouring planes, measured along the direction $v^\alpha$ at the point $r'$; $n^\alpha$ is a unit four vector with direction $\gamma^\alpha$. This may be seen by considering the surface element in the rest system of the $p$-th particle. From (2.3), we see that

$$\mathrm{d}s = \frac{v_\alpha(v^\alpha\cdot\mathrm{d}u)}{1-\dot{v}_\alpha\cdot\gamma^\alpha}\,,$$

and therefore

$$\mathrm{d}u = (1-(\dot{v},\gamma))\,\mathrm{d}s\,, \tag{3.10}$$

where we use the notation,

$$(\dot{v},\gamma) = \dot{v}_\alpha\cdot\gamma^\alpha\,.$$

Consequently, the surface integral over the $p$-th singularity will reduce to

$$\overset{*}{\underset{n}{h}}{}^{\mu\nu}{}_{,\nu} = (2\pi)^{-1}\sum_{p,t}\frac{(-)^t}{t!}\int\limits_n^p A^{\mu\nu}\cdot n_\nu\cdot D(r-r')_{,(\alpha)_t}\gamma^{(\alpha)_t}\cdot(1-(\dot{v},\gamma))\,\mathrm{d}^2S\cdot\mathrm{d}s = \tag{3.11}$$

$$= \sum_{p,t}\int\limits_p^p \overset{n}{k}{}^{\mu;(\alpha)_t}\cdot D(r-z)_{,(\alpha)_t}\cdot\mathrm{d}s\,,$$

where the $k$'s are defined by the equations,

$$\overset{p}{\underset{n}{k}}{}^{\mu;(\alpha)_t} = \frac{(-)^t}{2\pi t!}\int\limits_n^p A^{\mu\nu}(s,\gamma)\cdot n_\nu\cdot\gamma^{(\alpha)_t}\cdot(1-(\dot{v},\gamma))\cdot\mathrm{d}^2S\,, \tag{3.12}$$

and are completely symmetric in their indices, $(\alpha)_t$.

So far we have not specified the complementary function, $\underset{n}{U}{}^{\mu\nu}$. This may be either singular at the world lines or regular throughout space. Before we can eliminate this regular function from the field, it is necessary to define the boundary conditions at infinity. The natural hypothesis to make is that either there is no radiation at infinity, or there is no incoming radiation. Provided that one of these assumptions is made we can eliminate the arbitrary non singular wave function by choosing the appropriate Green's function. Under these circumstances, the complementary function for the field equations, (2.12), may be reduced to a line integral. To see this, we notice that, by the appropriate Green's theorem, the complementary function may be written as the integral of a function, linear in the $D(r-r')$ and its normal derivative, over the surfaces, $S^p$. By the methods of the previous paragraphs, this may

be reduced to a line integral along the world lines in the same way as (3.6) was reduced to (3.11).

If the physical system is such that there is radiation coming in from infinity, it can be introduced in the first approximation by the addition of an arbitrary non-singular function — it will not affect the following arguments. Consequently, we shall only look for a solution of (2.12) such that the co-ordinate conditions are satisfied, there being no point in looking for the most general solution in every approximation. Any arbitrariness in the $n$-th approximation is already present in the first approximation, so that the physical character of the solution is determined by $\underset{1}{h}^{\mu\nu}$.

Because of these considerations, we shall write the complementary function as a line integral,

$$\underset{n}{U}^{\mu\nu} = \sum_{t\,p} \int^{p} \overset{p}{\underset{n}{M}}{}^{\mu\nu;(\alpha)_t} \cdot D(r-z)_{,(\alpha)_t} \cdot \mathrm{d}s\,, \tag{3.13}$$

where the $M$'s are completely symmetric in the $\alpha$'s, as well as in $\mu$ and $\nu$. If we take the divergence of (3.13), we have

$$\underset{n}{U}^{\mu\nu}{}_{,\nu} = \sum_{t\,p} \int^{p} \overset{p}{\underset{n}{M}}{}^{\mu\nu;(\alpha)_t} \cdot D(r-z)_{,\nu,(\alpha)_t} \cdot \mathrm{d}s\,, \tag{3.14}$$

so that the co-ordinate conditions may be satisfied, provided that

$$\sum_{t\,p} \int^{p} (\overset{p}{\underset{n}{M}}{}^{\mu\nu;(\alpha)_t} + \overset{p}{\underset{n}{k}}{}^{\mu;\nu(\alpha)_t}) \cdot D(r-z)_{,\nu,(\alpha)_t} \cdot \mathrm{d}s + \sum_{p} \int^{p} \overset{p}{\underset{n}{k}}{}^{\mu} \cdot D(r-z) \cdot \mathrm{d}s = 0\,, \tag{3.15}$$

where the particle tensors, $M^{\cdot\cdot}$, may be chosen as arbitrary functions of the world lines parameter, $s$. We shall chose particular solutions for these parameters so that this equation is satisfied. When this is done, we shall have the complete solution of the field equations and co-ordinate conditions in the $n$-th approximation if we add any solution of the wave equation which also satisfies the co-ordinate conditions, onto this expression.

First, we observe that

$$\begin{aligned} \int f(s) \cdot v^{\varrho} \cdot F(r-z)_{,\varrho} \cdot \mathrm{d}s &= -\int f(s) \cdot v^{\varrho} \cdot \frac{\partial}{\partial z^{\varrho}} \cdot F(r-z) \cdot \mathrm{d}s = \\ &= -\int \frac{\mathrm{d}}{\mathrm{d}s} F(r-z) \cdot f(s) \cdot \mathrm{d}s = \int \dot{f} \cdot F(r-z) \cdot \mathrm{d}s\,, \end{aligned} \tag{3.16}$$

where $F(r-z)$ is any function that vanishes sufficiently rapidly in a time

like direction. We define $\overset{p}{W}{}^{\mu\nu}_{n}$ by the equation,

$$W^{\mu\nu} = \int^{p} (A^{\varrho} v^{\mu} v^{\nu} - A^{\mu} v^{\varrho} v^{\nu} - A^{\nu} v^{\varrho} v^{\mu}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,. \tag{3.17}(*)$$

It is symmetric in $\mu$ and $\nu$, and

$$W^{\mu\nu}{}_{,\nu} = -\int (A^{\mu} v^{\nu} v^{\varrho}) D(r - z)_{,\varrho\nu} \cdot \mathrm{d}s = \tag{3.18}$$

$$= -\int \ddot{A}^{\mu} \cdot D(r - z) \cdot \mathrm{d}s - \int A^{\mu} \cdot v \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,,$$

so that, if we define $A^{\mu}$ by the equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} (A^{\mu}) = k^{\mu} \,, \tag{3.19}$$

we can eliminate the monopole term in (3.15) by adding $W^{\mu\nu}$ onto the original solution. The term in $D_{,\nu}$ will introduce a further complication into the theory, since, if we equate its coefficient to zero in (3.15),

$$M^{\mu\nu} + k^{(\mu\nu)} - A^{\mu} \cdot \dot{v}^{\nu} = 0 \,, \tag{3.20}$$

we cannot satisfy this equation with a *symmetric* $M^{\mu\nu}$. However, if we write

$$Q^{\mu\nu} = \int (F^{\mu\varrho} \cdot v_{\nu} \, \mathrm{c} \, F^{\nu\varrho} \cdot v^{\mu}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,, \tag{3.21}$$

where $F^{\mu\nu}$ is antisymmetric in $\mu$ and $\nu$, we see that

$$Q^{\mu\nu}{}_{,\nu} = \int \frac{\mathrm{d}}{\mathrm{d}s} (F^{\mu\varrho}) \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s \,, \tag{3.22}$$

This shows that we can eliminate the antisymmetric part of (3.20) by adding $Q_{\mu\nu}$ onto the original solution, provided that

$$\frac{\mathrm{d}}{\mathrm{d}s} (F^{\mu\nu}) = -k^{[\mu,\nu]} + A^{[\mu} \cdot \dot{v}^{\nu]} \,, \tag{3.23}(**)$$

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(*) For the sake of clarity, we shall omit the particle index, $p$, and the order index, $n$, for the rest of this section.

(**) We shall use the standard notation, $2.A^{[\mu}B^{\nu]} = A^{\mu}B^{\nu} - A^{\nu}B^{\mu}$, and $2.A^{(\mu}B^{\nu)} = A^{\mu}B^{\nu} + A^{\nu}B^{\mu}$.

and we can eliminate the symmetric part of (3.20) by choosing an $M^{\mu\nu}$ such that

$$M^{\mu\nu} = -k^{(\mu:\nu)} + A^{(\mu}\cdot\dot{v}^{\nu)}\,. \tag{3.24}$$

The higher order multipoles in (3.15) offer no problems whatsoever as they may be eliminated by adding to the field

$$\sum_t \int (k^{\varrho:\mu\nu(\lambda)_t} - k^{\mu:\varrho\nu(\lambda)_t} - k^{\nu:\varrho\mu(\lambda)_t}) D(r-z)_{,\varrho(\lambda)_t}\cdot \mathrm{d}s\,, \tag{3.25}$$

which is symmetric in $\mu$ and $\nu$, because the $k$'s are symmetric in all their indices after the colon. Finally, we obtain a solution of the equations in the $n$'th approximation by adding onto the original integral expression, (3.1), the additional terms, (3.17), (3.21), (3.24), and (3.25). There are, of course, other singular function that we can add to (3.1) to remove the monopole and dipole terms, but we should still have to solve differential equations like (3.19) and (3.23) to satisfy the coordinate conditions. The reason for this is that we cannot remove the monopole, $k^{\mu}$, in general by adding a monopole term to the original field, and so we have to add a dipole. When we insert this into the co-ordinate conditions we obtain a quadrupole expression, and the only way that this can be reduced to a monopole is through the application of equation (3.16) twice. Consequently, we obtain the second derivatives of the dipole constants. The reason that this reduction of a quadrupole to a monopole is possible is that the derivatives of the Green's function are not linearly independant under the integral sign but are related by (3.16) as well as $\eta^{\alpha\beta}\cdot D(r-z)_{,\alpha\beta} = 0$.

> *Theorem 2.* The field equations and co-ordinate conditions may be satisfied in the $n$'th approximation, provided that they are satisfied in every previous approximation. In general, however, there will be a second order differential equation to satisfy, and so the solution will be finite in space like directions only.

The first part of the theorem has been proved, as we have found a particular solution of the field equations which satisfies the co-ordinate conditions. To prove the second part we observe that $A^{\mu}$ must satisfy a second order differential equation and so it will diverge at infinity on the world line. This is the case in the quasi-static approximation where the second order dipole moment may be interpreted as the distance between the physical world line and the world line obtained by integrating the equations of motion in the fourth approximation. The solution will be finite only if the system is periodic.

The answer to this is that the individual $\underset{n}{A}$'s and $\underset{n}{F}$'s are not physical

tensors, but have only been introduced so that the coordinate conditions may be satisfied *exactly* in each approximation. This is a rather naive situation as the field equations have only been satisfied approximately, *i.e.* to the $n$'th approximation. What is wanted is an approximation method in which the coordinate conditions are satisfied to the same approximation as the field equations, but no further. We shall derive a reiteration method in the next chapter which does this without it being necessary to introduce the non-physical $\underset{n}{A}$'s and $\underset{n}{F}$'s. It must first be stressed, however, that these differential equation do *not* arise because we have chosen the De Donder coordinate conditions. Unless these conditions are satisfied, *exactly*, the $n$'th approximation equations of Einstein are not satisfied since we have used the coordinate conditions to reduce Einstein's equations to the simpler Fock equations. However, if the co-ordinate conditions are satisfied to the $n$'th approximation, the solution we derive will satisfy Einstein's gravitational equations to the $n$'th approximation and that is all we should expect from an approximation method!

## 4. – The equations of motion.

If we sum equations (3.19) and (3.23) over all approximation orders, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}(A^\mu) = \sum_{n=1}^{\infty} \overset{p}{\underset{n}{k}}{}^{\mu}(\overset{q}{\underset{r}{A}}{}^{\alpha}, \overset{q}{\underset{r}{F}}{}^{\alpha\beta};\ r < n) \overset{\text{def.}}{=} \overset{p}{k}{}^{\mu}(\overset{q}{\underset{r}{A}}{}^{\nu}, \overset{q}{\underset{r}{F}}{}^{\alpha\beta}), \tag{4.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\overset{p}{F}{}^{\mu\nu}) = \sum_{n=1}^{\infty} (\overset{p}{\underset{n}{k}}{}^{[\nu,\mu]} + \overset{q}{\underset{n}{A}}{}^{[\mu}\overset{p}{v}{}^{\nu]}) \overset{\text{def.}}{=} \overset{p}{k}{}^{[\nu,\mu]} + \overset{p}{A}{}^{[\mu}\overset{p}{v}{}^{\nu]}, \tag{4.2}$$

where we have defined

$$\overset{p}{A}{}^{\mu} = \sum_{n=1} \overset{p}{\underset{n}{A}}{}^{\mu}, \qquad \overset{p}{F}{}^{\mu\nu} = \sum_{n=1} \overset{p}{\underset{n}{F}}{}^{\mu\nu}. \tag{4.3}$$

What we should like to say is that the individual $n$'th order constants have no physical meaning, but only the total $\overset{p}{A}{}^{\mu}$ and $\overset{p}{F}{}^{\mu\nu}$, defined in (4.3). Provided that the right hand sides of equations (4.1) and (4.2), as well as the limiting field, $\underset{(\infty)}{h}{}^{\mu\nu}$, are functions of the *total* spin and dipole moments and not of the individual $n$-th order parameters, we could interpret these equations as the *physical* equations of motion and spin. Under these circumstances, we should not be interested in the solutions of (3.19) and (3.23) at all, but only in the solution of (4.1-2).

Now, it can be proved that it is only the total spin and dipole parameters that appear in the limiting field and the equations of motion. The reason that we have had to introduce the $\underset{n}{A}$'s, etc., is that we have been trying to satisfy the coordinate conditions *exactly* in each approximation. What we shall now prove is that, if we weaken this condition so that the field to the $n$'th approximation does not satisfy these conditions exactly, but only to the $n$'th approximation, we shall not need to expand out these parameters at all. It will be found that the co-ordinate conditions will be satisfied to the $n$'th order provided that *three* equations of spin and four equations of motion and mass are satisfied to the same order.

We shall define the reiteration $\underset{(n)}{k}$'s by the equations,

$$\underset{(n)}{\overset{p}{k}}{}^{\mu} = \sum_{r=1}^{n} \underset{r}{\overset{p}{k}}{}^{\mu}, \qquad \underset{(n)}{\overset{p}{k}}{}^{\mu\nu} = \sum_{r=1}^{n} \underset{r}{\overset{p}{k}}{}^{[\mu\nu]}, \tag{4.4 (*)}$$

where the $\underset{r}{k}$'s will be defined by induction. They correspond to the $k$'s of the previous section but they do not contain the individual $\underset{n}{A}$'s or $\underset{n}{F}$'s. This does not mean that we shall only consider particles whose spin and dipole constants are zero, since the *physical* constants may be introduced into the first approximation, and both the reiteration field and the $k$'s will be functions of these first order constants. What we shall not do is expand these physical tensors into non-physical components of different orders.

We shall now prove, by induction, that it is possible to solve the approximation equations in such a way that $\underset{(n)}{\mathfrak{g}}{}^{\mu\nu}{}_{,\nu}$ is a function in which every term consists of a $\underset{(r)}{k}$ multiplied by a function of the $(n-r)$-th approximation order. When we equate the sum of the $k$'s to zero, for each particle,

$$\underset{(\infty)}{\overset{p}{k}}{}^{\mu} = \sum_{n=1}^{\infty} \underset{n}{\overset{p}{k}}{}^{\mu} = 0, \qquad \underset{(\infty)}{\overset{p}{k}}{}^{\mu\nu} = \sum_{n=1}^{\infty} \underset{n}{\overset{p}{k}}{}^{\mu\nu} = 0, \tag{4.5}$$

we shall have

$$\underset{(r)}{\overset{p}{k}}{}^{\cdots} = -\sum_{r+1}^{\infty} \underset{r}{\overset{p}{k}}{}^{\cdots}, \tag{4.6}$$

where every term on the right hand side is of the $(r+1)$-th, or higher, order. Consequently, every term in $\underset{(n)}{\mathfrak{g}}{}^{\mu\nu}{}_{,\nu}$ will be zero *to the* $n$-th *order*, and so the co-ordinate conditions will be satisfied to that order. Hence, provided that the

(*) In the later part of this section, we shall show that $\underset{n}{k}{}^{\mu\nu}v_{\mu} = 0$ and so there are only three independent components.

solution is convergent, the co-ordinate conditions will be satisfied by the limiting field, whenever equations (4.5) are satisfied.

FOCK has shown that the Einstein tensor may be reduced to

$$2\mathfrak{G}^{\mu\nu} = -\Box h^{\mu\nu} + 2\Lambda^{\mu\nu} + Z^{\mu\nu}, \tag{4.7}$$

where

$$Z^{\mu\nu} = h^{\varkappa\beta}{}_{,\beta} \cdot P^{\mu\nu}_{\varkappa} + h^{\varkappa\beta}{}_{,\beta\varrho} \cdot Q^{\mu\nu\varrho}_{\varkappa}. \tag{4.8}$$ (*)

The Bianchi identities may be written as

$$\begin{aligned} &-\Box h^{\mu\nu}{}_{,\nu} + 2\Lambda^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\varkappa\beta}(-\Box h^{\varkappa\beta} + 2\Lambda^{\varkappa\beta}) \\ &\quad = -h^{\varkappa\beta}{}_{,\beta}(P^{\mu\nu}_{\varkappa}{}_{,\nu} + \Gamma^{\mu}_{\varrho\nu} P^{\varrho\nu}_{\varkappa}) - h^{\varkappa\beta}{}_{,\beta\gamma}(P^{\mu\gamma}_{\varkappa} + Q^{\mu\nu\gamma}_{\varkappa}{}_{,\nu} + \Gamma^{\mu}_{\varrho\nu} Q^{\varrho\nu\gamma}_{\varkappa}) - h^{\varkappa\beta}{}_{,\beta\varrho\nu} Q^{\mu\nu\varrho}_{\varkappa}, \end{aligned} \tag{4.9}$$

where the important thing to notice is that the right hand side contains the divergence of the field tensor as a linear factor in each term. If we insert the $\lambda$-series, (2.11), into this identity, and then equate the coefficient of $\lambda^n$ to zero, we have

$$2\underset{n}{\Lambda}{}^{\mu\nu}{}_{,\nu} + \sum_{r>0} \underset{r}{\Gamma}{}^{\mu}_{\varkappa\beta}(-\Box \underset{n-r}{h}{}^{\varkappa\beta} + 2\underset{n}{\Lambda}{}^{\alpha\beta}_{r}) = -\sum_{r=1}^{n-1} \underset{n-r}{h}{}^{\alpha\beta}{}_{,\beta}(\underset{r}{P}{}^{\mu\nu}_{\alpha,\nu} + \underset{r}{\Gamma}{}^{\mu}_{\varrho\nu} \underset{r}{P}{}^{\varrho\nu}_{\varkappa}) - \ldots, \tag{4.10}$$

where there are six terms on the R.H.S., all linear in $\underset{r}{h}{}^{\alpha\beta}{}_{,\beta}$ or its derivatives. It might be wondered why we have omitted the first term in (4.9) from (4.10). The reason is that, since (4.9) is an identity, it must cancel with the terms on the R.H.S. which are linear in the derivatives of $\underset{(n)}{h}{}^{\varkappa\beta}$. This is also the reason why we have started the summation in (4.10) from $r = 1$ rather than from $r = 0$.

When we sum these equations from 1 to $n$, we have

$$2\underset{(n)}{\Lambda}{}^{\mu\nu}{}_{,\nu} + \sum \underset{n-r}{\Gamma}{}^{\mu}_{\varkappa\beta}(-\Box \underset{(r)}{h}{}^{\varkappa\beta} + 2\underset{(r)}{\Lambda}{}^{\varkappa\beta}) = -\sum_{r=1} \underset{(n-r)}{h}{}^{\alpha\beta}{}_{,\beta} \cdot \underset{r}{P}{}^{\mu\nu}_{\alpha,\nu} - \ldots, \tag{4.11}$$

where we must be careful to distinguish between the reiteration and approximation indices, $(t)$ and $t$ respectively.

Now, let us assume that the field equations are satisfied in all reiteration orders up to the $(n-1)$-th. Also, let us assume, as part of the induction hypothesis, that $\underset{(r)}{h}{}^{\mu\nu}{}_{,\nu}$ is a function in which every term contains a $\overset{\cdot\cdot}{k}{}^{s}_{t}$, or its derivative with respect to $s$, multiplied by a function of approximation

(*) See appendix A where $P^{\mu\nu}_{\alpha}$ and $Q^{\mu\nu\varrho}_{\varkappa}$ are defined completely. All that we are interested in here is that $Z^{\mu\nu}$ should be linear in $h^{\mu\nu}{}_{,\nu}$.

order $r-t$. If we call such a function an $O_r$-function, then we see, from (4.11), that $\underset{(n)}{A}{}^{\mu\nu}{}_{,\nu}$ is also an $O_n$-function. From the last two sections, we see that

$$\underset{(n)}{h}{}^{\mu\nu}{}_{,\nu} = +(2\pi)^{-1}\int_V \underset{(n)}{A}{}^{\mu\nu}{}_{,\nu}(r')\cdot D(r-r')\cdot \mathrm{d}^4r' + \text{multipoles}\,. \tag{4.12}$$

The first term on the R.H.S. is an $O_n$-function. In the last section we saw that we could chose the multipoles in such a way that the coefficient of all higher order multipoles were zero and this only involved algebraic processes through (3.25). Also, we can remove the symmetric part of the dipole field from (4.12) as in Sect. 3. The coefficients of the monopoles and the antisymmetric part of the dipole field define the reiteration $\overset{p}{\underset{(n)}{k}}{}^{\cdot\cdot}$, by induction. This shows that the $n$'th order reiteration field will satisfy the induction hypothesis, *i.e.* it will be an $O_n$-function, provided that all previous reiteration field do. Consequently, we have proved by induction that the divergence of the reiteration field will be an $O_n$-function.

It might be thought that our proof depends on the way that we have integrated the field equations, but this is not so. Any solution of the field equations, satisfying the appropriate boundary conditions, outside the surfaces may be obtained from any other by adding a set of multipoles, since the difference satisfies the wave equation. Furthermore, if we use analytic continuation to extend this solution into the interior of the surfaces, we can obtain an arbitrary solution of the field equations.

*Theorem 3.* Provided that the coordinate conditions are satisfied approximately, in the sense explained above, for all lower order reiteration fields, we may integrate the field equations arbitrarily in the $n$'th approximation and then satisfy the coordinate conditions to the $n$'th approximation by the following procedure:

Firstly, we add this $\overset{*}{\underset{n}{h}}{}^{\mu\nu}$ onto the previously calculated reiteration field to give a solution of the $n$'th order reiteration equations, and then we calculate its divergence, $\overset{*}{\underset{(n)}{h}}{}^{\mu\nu}{}_{,\nu}$. We then use the equations of motion and spin to the $(n-1)$-th approximation to reduce this to a sum of multipoles. The higher order multipoles are eliminated as in (3.25) and the symmetric part of the dipole terms as in (3.20-4). This leaves us with a monopole and the antisymmetric part of the dipole field which define the $n$'th order reiteration $k$'s

$$\underset{(n)}{h}{}^{\mu\nu}{}_{,\nu} = \sum_p \int_p (\overset{p}{\underset{(n)}{k}}{}^{\mu}\cdot D(r-z) + \overset{p}{\underset{(n)}{k}}{}^{\mu\nu} D(r-z)_{,\nu})\cdot \mathrm{d}s + O_n\text{-function}\,. \tag{4.13}$$

The coordinate conditions will be satisfied to the $n$'th order, provided that

$$\overset{1}{\underset{(n)}{k}}{}^{\nu}, \ \overset{n}{\underset{(n)}{k}}{}^{\mu\nu} = 0 , \quad \text{to the } n\text{'th order.} \tag{4.14}$$

This follows because it implies that $\overset{n}{\underset{(r)}{k}}{}^{\cdot\cdot}$ is zero to the $r$'th approximation for all $r \leqslant n$.

The interpretation of these equations will depend on the first order solution chosen. Since the source function, $\overset{1}{A}{}^{\mu\nu}$, is zero, the field $\overset{1}{h}{}^{\mu\nu}$ must satisfy the wave equation. We shall now calculate the most general pole-dipole solution of the wave equation, such that the coordinate conditions are satisfied approximately in the sense above. We shall write the first order solution as

$$\overset{1}{h}{}^{\mu\nu} = \int M^{\mu\nu} \cdot D(r - z) \cdot \mathrm{d}s + \int M^{\mu\nu:\varrho} \cdot D(r - z)_{,\varrho} \cdot \mathrm{d}s . \tag{4.15}$$

It should be noticed that the dipole and quadrupole moments are related by the equations,

$$\int \eta^{\alpha\beta} \cdot D(r - z)_{,\alpha\beta} \cdot \mathrm{d}s = 0 , \tag{4.16}$$

$$\int f(s) \cdot v^{\alpha} \cdot D(r - z_{,\alpha\mu}) = \int \dot{f} \cdot D(r - z)_{,\mu} \cdot \mathrm{d}s . \tag{4.16'}$$

Equation (4.16)' follows from (3.16).

We shall look for the most general solution such that

$$\overset{1}{h}{}^{\mu\nu} = \int k^{\mu} \cdot D(r - z)\, \mathrm{d}s + \int k^{\mu\nu} \cdot D(r - z)_{,\nu} \cdot \mathrm{d}s . \tag{4.17}$$

First of all we observe that we may write $M^{\mu\nu:\varrho}$ as

$$\begin{aligned} M^{\mu\nu:\varrho} &= (M^{\mu\nu:\varrho} - M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho}) + M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho} , \\ &= \overset{*}{M}{}^{\mu\nu:\varrho} + M^{\mu\nu:\alpha} v_{\alpha} v^{\varrho} . \end{aligned} \tag{4.18}$$

By using (3.16), we may reduce the field corresponding to the second term in (4.18) to a monopole field. Consequently, since the first term satisfies $\overset{*}{M}{}^{\mu\nu:\varrho} \cdot v_{\varrho} = 0$, we may assume that

$$v_{\alpha} M^{\mu\nu:\alpha} = 0 , \tag{4.19}$$

without loss of generality.

Also, we have

$$k^{\mu\nu}_{\smile} \cdot v_{\nu} = 0 \,, \tag{4.20}$$

this equation being true for all the approximation and reiteration $k$'s of this section. To see this, let us suppose that we have reduced the divergence of the field tensor in the $n$'th reiteration to (4.13). We may write the dipole expression as

$$\begin{aligned} k^{\mu\nu}_{\smile} &= (k^{\mu\nu}_{\smile} - k^{\mu\nu}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu}) - (k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu}) + 2k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} \,, \\ &= \overset{*}{k}{}^{\mu\nu}_{\smile} - W^{\mu\nu}_{\smile} + 2k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} \,, \end{aligned} \tag{4.21}$$

where

$$\begin{cases} \overset{*}{k}{}^{\mu\nu}_{\smile} = k^{\mu\nu}_{\smile} - k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu} \,, \\ W^{\mu\nu}_{\smile} = k^{\mu\alpha}_{\smile} v_{\alpha} v^{\nu} + k^{\nu\alpha}_{\smile} v_{\alpha} v^{\mu} \,. \end{cases} \tag{4.22}$$

Since $W^{\mu\nu}_{\smile}$ is symmetric in $\mu$ and $\nu$, we can remove it from the field by the addition of a monopole field to the field tensor, as in (3.20-4). By the use of (3.16), we can reduce the field corresponding to the third term on the R.H.S. of (4.21) to a monopole field. This leaves us with a similar expression to (4.13) except that $k^{\mu\nu}_{\smile}$ is replaced by $\overset{*}{k}{}^{\mu\nu}_{\smile}$, which satisfies (4.20). This shows that there are only *three independent* differential equations corresponding to the independent components of $k^{\mu\nu}_{\smile}$.

We shall now proced with the calculation of the most general pole-dipole field. From (4.15) we see that

$$\underset{1}{h}{}^{\mu\nu}{}_{,\mu} = \int (M^{\mu\nu}_{\smile} \cdot D(r-z)_{,\nu} + M^{\mu\nu;\varrho}_{\smile} \cdot D(r-z)_{,\varrho\nu}) \cdot \mathrm{d}s \,. \tag{4.23}$$

Equation (4.17) and (4.23) do not imply that $M^{\mu\nu;\varrho}$ is zero, since the multipoles are connected by (4.16) and (4.16)′. Instead, we must have

$$M^{\mu\nu;\varrho}_{\smile} = B^{\mu} \cdot \eta^{\nu\varrho} + C^{\mu\nu} \cdot v^{\varrho} + D^{\mu\varrho} \cdot v^{\nu} + F^{\mu;\nu\varrho}_{\smile} \,, \tag{4.24}$$

where $F^{\mu;\nu\varrho}_{\smile}$ is antisymmetric in $\nu$ and $\varrho$, and the parameters on the right hand side have to be determined. Since the L.H.S. is symmetric in $\mu$ and $\nu$, we have

$$F^{\mu;\nu\varrho}_{\smile} - F^{\nu;\mu\varrho}_{\smile} = -B^{\mu} \cdot \eta^{\nu\varrho} + B^{\nu} \cdot \eta^{\mu\varrho} - C^{[\mu\nu]} v^{\varrho} + D^{\nu\varrho} v^{\mu} - D^{\mu\varrho} v^{\nu} \,. \tag{4.25}$$

Therefore, by interchanging the coefficients, we have

$$\begin{aligned} F^{\mu;\nu\varrho}_{\smile} &= \tfrac{1}{2}(F^{\mu;\nu\varrho}_{\smile} - F^{\nu;\mu\varrho}_{\smile}) + \tfrac{1}{2}(F^{\varrho;\nu\mu}_{\smile} - F^{\nu;\varrho\mu}_{\smile}) + (F^{\varrho;\mu\nu}_{\smile} - F^{\mu;\varrho\nu}_{\smile}) = \\ &= B^{\nu}\eta^{\mu\varrho} - B^{\varrho}\eta^{\mu\nu} + C^{[\nu\varrho]}v^{\mu} - C^{[\mu\nu]}v^{\varrho} + C^{[\mu\varrho]}v^{\nu} + D^{(\mu\nu)}v^{\varrho} + D^{[\nu\varrho]}v^{\mu} - D^{(\mu\varrho)}v^{\nu} \,, \end{aligned} \tag{4,26}$$

Consequently, we have that

$$M^{\underset{\smile}{\mu\nu}:\varrho} = (B^{\mu}\eta^{\nu\varrho} - B^{\varrho}\delta^{\mu\nu} + B^{\nu}\eta^{\mu\varrho}) + C^{[\nu\varrho]}v^{\mu} + C^{[\mu\varrho]}v^{\nu} + C^{(\mu\nu)}v^{\varrho} + \\ + D^{[\nu\varrho]}v^{\mu} + D^{[\mu\varrho]}v^{\nu} + D^{(\mu\nu)}v^{\varrho} . \tag{4.27}$$

The first term on the R.H.S. corresponds to the usual allowable coordinate transformations so we shall take $B^{\mu} = 0$. If we substitute

$$E^{\underset{\smile}{\mu\nu}} = C^{[\mu\nu]} + D^{[\mu\nu]} , \qquad G^{\underset{\smile}{\mu\nu}} = C^{(\mu\nu)} + D^{(\mu\nu)} , \tag{4.28}$$

which are obviously antisymmetric and symmetric, respectively, we may write (4,27) as

$$M^{\underset{\smile}{\mu\nu}:\varrho} = v^{\mu}E^{\underset{\smile}{\nu\varrho}} + v^{\nu}E^{\underset{\smile}{\mu\varrho}} + v^{\vartheta}G^{\underset{\smile}{\mu\nu}} . \tag{4.29}$$

If we multiply this equation by $v_{\varrho}$ and use (4.19), we have

$$G^{\underset{\smile}{\mu\nu}} = - v_{\lambda}E^{\underset{\smile}{\mu\lambda}}v^{\nu} - v_{\alpha}E^{\underset{\smile}{\nu\alpha}}v^{\mu} , \tag{4.30}$$

and so we may rewrite the dipole tensor as

$$M^{\underset{\smile}{\mu\nu}:\varrho} = v^{\mu}E^{\underset{\smile}{\nu\varrho}} + v^{\nu}E^{\underset{\smile}{\mu\varrho}} - (v_{\lambda}E^{\underset{\smile}{\mu\lambda}}v^{\nu} + v_{\lambda}E^{\underset{\smile}{\nu\lambda}}v^{\mu})v^{\varrho} , \tag{4.31}$$

$$= 4v^{\mu}S^{\underset{\smile}{\nu\varrho}} + 4v^{\nu}S^{\underset{\smile}{\mu\varrho}} + 4D^{\varrho}v^{\mu}v^{\nu} ,$$

where

$$4S^{\underset{\smile}{\mu\nu}} = E^{\underset{\smile}{\mu\nu}} - E^{\underset{\smile}{\mu\lambda}}v_{\lambda}v^{\nu} + E^{\underset{\smile}{\nu\lambda}}v_{\lambda}v^{\mu} ,$$

$$4D_{\mu} = - 2E^{\underset{\smile}{\mu\lambda}}v_{\lambda} .$$

From their definitions, we see that

$$v_{\lambda}S^{\underset{\smile}{\mu\lambda}} = v_{\lambda}D^{\alpha} = 0 . \tag{4.32}$$

If we use (4.31) in (4.15), we have

$$\overset{1}{h}{}^{\mu\nu}{}_{,\nu} = \int \left( M^{\underset{\smile}{\varrho\nu}} + 4\frac{\mathrm{d}}{\mathrm{d}s}(D^{\nu}v^{\mu}) + 4\dot{S}^{\underset{\smile}{\mu\nu}} \right) D(r-z)_{,\nu}\,\mathrm{d}s . \tag{4.33}$$

Since this must be equal to the R.H.S. of (4.17),

$$M^{\underset{\smile}{\mu\nu}} + \frac{\mathrm{d}}{\mathrm{d}s}(4D^{\nu}v^{\mu}) + 4\dot{S}^{\underset{\smile}{\mu\nu}} = k^{\underset{\smile}{\mu\nu}} + H^{\mu}v^{\nu} , \tag{4.34}$$

where $H^\mu$ is an arbitrary parameter. Taking the antisymmetric part of this equation,

$$2\frac{\mathrm{d}}{\mathrm{d}s}(D^\nu v^\mu - D^\mu v^\nu) + 4\dot{S}^{\mu\nu} + \tfrac{1}{2}(H^\nu v^\mu - H^\mu v^\nu) = k^{\mu\nu}, \tag{4.35}$$

and multiplying by $v_\nu$,

$$\tfrac{1}{2}\cdot H^\mu - \tfrac{1}{2}\cdot H^\alpha v_\alpha v^\mu = 4S^{\mu\alpha}v_\alpha - 2\dot{D}^\mu, \tag{4.36}$$

$$k^{\mu\nu} = 4\dot{S}^{\mu\nu} - 4(S^{\mu\alpha}v_\alpha v^\nu - S^{\nu\alpha}v_\alpha v^\mu) + 2(D^\nu \dot{v}^\mu - D^\mu \dot{v}^\nu).$$

Consequently,

$$H^\mu = mv^\mu + 8\dot{S}^{\mu\alpha}v_\alpha - 4\dot{D}^\mu, \tag{4.37}$$

and so

$$M^{\mu\nu} = 4mv^\mu v^\nu + 4(\dot{S}v_\alpha v^\nu + \dot{S}^{\nu\alpha}v_\alpha v^\mu) - 4(\dot{D}^\mu v^\nu + \dot{D}^\nu v^\mu) - 2(D^\mu \dot{v}^\nu + D^\nu \dot{v}^\mu), \tag{4.38}$$

where $m$ is an arbitrary parameter: $4m = H^\alpha v_\alpha$.

This shows that we can characterise the most general pole-dipole solution in the first approximation, by a mass parameter, a spin tensor, and the dipole moment of the rest mass, satisfying (4.32). This corresponds to the results of PAPAPETROU for a test body, though his results are slightly different in form because he defined the particle moments differently.

We have for the first order field,

$$\underset{1}{h}{}^{\mu\nu} = \int (4mv^\mu v^\nu - 4(\dot{D}^\mu v^\nu + \dot{D}^\nu v^\mu) - 2(D^\mu \dot{v} + D^\nu \dot{v}^\mu) - \tag{4.39}$$

$$- 4S^{\mu\alpha}\dot{v}_\alpha v^\nu - 4S^{\nu\alpha}\dot{v}_\alpha v^\mu)\, D(r-z)\,\mathrm{d}s + 4\int (S^{\mu\varrho}v^\nu + S^{\nu\varrho}v^\mu + D^\varrho v^\mu v^\nu) D(r-z)_{,\varrho}\,\mathrm{d}s,$$

We have introduced the factor of 4 so that the gravitational constant is unity. The particle tensors must satisfy (4.32).

If we take the divergence of (4.39), we verify that (4.17) holds with the first order reiteration $k$'s as follows:

$$\underset{1}{k}{}^\mu = 4\frac{\mathrm{d}}{\mathrm{d}s}(mv^\mu) - 4\ddot{D}^\mu - 8\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\alpha}\dot{v}^\alpha), \tag{4.40}$$

$$\underset{1}{k}{}^{\mu\nu} = 4\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\nu}) + 4(S^{\mu\varrho}\dot{v}_\varrho v^\nu - S^{\nu\varrho}\dot{v}_\varrho v^\mu) + 2(D^\nu \dot{v}^\mu - D^\mu \dot{v}^\nu). \tag{4.41}$$

where it will be observed that $\overset{\mu\nu}{k_1}$ satisfies (4.20), and so has only three independent components. In the previous section we were only interested in showing that the field equations and coordinate conditions could be satisfied in each approximation so we did not reduce the $\overset{[\mu\nu]}{k_n}$ to functions satisfying (4.20). In this section we are interested in the physical interpretation of the results. The reason that we have been able to satisfy (4.20) is that the multipoles are not independent, but are related by (4.16) and (4.16)'. This leaves only three independent spin *differential* equations.

Let us now suppose that we have found the reiateration $k$'s to the $n$'th order. When we equate them to zero, and use (4.40) and (4.41),

$$4\frac{\mathrm{d}}{\mathrm{d}s}(mv^\mu) - 4\ddot{D}^\mu - 8\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\alpha}\dot{v}_\alpha) = -\sum_{r=2}\overset{\mu}{k_r}, \tag{4.42}$$

$$4\frac{\mathrm{d}}{\mathrm{d}s}(S^{\mu\nu}) + 4(S^{\mu\alpha}\dot{v}_\alpha v^\nu - S^{\nu\alpha}\dot{v}_\alpha v^\mu) + 2(D^\nu\dot{v}^\mu - D^\mu\dot{v}^\nu) = -\sum_{r=2}^{n}\overset{\mu\nu}{k_r}, \tag{4.43}$$

where

$$S^{\mu\nu}v_\nu = D^\nu v_\mu = 0\,. \tag{4.44}$$

If we multiply (4.42) by $v_\mu$ and use the differential of (4.44) and also the equation, $\dot{v}_\nu v^\nu = 0$, we have

$$4\dot{m} - 4\ddot{D}^\mu v_\mu + 8v_\mu\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_\alpha) = 4\dot{m} - 4\ddot{D}^\alpha v_\alpha + 8\frac{\mathrm{d}}{\mathrm{d}s}(\dot{S}^{\mu\alpha}v_\alpha v_\mu) - 8S^{\mu\alpha}\dot{v}_\mu\dot{v}_\alpha, \tag{4.45}$$

$$= -\sum_{r=2}\overset{\alpha}{k_r}v_\mu\,,$$

and therefore

$$4\dot{m} = 4\ddot{D}^\nu v_\nu - \sum_{r=2}^{n}k^\nu v_\nu\,. \tag{4.45}$$

which is the equation of mass or energy. The remaining three equations of (4.42) give the equations of motion, while (4.43) correspond to the classical equations of angular momentum.

These equations are equivalent to those obtained by PAPAPETROU ([6]) for spinning test particles in general relativity. They are also equivalent to those derived by MATHISSON ([7]), LUBANSKI ([8]), and HÖNL and PAPAPETROU ([9,10]) for

([5]) A. PAPAPETROU: *Proc. Roy. Soc.*, **209**, 248 (1951).
([6]) M. MATHISSON: *Acta Phys. Polon.*, **6**, 167 (1937).
([7]) H. LUBANSKI: *Acta Phys. Polon.*, **6**, 356 (1937).
([8]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **112**, 512 (1939).
([9]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **114**, 478 (1939).
([10]) H. HÖNL and A. PAPAPETROU: *Zeits. Phys.*, **116**, 153 (1940).

spinning test particles in special relativity. What is interesting is that they should arise from the structure of the field equations outside the particles, rather than the precise form of the assumed energy momentum tensor. In a later paper, we shall calculate the equations of motion and spin for a particle in an external field arising from a set of particels.

Also, we shall show that essentially the same equations arise in the quasi-static approximation. The analysis is slightly different in that it is impossible to satisfy the coordinate conditions *exactly* in the $n$'th approximation unless certain differential equations are satisfied by the particle parameters *to the previous approximations*. As we have seen, the differential equations arising in the $n$'th covariant approximation involve the particle parameters of the same order. The reason for this is that derivatives with respect to the non-covariant time coordinate in the quasi-static approximation are considered to raise the approximation order of the function differentiated.

It will be noticed that (4.42-3) are inconsistent if we equate the spin tensor to zero. This is also true of the quasi-static approximation. As is to be expected, a dipole particle in an external field will start to spin under the applied couple. It is not possible, in general, to satisfy the coordinate conditions exactly in each approximation by introducting a dipole moment, unless a corresponding spin tensor is also introduced. In the quasi-static approximation there are three extra surface integral conditions to be satisfied as well as those found by EINSTEIN and INFELD. These correspond to the equations of spin.

## APPENDIX

Fock has show that, if we introduce the following functions,

$$\pi^{\mu,\alpha\beta} = \frac{1}{2g}\left(\mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta}{}_{,\nu} + \mathfrak{g}^{\beta\nu}\mathfrak{g}^{\mu\alpha}{}_{,\nu} - \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\beta\alpha}{}_{,\nu}\right),$$

$$\pi^{\mu\nu}{}_{\lambda} = g_{\mu\alpha}g_{\nu\beta}\pi^{\lambda,\alpha\beta},$$

$$y_{\mu} = \left(\log(-g)^{\frac{1}{2}}\right)_{,\mu}, \qquad y^{\mu} = g^{\mu\alpha}{}_{,\alpha},$$

$$\Gamma^{\mu} = -(-g)^{-\frac{1}{2}}\mathfrak{g}^{\mu\varrho}{}_{,\varrho},$$

$$\Gamma_{\mu\nu} = \tfrac{1}{2}\left(g_{\mu\alpha}\Gamma^{\alpha}{}_{,\nu} + g_{\nu\alpha}\Gamma^{\alpha}{}_{,\mu} + g_{\mu\nu,\alpha}\Gamma^{\alpha}\right),$$

$$\Gamma^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\Gamma_{\alpha\beta},$$

the Einstein tensor-density may be reduced to the following,

$$\mathfrak{G}^{\mu\nu} = -\tfrac{1}{2}(-g)^{-\frac{1}{2}}\mathfrak{g}^{\varkappa\beta}\mathfrak{g}^{\mu\nu}{}_{,\alpha\beta} + \pi^{\mu,\varkappa\beta}\pi_{\alpha\beta}{}^{\nu}(-g)^{-\frac{1}{2}} + \tfrac{1}{4}\mathfrak{g}^{\mu\nu}y_{\alpha}y^{\alpha} - \tfrac{1}{2}(-g)^{\frac{1}{2}}y^{\mu}y^{\nu} -$$

$$-\tfrac{1}{4}(-g)^{-\frac{1}{2}}\cdot\mathfrak{g}^{\mu\nu}\pi_{\alpha\beta}{}^{\varrho}\cdot\mathfrak{g}^{\alpha\beta}{}_{,\varrho} - (-g)^{\frac{1}{2}}\Gamma^{\mu\nu} - \tfrac{1}{2}(-g)^{\frac{1}{2}}(\Gamma^{\mu}y^{\nu} + \Gamma^{\nu}y^{\mu}) + \tfrac{1}{2}\mathfrak{g}^{\mu\nu}(g^{\alpha\beta}\Gamma_{\alpha\beta} + \Gamma^{\alpha}y_{\alpha}).$$

For our purposes, it is sufficient that it should be possible to write this as

$$2\mathfrak{G}^{\mu\nu} = -h^{\mu\nu} + 2A^{\mu\nu} + Z^{\mu\nu},$$

where

$$Z^{\mu\nu} = h^{\alpha\beta}{}_{,\beta}P^{\mu\nu}_{\mu} + h^{\alpha\beta}{}_{,\beta\varrho}Q^{\mu\nu:\varrho}_{\alpha}.$$

$Z^{\mu\nu}$ consists of the last three terms in the expression of $\mathfrak{G}^{\mu\nu}$. This is so because the last three terms are all linear in $\Gamma^{\alpha}$, or its derivative.

## RIASSUNTO (*)

Si è analizzato il metodo d'approssimazione covariante di Lorentz per il campo esterno a un sistema di particelle localizzate. Si è trovato che oltre le solite equazioni del moto e dell'energia ottenute da Einstein, Infeld e Hoffman per l'approssimazione quasi statica ne esistono altre tre, le equazioni dello spin, che debbono essere soddisfatte dalle costanti strutturali di ogni particella. Tali equazioni appaiono anche nella approssimazione quasi statica come condizioni degli integrali di superficie. Si dimostra inoltre che non è necessario sviluppare in serie le costanti di massa, dipolo o spin, essendo le costanti introdotte nella prima approssimazione quelle delle particelle fisiche. Solo le equazioni differenziali soddisfatte da queste costanti cambiano nelle approssimazioni degli ordini superiori.

(*) *Traduzione a cura della Redazione.*