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Classical Field Theory in the Space of Reference Frames.

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Summary. — The formalism of classical field theory is generalized by replacing the space-time manifold $\mathscr{M}$ by the ten-dimensional manifold $\mathscr{S}$ of all the local reference frames. The geometry of the manifold $\mathscr{S}$ is determined by ten vector fields corresponding to ten operationally defined infinitesimal transformations of the reference frames. The action principle is written in terms of a differential 4-form in the space $\mathscr{S}$ (the Lagrangian form). Densities and currents are represented by differential 3-forms in $\mathscr{S}$. The field equations and the connection between symmetries and conservation laws (Noether's theorem) are derived from the action principle. Einstein's theory of gravitation and Maxwell's theory of electromagnetism are reformulated in this language. The general formalism can also be used to formulate theories in which charge, energy and momentum cannot be localized in space-time and even theories in which a space-time manifold cannot be defined exactly in any useful way.

1. – Introduction.

The primitive concepts on which the usual field theories are based are the space-time manifold $\mathscr{M}$ and a set of observables which depend on a point of $\mathscr{M}$, namely the fields. One has to remark, however, that, if an observable is described by a given component of a tensor field, it is not completely specified by giving a point of space-time; one has to give a local reference frame. It seems, therefore, that the space $\mathscr{S}$ of all the local reference frames has a more fundamental physical meaning than the space-time $\mathscr{M}$.

This idea has been discussed in detail by LURÇAT (1). His work was mo-

(1) F. LURÇAT: *Physica*, **1**, 95 (1964).

tivated by the need of taking into account the dynamical role of the spin of elementary particles, suggested mainly by the discovery of Regge trajectories. If we disregard gravitation, the space $\mathscr{S}$ is isomorphic to the manifold of the Poincaré group (without the group structure). Lurçat's program was to build a quantum field theory on this group.

The concept of local reference frame can be introduced in two different ways. One can start from the space-time manifold $\mathscr{M}$ and define mathematically a local reference frame as a basis in the vector space tangent to $\mathscr{M}$ at a given point. Alternatively, one can give a direct physical definition of a local reference frame by means of a physical object with respect to which positions, directions, time and velocities are determined. Then one can possibly define mathematically the space-time manifold $\mathscr{M}$ in terms of the space $\mathscr{S}$ of all the local reference frames. When we say that the space $\mathscr{S}$ is more fundamental than the space $\mathscr{M}$, we mean that the second way is preferable from the physical point of view.

In the present paper we want to develop a field theory on the space $\mathscr{S}$ disregarding quantum effects, but taking gravitation into account. In analogy with Einstein's general relativity, the gravitational field is described as a geometric property of the space $\mathscr{S}$. Therefore, we have to find a geometric structure of the space $\mathscr{S}$ which plays a role similar to the pseudo-Riemannian metric of the space-time $\mathscr{M}$ of general relativity.

This geometric structure is suggested by an operational analysis of the physical space-time concepts ([2,3]), based on the requirement ([4]) that the primitive concepts of a theory should represent « procedures », namely prescriptions according to which one performs physical operations.

The prescriptions which form a procedure necessarily refer to some pre-existent physical objects which specify a local reference frame. One can consider « measurement procedures » the aim of which is to obtain a numerical result and « transformation procedures » the aim of which is to build a new reference frame starting from a pre-existent frame. According to the program described in ref. ([3]), the geometric concepts of physics should be defined in terms of transformation procedures. The measurement procedures define the observables.

Note that a local reference frame cannot be defined operationally, namely in terms of procedures. A transformation procedure can only define a relation between two local reference frames. Therefore, if we apply rigorously the operational point of view, the local reference frames should not appear as terms of the theory. The same argument holds with better reason for the points of space-time.

([2]) M. Toller: *Int. Journ. Theor. Phys.*, **12**, 349 (1975).
([3]) M. Toller: *Nuovo Cimento*, **40** B, 27 (1977).
([4]) R. Giles: *Journ. Math. Phys.*, **11**, 2139 (1970).

In the present paper, nevertheless, we shall use the space $\mathscr{S}$. This is possible because we assume that the objects which form a reference frame have very special properties: they do not interact with the other physical objects, apart from the very weak interaction necessary to transmit some information. Moreover, the operations used to construct a local frame do not interfere with other physical operations.

If we accept these assumptions, we may imagine that all the possible reference frames have really been constructed and labelled by means of a set of real numbers and, therefore, we may consider the space $\mathscr{S}$. This point of view, which we call classical space-time theory, has to be abandoned if we take into account the fact that the reference frames are formed by real physical objects which interact with all the surrounding objects, for instance the objects under investigation or the objects which form other reference frames. Then we should construct a quantum space-time theory.

Given the space $\mathscr{S}$, a transformation procedure defines a mapping of $\mathscr{S}$ into itself. In fact, if we perform a given transformation procedure starting from a given local reference frame, we obtain a new local reference frame. Note that we are neglecting the unavoidable statistical errors. It is important to remark that a transformation procedure does not define a mapping of the space-time manifold $\mathscr{M}$ into itself, because, in order to perform the operations prescribed by the procedure, it is not sufficient to specify a space-time point; one needs a local reference frame. We see that the space $\mathscr{S}$ is more strictly related to the fundamental concept of transformation procedure than the space-time $\mathscr{M}$.

It may happen that two different transformation procedures define the same mapping of $\mathscr{S}$ into itself. Then we say that the two transformation procedures are equivalent. It is convenient to consider the equivalence classes of transformation procedures, which we call « transformations ». A transformation is uniquely individuated by the corresponding mapping of $\mathscr{S}$ into itself.

In the following we shall consider only infinitesimal transformations, which correspond to an infinitesimal displacement of every point of $\mathscr{S}$ and can be represented by vector fields on $\mathscr{S}$. We obtain in this way a set of vector fields on $\mathscr{S}$ which can be defined operationally in terms of transformation procedures. These vector fields define the geometry of the space $\mathscr{S}$ in the same way as the metric tensor defines the geometry of the space $\mathscr{M}$. Note that also the metric tensor can be defined operationally in terms of procedures which have the aim of measuring lengths and time intervals.

The geometry of the space $\mathscr{S}$ is developed in sect. **2**. In sect. **3** we study the relation between the formalism given in the present paper and the usual formalism based on the space-time manifold $\mathscr{M}$. The manifold $\mathscr{M}$ can be defined in a natural way only if certain conditions are satisfied. We stress that the theory makes sense also if the manifold $\mathscr{M}$ cannot be defined.

In sect. **4** we introduce the action integral, which can be performed on an arbitrary four-dimensional surface in $\mathscr{S}$. It follows that the Lagrangian density has to be replaced by a differential 4-form, which we call the « Lagrangian form ». From the action principle we derive the field equations both for the fields which describe matter and for the vector fields which describe the geometry of $\mathscr{S}$. In sect. **5** we derive from the action principle the connection between symmetry properties of $\mathscr{S}$ and conservation laws. We derive also the more general continuity equations which hold in the absence of symmetry and contain source terms.

We get in this way a complete scheme of classical field theory. An interesting feature of this theory is that the density and the flow of energy and of momentum are represented by differential 3-forms on $\mathscr{S}$. As we shall see, this formalism permits to describe situations in which energy and momentum cannot be localized in space-time. The same remark holds for the electric charge.

In sect. **6** we reformulate Einstein's theory of gravitation as a field theory on $\mathscr{S}$. This reformulation is not unique and the problem requires further investigation. In sect. **7** we treat the electromagnetic field from a geometric point of view. This can be done by generalizing the concept of reference frame.

The examples studied in sect. **6** and **7** show that there is a large freedom in the choice of the Lagrangian forms and that some new physical principle is needed in order to make the « right » choice. Some suggestion about this problem is given in sect. **8**.

2. – The geometry of the space of reference frames.

Following the program sketched in the introduction, we consider the space $\mathscr{S}$ of all the local reference frames. In special relativity, if we fix an arbitrary frame of reference, all the other frames can be obtained from it by means of a uniquely defined transformation of the orthochronous Poincaré group. It follows that in this case $\mathscr{S}$ has the structure of a manifold isomorphic to the manifold of the orthochronous Poincaré group (1). As we want to take into account the ideas of general relativity, we assume only that $\mathscr{S}$ is an infinitely differentiable ten-dimensional manifold.

An infinitesimal transformation transforms every reference frame s into another reference frame s' very near to s and, therefore, it can be represented mathematically by a vector field on the manifold $\mathscr{S}$. The vector fields which represent infinitesimal transformations generate a subspace $\mathscr{T}$ in the linear space of all the vector fields on $\mathscr{S}$. In special relativity, $\mathscr{T}$ is just the space of right invariant vector fields on the Poincaré group, which are the generators of the left translations and form the Lie algebra of the group. In our general scheme, we assume only that $\mathscr{T}$ is a ten-dimensional vector space.

For every point $s \in \mathscr{S}$ we can consider the tangent space $T_s(\mathscr{S})$ and the linear mapping $\mathscr{T} \to T_s(\mathscr{S})$, which assigns to a vector field belonging to $\mathscr{T}$ its value at the point s. We assume that this mapping is an isomorphism of vector spaces. We obtain in this way an isomorphism between the tangent vector bundle $T(\mathscr{S})$ and the trivial vector bundle $\mathscr{S} \times \mathscr{T}$. The vector fields belonging to $\mathscr{T}$ are the sections of $T(\mathscr{S})$ which correspond to the constant sections of $\mathscr{S} \times \mathscr{T}$. We may say that the geometric properties of the space $\mathscr{S}$ are, at least partially, described by this trivialization of $T(\mathscr{S})$.

It is important to remark that the space $\mathscr{T}$ and the space $\mathscr{S}$ play a very different role in a physical theory. In fact, the elements of $\mathscr{T}$ (infinitesimal transformations) can be identified completely in terms of physical procedures, namely they have an operational meaning. On the contrary, the elements of $\mathscr{S}$ (local frames of reference) cannot be identified purely by means of physical procedures. In fact, in order to perform the operations prescribed by a procedure, one needs a pre-existing frame of reference: only the relation between two frames of reference has an operational meaning, while a single frame of reference cannot be identified operationally.

It follows that the physical laws, which are statements about procedures, cannot distinguish *a priori* (namely without performing an experiment) between different elements of $\mathscr{S}$. In other words, the physical laws must be « homogeneous » in the space $\mathscr{S}$. This is just the relativity principle. A similar argument does not hold for the space $\mathscr{T}$: for instance, infinitesimal time translations and infinitesimal rotations may appear in a completely different way in a physical law. As the elements of $\mathscr{T}$ identify a direction near every point of $\mathscr{S}$, we may say that the physical laws do not need to be « isotropic » in the space $\mathscr{S}$.

In particular, we remark that not all the elements of $\mathscr{T}$ can be interpreted as infinitesimal transformations. For instance, only positive time translations can be realized physically (3). We indicate by $\mathscr{T}^+$ the set of the elements of $\mathscr{T}$ which can be realized as physical transformations. If A and B represent infinitesimal transformations, the composition of these two transformations is represented by the vector $A + B$ (disregarding terms of second order). It follows that $\mathscr{T}^+$ has the property

$$\mathscr{T}^+ + \mathscr{T}^+ \subset \mathscr{T}^+ . \tag{2.1}$$

As $\mathscr{T}^+$ is also invariant with respect to the multiplication by a positive number, it is a convex wedge.

The detailed structure of the space $\mathscr{T}$ depends on the particular theory and is described by the Lagrangian form, as we shall see in the following sections. In this section we consider it just as a linear space and we study the geometric properties of the space $\mathscr{S}$ which follow just from the trivialization of its tangent bundle. We get a structure which can be considered as a generalization of the structure of Lie group. The basic concepts necessary for this investigation can be found, for instance, in ref. [5]. In view of future applications, we assume that $\mathscr{S}$ is a C^∞ n-dimensional manifold and that $\mathscr{T}$ is a n-dimensional vector space, where n may be different from ten. We assume also that all the fields are C^∞.

If A is a vector field on $\mathscr{S}$, we indicate by $\boldsymbol{L}_A$ the corresponding first-order differential operator acting on scalar fields (Lie derivative). If A is a vector field and ω is a differential 1-form, we indicate by $\boldsymbol{i}_A\omega$ their inner product, which is a scalar field. If f is a scalar field, we have

$$\boldsymbol{L}_A f = \boldsymbol{i}_A \mathrm{d}f \,. \tag{2.2}$$

We recall that the observables of a theory are operationally defined in terms of measurement procedures. Given a reference frame, following the prescriptions of a measurement procedure one obtains a real number, which depends on the state of the system and on the reference frame. If we fix the state of the system, an observable can be represented mathematically by a scalar field $f(s)$ on the space $\mathscr{S}$. Note that, in order to represent observables, it is not necessary to introduce vector or tensor fields on $\mathscr{S}$.

The composition of an observable f and an infinitesimal transformation A is a new observable f' which can be measured by measuring the observable f in the frame of reference s' obtained from s by means of the infinitesimal transformation A. In other words, we have

$$f'(s) = f(s') = f(s) + \boldsymbol{L}_A f(s) \,. \tag{2.3}$$

If we introduce in the space $\mathscr{T}$ a basis formed by the vector fields A_α $(\alpha = 0, 1, \ldots, n-1)$, we can write every vector field on $\mathscr{S}$ uniquely in the form

$$A = a^\alpha(s) A_\alpha \,, \tag{2.4}$$

where the coefficients $a^\alpha(s)$ are scalar fields. The summation over repeated indices is understood. Of course, the vector field A belongs to $\mathscr{T}$ if an only if the coefficients a^α are constant. In the following we shall always consider the components of vectors and tensors with respect to the basis A_α rather than with respect to the «natural» basis defined by a set of co-ordinates on the space $\mathscr{S}$. For simplicity of notation we indicate by $\boldsymbol{L}_\alpha$ the Lie derivative corresponding to the vector field A_α. Then we have

$$\boldsymbol{L}_A = a^\alpha(s) \boldsymbol{L}_\alpha \,. \tag{2.5}$$

(5) Y. Choquet-Bruhat: *Géométrie différentielle et systèmes extérieurs* (Paris, 1968).

The commutator of two first-order differential operators is a first-order differential operator and, therefore, we have

$$[\boldsymbol{L}_\alpha, \boldsymbol{L}_\beta] = \boldsymbol{L}_\alpha \boldsymbol{L}_\beta - \boldsymbol{L}_\beta \boldsymbol{L}_\alpha = F^\gamma_{\alpha\beta}(s)\,\boldsymbol{L}_\gamma\,, \tag{2.6}$$

where $F^\gamma_{\alpha\beta}(s)$ are functions on $\mathscr{S}$ which depend on the choice of the basis in $\mathscr{T}$ as the components of a tensor of rank three. Of course, we have

$$F^\gamma_{\alpha\beta} = -\,F^\gamma_{\beta\alpha}\,. \tag{2.7}$$

In special relativity, the quantities $F^\gamma_{\alpha\beta}$ do not depend on s and they are just the structure constants of the Lie algebra of the Poincaré group. We call them the structure coefficients of the space $\mathscr{S}$. It is important to remark that, after a choice of a basis in the space $\mathscr{T}$ and after an operational definition of the corresponding infinitesimal transformations, the structure coefficients $F^\gamma_{\alpha\beta}$ are measurable quantities and every statement about these coefficients has a physical meaning.

From the Jacobi identity

$$[\boldsymbol{L}_\alpha, [\boldsymbol{L}_\beta, \boldsymbol{L}_\gamma]] + [\boldsymbol{L}_\beta, [\boldsymbol{L}_\gamma, \boldsymbol{L}_\alpha]] + [\boldsymbol{L}_\gamma, [\boldsymbol{L}_\alpha, \boldsymbol{L}_\beta]] = 0\,, \tag{2.8}$$

using the formula

$$[\boldsymbol{L}_A, f\boldsymbol{L}_B] = (\boldsymbol{L}_A f)\,\boldsymbol{L}_B + f[\boldsymbol{L}_A, \boldsymbol{L}_B]\,, \tag{2.9}$$

we get after some calculation the fundamental formula

$$\boldsymbol{L}_\alpha F^\delta_{\beta\gamma} + \boldsymbol{L}_\beta F^\delta_{\gamma\alpha} + \boldsymbol{L}_\gamma F^\delta_{\alpha\beta} = F^\eta_{\alpha\beta} F^\delta_{\eta\gamma} + F^\eta_{\beta\gamma} F^\delta_{\eta\alpha} + F^\eta_{\gamma\alpha} F^\delta_{\eta\beta}\,. \tag{2.10}$$

As we shall see in the following, a surprising number of physically relevant relations can be obtained as special cases of this equation.

In order to give another derivation of these formulae, we introduce in the dual $\mathscr{T}^*$ of the space $\mathscr{T}$ a dual basis formed by the differential 1-forms ω^α ($\alpha = 0, 1, \ldots, n-1$). If, for simplicity of notation, we indicate by $\boldsymbol{i}_\alpha$ the inner product operator corresponding to the vector field A_α, we have

$$\boldsymbol{i}_\alpha \omega^\beta = \delta^\beta_\alpha\,, \tag{2.11}$$

where δ^β_α is the Kronecker symbol. Then from eq. (2.2) we get

$$\mathbf{d}f = (\boldsymbol{L}_\alpha f)\,\omega^\alpha\,. \tag{2.12}$$

Then we can write

$$(2.13) \qquad 0 = \mathbf{d}\mathbf{d}f = (\boldsymbol{L}_\beta \boldsymbol{L}_\alpha f)\omega^\beta \wedge \omega^\alpha + (\boldsymbol{L}_\alpha f)\,\mathbf{d}\omega^\alpha$$

and, therefore, using eq. (2.6),

$$(2.14) \qquad (\boldsymbol{L}_\gamma f)\,\mathrm{d}\omega^\gamma = \tfrac{1}{2}(\boldsymbol{L}_\alpha \boldsymbol{L}_\beta f - \boldsymbol{L}_\beta \boldsymbol{L}_\alpha f)\omega^\beta \wedge \omega^\alpha = \tfrac{1}{2} F^\gamma_{\alpha\beta}(\boldsymbol{L}_\gamma f)\omega^\beta \wedge \omega^\alpha .$$

As the quantities $\boldsymbol{L}_\gamma f$ are arbitrary in every point of $\mathscr{S}$, we get the Maurer-Cartan formula

$$(2.15) \qquad \mathbf{d}\omega^\gamma = -\tfrac{1}{2} F^\gamma_{\alpha\beta}\omega^\alpha \wedge \omega^\beta .$$

As a consequence we have

$$(2.16) \qquad \mathbf{d}(F^\gamma_{\alpha\beta}\omega^\alpha \wedge \omega^\beta) = 0 ,$$

which is equivalent to eq. (2.10).

In order to get more powerful notations, we recall that the inner product operators $\boldsymbol{i}_A$ can be extended to arbitrary differential forms, in such a way that, if α is a k-form, we have

$$(2.17) \qquad \boldsymbol{i}_A(\alpha \wedge \beta) = (\boldsymbol{i}_A \alpha) \wedge \beta + (-1)^k \alpha \wedge (\boldsymbol{i}_A \beta) .$$

The Lie derivative $\boldsymbol{L}_A$ can be extended to arbitrary tensor fields. In particular, if B is a vector field, the formulae

$$(2.18) \qquad \boldsymbol{L}_A B = [A, B] = -\boldsymbol{L}_B A = C$$

are equivalent to the formula

$$(2.19) \qquad [\boldsymbol{L}_A, \boldsymbol{L}_B] = \boldsymbol{L}_C .$$

If α is a differential form, we have

$$(2.20) \qquad \boldsymbol{L}_A \alpha = \mathbf{d}\boldsymbol{i}_A \alpha + \boldsymbol{i}_A \mathbf{d}\alpha .$$

Then we see that eq. (2.6) can be written in the form

$$(2.21) \qquad \boldsymbol{L}_\alpha A_\beta = F^\gamma_{\alpha\beta} A_\gamma .$$

From eqs. (2.11), (2.15) and (2.20), we get

$$(2.22) \qquad \boldsymbol{L}_\alpha \omega^\beta = \boldsymbol{i}_\alpha \mathbf{d}\omega^\beta = -F^\beta_{\alpha\gamma}\omega^\gamma .$$

3. – Theories with a space-time manifold.

In order to clarify the geometric concepts of the preceding section and their connection with the more familiar geometric structures, now we consider the special case in which it is possible to introduce a four-dimensional space-time manifold $\mathscr{M}$. We assume that there is a mapping π of $\mathscr{S}$ onto $\mathscr{M}$ and, if $x = \pi(s)$, we say that the point x of $\mathscr{M}$ is the origin of the local reference frame s.

We shall define the geometric properties of the space $\mathscr{M}$ in terms of the geometric properties of the space $\mathscr{S}$ without introducing new degrees of freedom. We shall only introduce some new structures in the vector space $\mathscr{T}$ which are constant, namely they do not depend on the point $s \in \mathscr{S}$.

For every point s we consider the tangent mapping $T_s(\pi)$ from the tangent space $T_s(\mathscr{S})$ to the tangent space $T_{\pi(s)}(\mathscr{M})$. It can be considered as a linear mapping from the space $\mathscr{T}$ to $T_{\pi(s)}(\mathscr{M})$. In accord with our program, we assume that the kernel of this mapping does not depend on s and we indicate it by $\mathscr{H}$. Note that the isomorphism of the linear space $\mathscr{T}/\mathscr{H}$ and $T_x(\mathscr{M})$ is not uniquely defined, as it depends on the choice of s in $\pi^{-1}(x)$.

An infinitesimal transformation belonging to $\mathscr{H}$ does not shift the origin of the local reference frame and, therefore, we call it a «homogeneous» infinitesimal transformation. $\mathscr{H}$ is $(n-4)$-dimensional and we choose the basis in $\mathscr{T}$ in such a way that

$$A_a \in \mathscr{H} \qquad \text{for } a \geqslant 4 \,. \tag{3.1}$$

In the following the Latin indices $a, b, c, \ldots, h$ take the values $4, \ldots, n-1$, while the Latin indices $i, j, k, l, m, n, \ldots$ take the values $0, \ldots, 3$. The Greek indices take the values $0, \ldots, n-1$ as in the preceding section. These conventions have to be taken into account also when we sum over the repeated indices.

The vectors $A_0, \ldots, A_3$ define a basis in the space $\mathscr{T}/\mathscr{H}$ and, therefore, for every $s \in \mathscr{S}$ they define a basis of the tangent space $T_{\pi(s)}(\mathscr{M})$ formed by the vectors $\hat{A}_0(s), \ldots, \hat{A}_3(s)$. Given a vector field $\hat{A}$ on $\mathscr{M}$, we can consider its components with respect to this basis and write

$$\hat{A}(x) = f^\iota(s)\hat{A}_\iota(s) \,, \qquad x = \pi(s) \,. \tag{3.2}$$

Note that the components f^i depend on s rather than on x. They must satisfy a «consistency condition», namely a differential equation which permits one to compute them on the whole fiber $\pi^{-1}(x)$ when they are known on one point of this fiber.

We say that a vector field A on $\mathscr{S}$ and a vector field $\hat{A}$ on $\mathscr{M}$ are related and we write $A \sim \hat{A}$, if for every $s \in \mathscr{S}$ we have

$$T_s(\pi)A(s) = \hat{A}(\pi(s)) . \tag{3.3}$$

For instance, the vector field $\hat{A}$ defined by eq. (3.2) is related to the field

$$A(s) = f^\alpha(s)A_\alpha(s) , \tag{3.4}$$

where $f^\alpha(s)$ for $\alpha \geqslant 4$ is arbitrary. As we see from this example, many different vector fields on $\mathscr{S}$ are related with a vector field on $\mathscr{M}$, but at most one vector field on $\mathscr{M}$ can be related to a vector field on $\mathscr{S}$. The relation $A \sim 0$ means that $A(s) \in \mathscr{H}$ for every $s \in \mathscr{S}$. Clearly we have

$$A_a \sim 0 . \tag{3.5}$$

If $\hat{f}$ is a function on $\mathscr{M}$, we can define a function $f = \pi^* \hat{f}$ on $\mathscr{S}$ given by

$$f(s) = \hat{f}(\pi(s)) . \tag{3.6}$$

If $A \sim \hat{A}$, we have

$$\boldsymbol{L}_A \pi^* \hat{f} = \pi^* \boldsymbol{L}_{\hat{A}} \hat{f} . \tag{3.7}$$

Conversely, if eq. (3.7) holds for any choice of the scalar field $\hat{f}$, we have $A \sim \hat{A}$. It follows that, if $A \sim \hat{A}$ and $B \sim \hat{B}$, it is

$$[A, B] \sim [\hat{A}, \hat{B}] . \tag{3.8}$$

In particular, we have

$$[A_a, A_b] \sim 0 \tag{3.9}$$

and we obtain the important formula

$$F^i_{ab} = 0 . \tag{3.10}$$

If the vector field A on $\mathscr{S}$ is related to some vector field $\hat{A}$ on $\mathscr{M}$, we have

$$[A_a, A] = \boldsymbol{L}_a A \sim 0 . \tag{3.11}$$

If A is given by eq. (3.4), eq. (3.11) takes the form

$$\boldsymbol{L}_a f^i = -F^i_{ak} f^k . \tag{3.12}$$

This is the consistency condition satisfied by the components of the vector field $\hat{A}$ which appear in eq. (3.2).

For every point $s \in \mathscr{S}$ the transpose of the tangent mapping $T_s(\pi)$ is an isomorphism of the cotangent space $T^*_{\pi(s)}(\mathscr{M})$ onto a subspace of $T^*_s(\mathscr{S})$ or, if we prefer, onto a subspace $\mathscr{H}_\perp$ of the dual $\mathscr{T}^*$ of $\mathscr{T}$. $\mathscr{H}_\perp$ is just the subspace of $\mathscr{T}^*$ orthogonal to $\mathscr{H}$. It follows that for every differential form $\hat{\eta}$ on $\mathscr{M}$ one can define univocally a differential form

$$\eta = \pi^* \hat{\eta} \tag{3.13}$$

on the space $\mathscr{S}$.

If we adopt the choice (3.1), the forms $\omega^0, \ldots, \omega^3$ define a basis in the space $\mathscr{H}_\perp$ and for every s the corresponding forms $\hat{\omega}^0(s), \ldots, \hat{\omega}^3(s)$ form a basis in the cotangent space $T^*_{\pi(s)}(\mathscr{M})$. It follows that every differential 1-form $\hat{\eta}$ on $\mathscr{M}$ can be written as

$$\hat{\eta} = f_i(s)\, \hat{\omega}^i(s) \, . \tag{3.14}$$

Note that also in this case the components f_i depend on s rather than on x and they have to satisfy a consistency condition. The corresponding form on $\mathscr{S}$ is

$$\eta = \pi^* \hat{\eta} = f_i(s)\, \omega^i \, . \tag{3.15}$$

The operation π^* has the properties

$$\mathbf{d}\pi^* \hat{\eta} \;\;= \pi^* \mathbf{d}\hat{\eta} \, , \tag{3.16}$$

$$\boldsymbol{i}_A \pi^* \hat{\eta} \;\;= \pi^* \boldsymbol{i}_{\hat{A}}\, \hat{\eta} \qquad \text{if } A \sim \hat{A} \, , \tag{3.17}$$

$$\boldsymbol{L}_A \pi^* \hat{\eta} = \pi^* \boldsymbol{L}_{\hat{A}}\, \hat{\eta} \qquad \text{if } A \sim \hat{A} \, . \tag{3.18}$$

It follows that the form η defined by eq. (3.13) satisfies the conditions

$$\boldsymbol{i}_a \eta \;\;= 0 \, , \tag{3.19}$$

$$\boldsymbol{L}_a \eta = 0 \, . \tag{3.20}$$

From eqs. (3.15) and (3.20) we obtain the following consistency condition for the components f_i of a differential 1-form on $\mathscr{M}$:

$$\boldsymbol{L}_a f_i = F^k_{ai} f_k \, . \tag{3.21}$$

Generalizing eqs. (3.12) and (3.21), we get the consistency condition for the components of an arbitrary tensor field on $\mathscr{M}$ with respect to the basis $\hat{A}_i(s)$ and to the corresponding dual basis $\hat{\omega}^i(s)$. In a similar way one can also treat an arbitrary spinor field on $\mathscr{M}$.

In particular we see that the quantities g^α_β which do not vanish only in the cases

$$g^0_0 = g^1_1 = g^2_2 = g^3_3 = 1 \tag{3.22}$$

can be considered as the components of a tensor on $\mathscr{M}$.

Now we consider a four-dimensional subspace $\mathscr{K}$ of $\mathscr{T}$ in such a way that $\mathscr{T}$ is the direct sum of $\mathscr{K}$ and $\mathscr{H}$. There is an isomorphism between $\mathscr{K}$ and $\mathscr{T}/\mathscr{H}$ and, therefore, for every $s \in \mathscr{S}$ there is an isomorphism between $\mathscr{K}$ and $T_{\pi(s)}(\mathscr{M})$. Then, if $x = \pi(s)$, for every infinitesimal displacement of x we can define univocally an infinitesimal displacement of s along a direction belonging to the subspace $\mathscr{K}$. This infinitesimal displacement of s defines a « parallel displacement » of the basis vectors $\hat{A}_i(s)$ and, therefore, a parallel displacement of any vector on the space $\mathscr{M}$. If this parallel displacement does not depend on the choice of the point $s \in \pi^{-1}(x)$, we have defined in this way a connection on $\mathscr{M}$.

In order to get the covariant derivatives of the vector field (3.2), we remark that, if we displace the point s in a direction belonging to the space $\mathscr{K}$, the basis vectors $\hat{A}_i(s)$ undergo a parallel displacement and we have just to perform the derivative of the functions $f^i(s)$ along this direction. If we choose the vectors A_i in the space $\mathscr{K}$, the covariant derivatives of the field (3.2) are given by

$$f^i_j = \boldsymbol{L}_j f^i \, . \tag{3.23}$$

In a similar way we get the covariant derivatives of an arbitrary tensor field.

Our definitions are consistent only if the components (3.23) satisfy the differential equation

$$\boldsymbol{L}_a f^i_j = - F^i_{ak} f^k_j + F^k_{aj} f^i_k \, . \tag{3.24}$$

After some calculation, we see that this condition is satisfied if we have

$$\boldsymbol{L}_j F^k_{ai} = - F^b_{aj} F^k_{bi} \, . \tag{3.25}$$

This condition ensures the consistency of the definition of the covariant derivatives of an arbitrary tensor field.

If f is a scalar field on $\mathscr{M}$ and f^j are the components of a vector field, we have

$$(\boldsymbol{L}_i \boldsymbol{L}_k - \boldsymbol{L}_k \boldsymbol{L}_i) f = F^l_{ik} \boldsymbol{L}_l f \, , \tag{3.26}$$

$$(\boldsymbol{L}_i \boldsymbol{L}_k - \boldsymbol{L}_k \boldsymbol{L}_i) f^j = F^l_{ik} \boldsymbol{L}_l f^j - F^a_{ik} F^j_{al} f^l \, . \tag{3.27}$$

If we recall the definition [5] of the torsion tensor S and of the Riemann curvature tensor R, we find the following expressions for their components:

$$S^l_{ik} = - F^l_{ik}\,, \tag{3.28}$$

$$R^j_{lik} = - F^j_{al} F^a_{ik}\,. \tag{3.29}$$

One can show, using eqs. (2.10), (3.10) and (3.25), that these quantities satisfy the consistency conditions required by their tensor character. From the same equations one can also derive the well-known formulae [5]

$$L_i R^r_{sjk} + L_j R^r_{ski} + L_k R^r_{sij} = F^t_{ij} R^r_{stk} + F^t_{jk} R^r_{sti} + F^t_{ki} R^r_{stj}\,, \tag{3.30}$$

$$L_i F^l_{jk} + L_j F^l_{ki} + L_k F^l_{ij} = - R^l_{kij} - R^l_{ijk} - R^l_{jki} + F^r_{ij} F^l_{rk} + F^r_{jk} F^l_{ri} + F^r_{ki} F^l_{rj}\,. \tag{3.31}$$

From eqs. (2.10) and (3.10) one can also obtain the formula

$$F^c_{ab} F^k_{ci} = L_a F^k_{bi} - L_b F^k_{ai} + F^r_{bi} F^k_{ar} - F^r_{ai} F^k_{br}\,. \tag{3.32}$$

This equation, together with eqs. (3.10), (3.25), (3.28) and (3.29), permits us to compute all the structure coefficients $F^\gamma_{\alpha\beta}$ starting from S^l_{ik}, R^j_{lik} and F^k_{ai}. These three quantities are not independent, but are connected by eqs. (3.30) and (3.31) and by the consistency conditions for the tensors S and R.

In order to introduce a pseudo-Riemannian metric in the space $\mathscr{M}$, we have just to consider a quadratic form $g_{\alpha\beta}$ in the vector space $\mathscr{T}$. We assume that the operations used to measure lenghts and time intervals can be deduced from the operations used to construct new frames of reference. The quantities $g_{\alpha\beta}$ describe the connection between these operations and, therefore, we assume that they are constant. As they are the components of a symmetric covariant tensor in $\mathscr{M}$, they must satisfy the equations

$$g_{\alpha\beta} = g_{\beta\alpha}\,, \tag{3.33}$$

$$g_{a\beta} = 0\,, \tag{3.34}$$

$$L_a g_{ik} = 0 = F^j_{ai} g_{jk} + F^j_{ak} g_{ij}\,. \tag{3.35}$$

The last formula can be considered as a condition on the structure coefficients F^i_{ak}.

It is easy to show that also the quantities g^{ik} defined by

$$g^{ij} g_{jk} = \delta^i_k \tag{3.36}$$

can be considered as the components of a tensor in $\mathscr{M}$. We can choose the basis in the space $\mathscr{T}$ in such a way that the nonvanishing components of $g_{\alpha\beta}$ and of $g^{\alpha\beta}$ are

$$g_{00} = g^{00} = 1\,, \qquad g_{11} = g^{11} = g_{22} = g^{22} = g_{33} = g^{33} = -1\,. \tag{3.37}$$

We consider also the Levi-Civita symbol e_{ijkl} completely antisymmetric and normalized by $e_{0123} = 1$. It satisfies the identity

$$e_{ijkl} f_m + e_{jklm} f_i + e_{klmi} f_j + e_{lmij} f_k + e_{mijk} f_l = 0\,. \tag{3.38}$$

From this identity and from the formula

$$F^k_{ak} = 0\,, \tag{3.39}$$

which is a consequence of eq. (3.35), we have

$$F^t_{ai} e_{tjkl} + F^t_{aj} e_{itkl} + F^t_{ak} e_{ijtl} + F^t_{al} e_{ijkt} = 0\,. \tag{3.40}$$

From this formula we see that the quantities e_{ijkl} can be considered as the components of a covariant tensor in $\mathscr{M}$ (disregarding reflections). We shall also consider the quantities

$$e^{ijkl} = -e_{ijkl}\,, \tag{3.41}$$

which are the components of a contravariant tensor in $\mathscr{M}$.

In the calculations of the following sections we shall often use the identities (3.38) and

$$\omega^i \wedge \omega^j \wedge \omega^k = -\tfrac{1}{6} e^{ijkp} e_{rstp} \omega^r \wedge \omega^s \wedge \omega^t\,, \tag{3.42}$$

$$\omega^i \wedge \omega^j \qquad = -\tfrac{1}{4} e^{ijpq} e_{rspq} \omega^r \wedge \omega^s\,. \tag{3.43}$$

Summarizing the results of the present section, we have seen that the very existence of a space-time manifold $\mathscr{M}$ defines the « vertical » subspace $\mathscr{H}$ of $\mathscr{T}$ spanned by the vectors $A_4, \ldots, A_{n-1}$ and requires the validity of the condition (3.10). If we choose also an « horizontal » subspace $\mathscr{K}$ of $\mathscr{T}$ spanned by the vectors $A_0, \ldots, A_3$, and we impose the condition (3.25), we can define a connection on $\mathscr{M}$ and, therefore, a torsion and a curvature tensor.

If we choose a quadratic form $g_{\alpha\beta}$ on the space $\mathscr{T}$ and we impose the conditions (3.34) and (3.35), we obtain a Riemann metric on the space $\mathscr{M}$. Note that the choice of this quadratic form determines, through eq. (3.34), the vertical subspace $\mathscr{H}$, but it is compatible with many different choices of the horizontal subspace $\mathscr{K}$, namely with many different connections on $\mathscr{M}$.

Finally we remark that, even with all the conditions imposed in the present section, the geometric structure we are considering has more degrees of freedom than the corresponding Riemann-Cartan space-time ([6]), due to the presence of the quantities F^k_{ai}. From eq. (3.35) we see that these quantities, if we fix the subscript a, form a matrix which generates an infinitesimal homogeneous Lorentz transformation. This matrix may depend on the point s of $\mathscr{S}$ if, by performing the same physical operations in different frames of reference due to the presence of a new kind of field, one obtains different linear transformations of the space tangent to $\mathscr{M}$ at the point $\pi(s)$.

4. – The action principle and the field equations.

In this section we want to develop a Lagrangian field theory in the n-dimensional space $\mathscr{S}$ with $n\geqslant 4$, without assuming the existence of a space-time manifold $\mathscr{M}$. In order to find the general form of the field equations and of the conservation laws, we start from an action principle of the kind

$$\delta\int_S \lambda = 0\,, \tag{4.1}$$

where S is an arbitrary four-dimensional surface in $\mathscr{S}$ with boundary ∂S and λ is a differential 4-form depending on the fields and on their derivatives, which we call the Lagrangian form. When $n = 4$, λ becomes a scalar density and the action principle (4.1) takes the usual form. In expression (4.1) we can vary the fields, keeping them fixed on ∂S, and we can vary also the surface S keeping its boundary ∂S fixed.

We consider as dynamical variables the vector fields A_α which describe the geometry of $\mathscr{S}$ and other fields which describe matter. We may assume that the matter fields are scalar in the space $\mathscr{S}$, otherwise we can replace them by their components in the frame of reference defined by the vector fields A_α. We assume for simplicity that there is only one scalar matter field f. Instead of the vector fields A_α we can introduce as dynamical variables the differential forms ω^α.

When we treat the action principle in n dimensions, we find new features and new field equations which do not appear in the four-dimensional case. First of all the action integral must be stationary when we deform the surface S keeping the boundary fixed and this gives the field equation

$$\mathbf{d}\lambda = 0\,. \tag{4.2}$$

([6]) F. W. Hehl, P. von der Heyde and G. D. Kerlick: *Rev. Mod. Phys.*, **48**, 393 (1976). This paper contains a large list of references about the generalizations of Einstein's gravitational theory.

This equation is trivially satisfied if $n = 4$. We shall show that, as a deformation of the integration surface S can be reinterpreted as a variation of all the fields, eq. (4.2) is a consequence of all the other field equations.

Then we can vary the matter field f. We assume that λ depends on f and on its Lie derivatives $\boldsymbol{L}_\alpha f$. If we put

$$\delta f = \varepsilon \,, \tag{4.3}$$

disregarding second-order terms we can write

$$\delta\lambda = \varepsilon\eta + \theta(\mathbf{d}\varepsilon) \,, \tag{4.4}$$

where

$$\eta = \frac{\partial\lambda}{\partial f} \tag{4.5}$$

is a differential 4-form and

$$\theta(\mathbf{d}\varepsilon) = \frac{\partial\lambda}{\partial \boldsymbol{L}_\alpha f}\, \boldsymbol{i}_\alpha \mathbf{d}\varepsilon \tag{4.6}$$

is a differential 4-form which depends linearly on the differential 1-form $\mathbf{d}\varepsilon$.

We introduce in a region of the manifold $\mathscr{S}$ the co-ordinates $x^0, \ldots, x^{n-1}$ and we assume that the surface S is given by the equations

$$x^\alpha = 0 \qquad \text{for } \alpha \geqslant 4 \,. \tag{4.7}$$

Then we assume that ε is given by

$$\varepsilon = g(x^0, x^1, x^2, x^3)\, x^4 \,, \tag{4.8}$$

where g is a differentiable function with compact support in R^4. Note that ε vanishes on the surface S. If we put

$$\theta(\alpha) = \frac{1}{24}\, \theta_{\alpha\beta\gamma\delta}(\alpha)\, \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta \wedge \mathbf{d}x^\gamma \wedge \mathbf{d}x^\delta \,, \tag{4.9}$$

the action principle takes the form

$$\int g\theta_{0123}(\mathbf{d}x^4)\, \mathrm{d}x^0\, \mathrm{d}x^1\, \mathrm{d}x^2\, \mathrm{d}x^3 = 0 \tag{4.10}$$

and, as g is arbitrary, we get the field equation

$$\theta_{0123}(\mathbf{d}x^4) = 0 \,. \tag{4.11}$$

In a similar way we see that, if the indices α, β, γ, δ, ε are all different, we have

$$\theta_{\alpha\beta\gamma\delta}(\mathbf{d}x^\varepsilon) = 0 \tag{4.12}$$

and one can see easily that all these equations are equivalent to the formula

$$\alpha\wedge\theta(\alpha) = 0 \tag{4.13}$$

or, more in general,

$$\alpha\wedge\theta(\beta) + \beta\wedge\theta(\alpha) = 0\,, \tag{4.14}$$

where α and β are arbitrary 1-forms. Also this field equation is trivially satisfied for $n = 4$. Using eq. (4.6), we can write it in the more explicit form

$$\omega^\alpha\wedge\frac{\partial\lambda}{\partial \boldsymbol{L}_\beta f} + \omega^\beta\wedge\frac{\partial\lambda}{\partial \boldsymbol{L}_\alpha f} = 0\,. \tag{4.15}$$

From eq. (4.14) we obtain

$$\begin{aligned} &0 = \boldsymbol{i}_\mu\big(\omega^\mu\wedge\theta(\alpha) + \alpha\wedge\theta(\omega^\mu)\big) = \\ &= n\theta(\alpha) - \omega^\mu\wedge\boldsymbol{i}_\mu\theta(\alpha) + (\boldsymbol{i}_\mu\alpha)\theta(\omega^\mu) - \alpha\wedge\boldsymbol{i}_\mu\theta(\omega^\mu) = (n-4+1)\theta(\alpha) - \alpha\wedge\boldsymbol{i}_\mu\theta(\omega^\mu)\,, \end{aligned} \tag{4.16}$$

and we see that we can write

$$\theta(\alpha) = \alpha\wedge\varrho\,, \tag{4.17}$$

where the differential 3-form ϱ is given by

$$\varrho = \frac{1}{n-3}\,\boldsymbol{i}_\mu\theta(\omega^\mu) = \frac{1}{n-3}\,\boldsymbol{i}_\mu\frac{\partial\lambda}{\partial \boldsymbol{L}_\mu f}\,. \tag{4.18}$$

If we take eq. (4.17) into account, the action principle can be written in the form

$$0 = \int_S (\varepsilon\eta + \mathbf{d}\varepsilon\wedge\varrho) = \int_{\partial S}\varepsilon\varrho + \int_S \varepsilon(\eta - \mathbf{d}\varrho) \tag{4.19}$$

and, as ε is an arbitrary function vanishing on ∂S, we get the field equation

$$\eta - \mathbf{d}\varrho = 0\,. \tag{4.20}$$

Using eqs. (2.20), (4.5) and (4.18), we can write this equation in the more explicit form

$$\boldsymbol{L}_\mu \frac{\partial \lambda}{\partial \boldsymbol{L}_\mu f} - (n-3)\frac{\partial \lambda}{\partial f} - \boldsymbol{i}_\mu \mathbf{d} \frac{\partial \lambda}{\partial \boldsymbol{L}_\mu f} = 0 . \tag{4.21}$$

If $n = 4$, the last term vanishes and we get the usual Euler-Lagrange equation. In conclusion, we have for every field a « normal » field equation of the kind (4.14) and a « tangential » field equation of the kind (4.20).

In order to find the equations for the geometric fields, we split the Lagrangian form λ into a part λ^G which contains only the geometric fields ω^α and their derivatives (namely the functions $F^\gamma_{\alpha\beta}$) and a part λ^M which describes matter and has the form

$$\lambda^M = \frac{1}{24} \lambda^M_{\alpha\beta\gamma\delta}(f, \boldsymbol{L}_\mu f)\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta . \tag{4.22}$$

The geometric fields appear explicity in the last factors and in the Lie derivatives of the matter field f. This is a kind of minimal coupling between matter and geometric fields. The forms η, θ and ϱ arise from the variation of the matter field f and, therefore, they do not depend on the geometric part of the Lagrangian form.

If we put

$$\delta\omega^\alpha = \varepsilon^\alpha_\beta \omega^\beta , \tag{4.23}$$

$$\delta A_\alpha = -\varepsilon^\beta_\alpha A_\beta \tag{4.24}$$

and we use eq. (4.3), we obtain

$$\delta(\boldsymbol{L}_\mu f) = \boldsymbol{L}_\mu \varepsilon - \varepsilon^\nu_\mu \boldsymbol{L}_\nu f = \boldsymbol{i}_\mu\big(\mathbf{d}\varepsilon - (\boldsymbol{L}_\nu f)\,\delta\omega^\nu\big) , \tag{4.25}$$

$$\delta(\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta) = \delta\omega^\nu \wedge \boldsymbol{i}_\nu(\omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma \wedge \omega^\delta) \tag{4.26}$$

and, therefore, using eqs. (4.5) and (4.6),

$$\delta\lambda^M = \varepsilon\eta + \theta\big(\mathbf{d}\varepsilon - (\boldsymbol{L}_\nu f)\,\delta\omega^\nu\big) + \delta\omega^\nu \wedge \boldsymbol{i}_\nu \lambda^M . \tag{4.27}$$

If we use also the field equations (4.17) and (4.20), we obtain

$$\delta\lambda^M = \mathbf{d}(\varepsilon\varrho) - (\boldsymbol{L}_\nu f)\,\delta\omega^\nu \wedge \varrho + \delta\omega^\nu \wedge \boldsymbol{i}_\nu \lambda^M . \tag{4.28}$$

For every vector field B we define the differential 3-form

$$\tau_B = (\boldsymbol{L}_B f)\varrho - \boldsymbol{i}_B \lambda^M \tag{4.29}$$

and we indicate by τ_ν the differential 3-form corresponding to the vector field A_ν.

Then eq. (4.28) can be written in the form

$$\delta\lambda^{\mathrm{M}} = \mathbf{d}(\varepsilon\varrho) - \delta\omega^\nu \wedge \tau_\nu \,. \tag{4.30}$$

If the matter field equations are satisfied, we can use this formula and write the action principle in the form

$$\delta\int_S \lambda^{\mathrm{G}} - \int_S \delta\omega^\nu \wedge \tau_\nu = 0 \,. \tag{4.31}$$

We see that the quantities τ_ν can be considered as the sources of the geometric field.

We assume that λ^{G} has the form

$$\lambda^{\mathrm{G}} = \frac{1}{24}\,\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta \,, \tag{4.32}$$

where the quantities $\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta}$ depend on the structure coefficients $F^\varrho_{\mu\nu}$. We

$$\delta\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta} = G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\,\delta F^\varrho_{\mu\nu} \,, \tag{4.33}$$

where the coefficients $G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}$ are antisymmetric with respect to the indices α, β, γ, δ and with respect to the indices μ, ν. By differentiation of eq. (2.15), using eq. (4.23), we get after some calculation

$$\delta F^\varrho_{\mu\nu} = \boldsymbol{L}_\nu\varepsilon^\varrho_\mu - \boldsymbol{L}_\mu\varepsilon^\varrho_\nu + \varepsilon^\varrho_\theta F^\theta_{\mu\nu} - \varepsilon^\theta_\mu F^\varrho_{\theta\nu} + \varepsilon^\theta_\nu F^\varrho_{\theta\mu} \,. \tag{4.34}$$

If the quantities $\delta\omega^\nu$ and, therefore, also the quantities ε^ν_μ vanish on the surface S, we have on this surface

$$\delta\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta} = -\,2G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\boldsymbol{L}_\mu\varepsilon^\varrho_\nu \,, \tag{4.35}$$

namely

$$\delta\lambda^{\mathrm{G}} = \sum_{\varrho\nu}\theta^\nu_\varrho(\mathbf{d}\varepsilon^\varrho_\nu) \,, \tag{4.36}$$

where

$$\theta^\nu_\varrho(\alpha) = -\frac{1}{12}\,G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}(\boldsymbol{i}_\mu\alpha)\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta \tag{4.37}$$

is a differential 4-form which depends linearly on the differential 1-form α. Proceeding as in the proof of eq. (4.14) we get the normal field equation

$$\beta\wedge\theta^\nu_\varrho(\alpha) + \alpha\wedge\theta^\nu_\varrho(\beta) = 0 \,. \tag{4.38}$$

This equation is equivalent to the requirement that the expression

$$G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\omega^\eta\wedge\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta \tag{4.39}$$

is completely antisymmetric with respect to the indices μ, ν, η.

From the normal field equation we have in particular

$$\boldsymbol{i}_\nu[G^{\mu\eta}_{\varrho\alpha\beta\gamma\delta}\omega^\nu\wedge\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta + G^{\nu\eta}_{\varrho\alpha\beta\gamma\delta}\omega^\mu\wedge\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta] = 0 \tag{4.40}$$

and, after some calculation,

$$(n-3)\,G^{\mu\eta}_{\varrho\alpha\beta\gamma\delta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta - 4G^{\alpha\eta}_{\varrho\alpha\beta\gamma\delta}\omega^\mu\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta = 0\ . \tag{4.41}$$

From this equation, using again the antisymmetry of the expression (4.39), we obtain

$$\boldsymbol{i}_\eta[(n-3)\,G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\omega^\eta\wedge\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta + 4G^{\alpha\eta}_{\varrho\alpha\beta\gamma\delta}\omega^\nu\wedge\omega^\mu\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta] = 0 \tag{4.42}$$

and performing the calculations

$$\begin{aligned}(n-3)(n-4)\,G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta + \\ + 4G^{\alpha\nu}_{\varrho\alpha\beta\gamma\delta}\omega^\mu\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta - 4G^{\alpha\mu}_{\varrho\alpha\beta\gamma\delta}\omega^\nu\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta + 12G^{\alpha\beta}_{\varrho\alpha\beta\gamma\delta}\omega^\nu\wedge\omega^\mu\wedge\omega^\gamma\wedge\omega^\delta = 0\ .\end{aligned} \tag{4.43}$$

From eqs. (4.41) and (4.43) we obtain finally

$$G^{\mu\nu}_{\varrho\alpha\beta\gamma\delta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta = G_{\varrho\gamma\delta}\omega^\mu\wedge\omega^\nu\wedge\omega^\gamma\wedge\omega^\delta\ , \tag{4.44}$$

where

$$G_{\varrho\gamma\delta} = \frac{12}{(n-2)(n-3)}\,G^{\alpha\beta}_{\varrho\alpha\beta\gamma\delta}\ . \tag{4.45}$$

From eqs. (4.32), (4.33) and (4.44) we have

$$\delta\lambda^{\mathrm{G}} = \frac{1}{24}\,G_{\varrho\gamma\delta}\,\delta F^\varrho_{\alpha\beta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta + \frac{1}{6}\,\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta}\,\delta\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta \tag{4.46}$$

and, using eq. (2.15),

$$\begin{aligned}\delta\lambda^{\mathrm{G}} = -\frac{1}{12}\,G_{\varrho\gamma\delta}\,(\delta\,\mathbf{d}\omega^\varrho + F^\varrho_{\alpha\beta}\,\delta\omega^\alpha\wedge\omega^\beta)\wedge\omega^\gamma\wedge\omega^\delta + \\ + \frac{1}{6}\,\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta}\,\delta\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta\ ,\end{aligned} \tag{4.47}$$

and after some calculation

$$(4.48)\qquad \delta\lambda^{G} = -\frac{1}{12}\mathbf{d}(G_{\varrho\gamma\delta}\,\delta\omega^{\varrho}\wedge\omega^{\gamma}\wedge\omega^{\delta}) - -\delta\omega^{\varrho}\wedge\frac{1}{12}[\mathbf{d}(G_{\varrho\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta}) + (G_{\theta\gamma\delta}F^{\theta}_{\varrho\beta} - 2\lambda^{G}_{\varrho\beta\gamma\delta})\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}]\,.$$

Substituting this expression in the action principle (4.31) we get immediately the following equation for the geometric field:

$$(4.49)\qquad -\frac{1}{12}\mathbf{d}(G_{\varrho\gamma\delta}\omega^{\gamma}\wedge\omega^{\delta}) - \frac{1}{12}(G_{\theta\gamma\delta}F^{\theta}_{\varrho\beta} - 2\lambda^{G}_{\varrho\beta\gamma\delta})\,\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta} = \tau_{\varrho}\,.$$

This equation can also be written in the form

$$(4.50)\qquad \tau_{\varrho} = \frac{1}{12}[-\boldsymbol{L}_{\beta}G_{\varrho\gamma\delta} + G_{\varrho\theta\delta}F^{\theta}_{\beta\gamma} - G_{\theta\gamma\delta}F^{\theta}_{\varrho\beta} + 2\lambda^{G}_{\varrho\beta\gamma\delta}]\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}\,.$$

We can write this equation in another way if we assume that the quantities $\lambda^{G}_{\alpha\beta\gamma\delta}$ are homogeneous functions of degree k of the quantities $F^{\varrho}_{\mu\nu}$. Then we can use the Euler theorem and write

$$(4.51)\qquad \lambda^{G} = \frac{1}{24k}G_{\theta\gamma\delta}F^{\theta}_{\alpha\beta}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}\,.$$

Note that this is not the original Lagrangian form, but it has been simplified by means of the normal field equations. Then eq. (4.50) can be written in the form

$$(4.52)\qquad \tau_{\varrho} = \frac{1}{12}\left[-\boldsymbol{L}_{\beta}G_{\varrho\gamma\delta} + G_{\varrho\theta\delta}F^{\theta}_{\beta\gamma} + \left(\frac{1}{k}-1\right)G_{\theta\gamma\delta}F^{\theta}_{\varrho\beta} + \frac{1}{k}G_{\theta\varrho\delta}F^{\theta}_{\beta\gamma}\right]\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}.$$

If we assume that the matter Lagrangian form λ^{M} is an homogeneous function of degree k of the quantities f and $\boldsymbol{L}_{\alpha}f$, we can also use Euler's theorem and we get

$$(4.53)\qquad \lambda^{M} = \frac{1}{k}(\theta(\mathbf{d}f) + f\eta)\,.$$

Using the field equation (4.17) and (4.20), we obtain

$$(4.54)\qquad \lambda^{M} = \frac{1}{k}(\mathbf{d}f\wedge\varrho + f\,\mathbf{d}\varrho) = \frac{1}{k}\mathbf{d}(f\varrho)\,.$$

We see that, as a consequence of the field equations, we have

$$\mathbf{d}\lambda^{\mathrm{M}} = 0 . \tag{4.55}$$

This formula holds also in other interesting cases and we shall use it in the next section.

5. – Conservation laws.

Many important observables (for instance the electric charge) concern a region of the three-dimensional space or more generally a region of a spacelike three-dimensional surface in the four-dimensional space-time $\mathscr{M}$. The most natural mathematical representation of these quantities is a differential 3-form in $\mathscr{M}$. Usually, this 3-form is represented in terms of a vector density J^i by means of the formula

$$\hat{\tau} = \frac{1}{6} J^i e_{ijkl}\, \mathbf{d}x^j \wedge \mathbf{d}x^k \wedge \mathbf{d}x^l . \tag{5.1}$$

If the differential form $\hat{\tau}$ is closed, namely if the continuity equation

$$\mathbf{d}\hat{\tau} = \frac{\partial J^i}{\partial x^i} \mathbf{d}x^0 \wedge \mathbf{d}x^1 \wedge \mathbf{d}x^2 \wedge \mathbf{d}x^3 = 0 \tag{5.2}$$

holds, the corresponding quantity is conserved. In fact, if the form (5.1) vanishes in the spacelike directions, its integral over a sufficiently large spacelike surface does not depend on the choice of the surface.

As we have seen in sect. **3**, from the differential form $\hat{\tau}$ in $\mathscr{M}$ we can get immediately a differential 3-form τ in the space $\mathscr{S}$. The integral of τ over a three-dimensional surface in $\mathscr{S}$ is equal to the integral of $\hat{\tau}$ on the projection of this surface on the space $\mathscr{M}$. If the quantity we are considering is conserved, also τ is closed.

We may also consider a differential 3-form τ in $\mathscr{S}$ which cannot be obtained from a differential form $\hat{\tau}$ in $\mathscr{M}$. This situation necessarily appears if we deal with a theory in which a space-time manifold $\mathscr{M}$ cannot be defined. If a 3-form of this general kind is closed, namely if

$$\mathbf{d}\tau = 0 , \tag{5.3}$$

and its support in $\mathscr{S}$ has suitable properties, the integral of τ over the three-dimensional surfaces belonging to a certain class does not depend on the surface. In this way we get a conserved quantity, which, however, cannot be localized in space. For instance, the quantity corresponding to a limited region in a spacelike surface could depend on the velocity of the observer.

We recall that in Einstein's general relativity the density and the flow of energy and momentum of matter are the source of the geometric (gravitational) field. Therefore, we assume that in the formalism we are considering these quantities are represented by differential 3-forms of the kind τ_B defined in the preceding section.

In order to find a conservation equation, we remark that from eqs. (4.30), (4.48) and (4.49) we have

$$\delta\lambda = \mathrm{d}\left(\varepsilon\varrho - \frac{1}{12}\, G_{\varrho\gamma\delta}\,\delta\omega^\varrho\wedge\omega^\gamma\wedge\omega^\delta\right). \tag{5.4}$$

This formula takes into account all the field equations.

Now we consider a special variation of the fields f and ω^α which is generated by the infinitesimal transformation of the space $\mathscr{S}$ defined by the vector field B. In this case we have, using eq. (2.20),

$$\delta\lambda = \boldsymbol{L}_B\lambda = \boldsymbol{i}_B\,\mathbf{d}\lambda + \mathbf{d}\boldsymbol{i}_B\lambda\,, \tag{5.5}$$

$$\delta f = \varepsilon = \boldsymbol{L}_B f\,, \tag{5.6}$$

$$\delta\omega^\alpha = \boldsymbol{L}_B\omega^\alpha\,. \tag{5.7}$$

From eqs. (5.4)-(5.7) we have

$$\boldsymbol{i}_B\,\mathbf{d}\lambda = \mathbf{d}\left[(\boldsymbol{L}_B f)\varrho - \frac{1}{12}\, G_{\varrho\gamma\delta}(\boldsymbol{L}_B\omega^\varrho)\wedge\omega^\gamma\wedge\omega^\delta - \boldsymbol{i}_B\lambda\right] = \mathbf{d}(\tau_B + \tau_B^{\mathrm{G}})\,, \tag{5.8}$$

where τ_B is defined by eq. (4.29) and

$$\tau_B^{\mathrm{G}} = -\frac{1}{12}\, G_{\varrho\gamma\delta}(\boldsymbol{L}_B\omega^\varrho)\wedge\omega^\gamma\wedge\omega^\delta - \boldsymbol{i}_B\lambda^{\mathrm{G}}\,. \tag{5.9}$$

If the vector field B vanishes on the boundary ∂S, from eq. (5.8) we have

$$\int_S \boldsymbol{i}_B\,\mathbf{d}\lambda = 0\,, \tag{5.10}$$

and from the arbitrariness of S and B we get eq. (4.2), as anticipated in the preceding section. In conclusion we have

$$\mathbf{d}(\tau_B + \tau_B^{\mathrm{G}}) = 0\,, \tag{5.11}$$

namely we have obtained a conserved quantity for every vector field B on the space $\mathscr{S}$.

This formula can also be obtained from the field equation (4.49), which, together with eq. (5.9), gives

$$\tau_\varrho + \tau_\varrho^{\mathrm{G}} = \mathbf{d}\left(-\frac{1}{12}\, G_{\varrho\gamma\delta}\,\omega^\gamma \wedge \omega^\delta\right). \tag{5.12}$$

If we start from eqs. (4.30), (5.6) and (5.7), we have

$$\boldsymbol{i}_B \mathbf{d}\lambda^{\mathrm{M}} + \mathbf{d}\boldsymbol{i}_B\lambda^{\mathrm{M}} = \boldsymbol{L}_B\lambda^{\mathrm{M}} = \delta\lambda^{\mathrm{M}} = \mathbf{d}[(\boldsymbol{L}_B f)\varrho] - (\boldsymbol{L}_B\omega^\nu)\wedge\tau_\nu \tag{5.13}$$

and, using the definition (4.29),

$$\mathbf{d}\tau_B = (\boldsymbol{L}_B\omega^\nu)\wedge\tau_\nu + \boldsymbol{i}_B\, \mathbf{d}\lambda^{\mathrm{M}}. \tag{5.14}$$

In the following we consider the most interesting case in which $\mathbf{d}\lambda^{\mathrm{M}}$ vanishes as a consequence of the field equations, as happens when λ^{M} is an homogeneous function of the matter fields and of their derivatives. Then we can write eq. (5.14) in the simplified form

$$\mathbf{d}\tau_B = (\boldsymbol{L}_B\omega^\nu)\wedge\tau_\nu . \tag{5.15}$$

If $B = A_\varrho$, using eq. (2.22) we can write

$$\mathbf{d}\tau_\varrho = - F^\mu_{\varrho\nu}\,\omega^\nu\wedge\tau_\mu . \tag{5.16}$$

From eq. (5.15) we see that the quantity τ_B is conserved if we have for all the values of ν

$$\boldsymbol{L}_B\omega^\nu = 0 , \tag{5.17}$$

namely if the infinitesimal transformation generated by the vector field B is an isomorphism of the geometric structure of the space $\mathscr{S}$. This is the usual connection between symmetry and conservation laws. Conditions (5.17) are equivalent to the conditions

$$\boldsymbol{L}_B A_\alpha = [B, A_\alpha] = -\boldsymbol{L}_\alpha B = 0 . \tag{5.18}$$

If $\mathscr{S}$ is a Lie group, there are n independent fields with this property, namely the left-invariant vector fields, which are the generators of the infinitesimal right translations. In this case there are n independent conserved quantities.

In order to get a physical interpretation of the quantities τ_ϱ, we disregard gravitation. If we choose a reference frame s^0, we can identify any other element s of $\mathscr{S}$ by means of the element (x, L) of the Poincaré group which transforms the reference frame s^0 into the reference frame s. The infinitesimal left translations generated by the vector fields εA_α are

(5.19) $$\varepsilon A_i : (x^k, L^r_s) \to (x^k - \varepsilon\delta^k_i, L^r_s) ,$$

(5.20) $$\varepsilon A_a : (x^k, L^r_s) \to (x^k + \varepsilon F^k_{aj} x^j, L^r_s + \varepsilon F^r_{at} L^t_s) .$$

Note that the structure constants F^r_{as} coincide with the generators of the infinitesimal homogeneous Lorentz transformations acting on four-vectors. We can also define ten vector fields B_α which generate the infinitesimal right translations in the following way:

(5.21) $$\varepsilon B_i : (x^k, L^r_s) \to (x^k - \varepsilon L^k_i, L^r_s) ,$$

(5.22) $$\varepsilon B_a : (x^k, L^r_s) \to (x^k, L^r_s + \varepsilon L^r_t F^t_{as}) .$$

Comparing eqs. (5.19)-(5.22), we obtain the relations

(5.23) $$B_i = L^k_i A_k ,$$

(5.24) $$B_a = R^b_a (A_b + F^k_{bj} x^j A_k) .$$

We have introduced for every homogeneous Lorentz transformation L^r_s the matrix R^b_a defined by

(5.25) $$L^r_t F^t_{as} = R^b_a F^r_{bt} L^t_s .$$

The matrices R^b_a form the adjoint representation of the homogeneous Lorentz group, which acts on the Lie algebra of this group. It is equivalent to the representation which acts on the antisymmetric tensors of second order.

If we indicate by $\tilde{\tau}_\varrho$ the differential 3-forms which correspond to the vector fields B_ϱ, from eqs. (5.23) and (5.24) we have

(5.26) $$\tilde{\tau}_i = L^k_i \tau_k ,$$

(5.27) $$\tilde{\tau}_a = R^b_a (\tau_b + F^k_{bj} x^j \tau_k).$$

The vector fields B_i generate space-time translations along the axes of the fixed frame of reference s^0, while the vector fields B_a generate homogeneous Lorentz transformations leaving two of the axes of s^0 fixed. Therefore, it is natural to assume that the differential forms (5.26) and (5.27) define conserved quantities which are just the components of the four-momentum and of the relativistic angular momentum with respect to the fixed frame of reference s^0.

Then, from eq. (5.26), we see that τ_k represents the density and the flow of a component of four-momentum with respect to the variable frame s. From eq. (5.27) we see that the relativistic angular momentum is composed of two parts, which we interpret as spin and orbital angular momentum. We see that τ_b describes the density and the flow of a component of spin angular momentum with respect to the variable reference frame s.

When we take gravitation into account, the Lie-group structure of $\mathscr{S}$ is lost and we cannot define the vector fields B_ϱ and the differential forms $\tilde{\tau}_\varrho$ any longer. However, we assume that the interpretation of the differential forms τ_ϱ that we have found is still valid. Then eq. (5.11) shows that the forms τ_ϱ^G can be interpreted as a description of the four-momentum and the spin angular momentum of the geometric field.

Equation (5.16) shows that the source of one of the quantities τ_ϱ is given by the product of a geometric field and another quantity τ_μ. This is in agreement with elementary field theory: for instance, the product of the total energy density by the gravitational field is a source of momentum. In order to better understand the meaning of eq. (5.16), we assume that the forms τ_ϱ can be written in the following way:

$$\tau_\varrho = \tfrac{1}{6} T^i_\varrho e_{irst} \omega^r \wedge \omega^s \wedge \omega^t . \tag{5.28}$$

After some calculation, using eq. (3.38), we obtain

$$\mathbf{d}\tau_\varrho = (\boldsymbol{L}_i - F_i) T^i_\varrho \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 + \\ + \tfrac{1}{6}[(\boldsymbol{L}_a - F_a) T^i_\varrho + F^i_{aj} T^j_\varrho] e_{irst} \omega^a \wedge \omega^r \wedge \omega^s \wedge \omega^t , \tag{5.29}$$

where

$$F_\alpha = F^j_{\alpha j} . \tag{5.30}$$

The conservation law (5.16) can be written in the form

$$\mathbf{d}\tau_\varrho = -F^\mu_{\varrho i} T^i_\mu \omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3 - \tfrac{1}{6} F^\mu_{\varrho a} T^i_\mu e_{irst} \omega^a \wedge \omega^r \wedge \omega^s \wedge \omega^t . \tag{5.31}$$

Comparing eqs. (5.29) and (5.31) we obtain the conservation laws in the following form:

$$(L_i - F_i) T^i_\varrho = -F^\mu_{\varrho i} T^i_\mu , \tag{5.32}$$

$$(L_a - F_a) T^i_\varrho = F^\mu_{a\varrho} T^i_\mu - F^i_{aj} T^j_\varrho . \tag{5.33}$$

If, moreover, we assume the validity of eqs. (3.25) and (3.35) and we put

$$T^i_a = \tfrac{1}{2} F^k_{al} g^{lj} T^i_{jk} , \tag{5.34}$$

$$T^i_{jk} + T^i_{kj} = 0 , \tag{5.35}$$

$$T^i_k = g^{ij} T_{jk} , \tag{5.36}$$

eq. (5.32) can be written in the form

$$(\boldsymbol{L}_i - F_i)\,T^i_k = F^j_{ik}T^i_j + \tfrac{1}{2}R^l_{mki}g^{mj}T^i_{jl}\,, \tag{5.37}$$

$$(\boldsymbol{L}_i - F_i)\,T^i_{jk} = -T_{jk} + T_{kj}\,. \tag{5.38}$$

These formulae are the correct generalization (6) of the familiar conservation equations

$$\boldsymbol{L}_i T^i_k = 0\,, \tag{5.39}$$

$$T_{ik} = T_{ki}\,, \tag{5.40}$$

which hold in the absence of torsion and of spin angular momentum.

If we assume that the quantities F^i_{ak} are constant, using eqs. (3.10), (3.25), (3.32) and (3.35), eq. (5.33) can be written as

$$\boldsymbol{L}_a T^i_k = F^l_{ak}T^i_l - F^i_{al}T^l_k\,, \tag{5.41}$$

$$\boldsymbol{L}_a T^i_{jk} = F^l_{aj}T^i_{lk} + F^l_{ak}T^i_{jl} - F^i_{al}T^l_{jk}\,. \tag{5.42}$$

These equations are just the consistency conditions which ensure that the quantities T^i_k and T^i_{jk} can be considered as tensors in the space $\mathscr{M}$. They are not equivalent to eq. (5.33) if the quantities F^i_{ak} are variable.

6. – An example of Lagrangian theory.

In this section we consider in detail a specific Lagrangian form λ^{G}, in order to show how the formalism developed in sect. **4** works. There is a large arbitrariness in the choice of the Lagrangian form and we shall discuss elsewhere the principles which should guide this choice. Here we start from the following Lagrangian, which describes a theory without torsion and without spin density, strictly related to Einstein's theory of gravitation:

$$\lambda^{\mathrm{G}} = \frac{1}{4\varkappa}F^j_{al}g^{lk}e_{rjks}F^r_{\alpha\beta}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^a\wedge\omega^s + \frac{1}{4\varkappa}F^j_{al}g^{lk}e_{rkts}F^r_{bj}\,\omega^a\wedge\omega^b\wedge\omega^t\wedge\omega^s\,. \tag{6.1}$$

From this equation we obtain

$$\begin{aligned}(\delta\lambda^{\mathrm{G}}_{\alpha\beta\gamma\delta})\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta = {}&\frac{6}{\varkappa}F^j_{al}g^{lk}e_{rjks}\,\delta F^r_{\alpha\beta}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^a\wedge\omega^s + \\ &+\frac{6}{\varkappa}F^r_{bc}g^{lk}e_{rjks}\,\delta F^j_{al}\,\omega^b\wedge\omega^c\wedge\omega^a\wedge\omega^s + \frac{6}{\varkappa}F^r_{pq}g^{lk}e_{rjks}\,\delta F^j_{al}\,\omega^p\wedge\omega^q\wedge\omega^a\wedge\omega^s + \\ &+\frac{6}{\varkappa}\left[2F^r_{bt}g^{lk}e_{rjks} - F^l_{br}g^{rk}e_{jkts} + F^r_{bj}g^{lk}e_{rkts}\right]\delta F^j_{al}\,\omega^a\wedge\omega^b\wedge\omega^t\wedge\omega^s\,.\end{aligned} \tag{6.2}$$

From eqs. (4.33) and (4.44), we see that, if the normal equations are satisfied, we can write

$$(6.3)\qquad (\delta\lambda^{G}_{\alpha\beta\gamma\delta})\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta = G_{\varrho\gamma\delta}\,\delta F^{\varrho}_{\alpha\beta}\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma\wedge\omega^\delta\,.$$

Comparing eqs. (6.2) and (6.3) in the case in which α, $\beta\geqslant 4$, we see that we must have

$$(6.4)\qquad G_{ras} = -G_{rsa} = \frac{3}{\varkappa} F^{j}_{al} g^{lk} e_{rjks}\,,$$

while the other components of this quantity must vanish. We see also that, as a consequence of the normal field equations, the last three terms in eq. (6.2) must vanish. We get in this way the following normal field equations:

$$(6.5)\qquad F^{r}_{be} = 0\,,$$

$$(6.6)\qquad F^{r}_{pq} e_{rjks} e^{upqs} = 0\,,$$

$$(6.7)\qquad F^{r}_{bt} g^{lk} e_{rjks} - F^{r}_{bs} g^{lk} e_{rjkt} - F^{l}_{br} g^{rk} e_{jkts} + F^{r}_{bj} g^{lk} e_{rkts} = 0\,.$$

Equation (6.5) is just eq. (3.10), which is related to the existence of a space-time manifold. Equations (6.6) and (6.7) after some calculation can be written in the simpler form

$$(6.8)\qquad F^{r}_{pq} = 0\,,$$

$$(6.9)\qquad F^{r}_{bl} g^{lk} + F^{k}_{bl} g^{lr} = 0\,.$$

Equation (6.8) means that we are dealing with a torsion-free theory. Equation (6.9) is equivalent to condition (3.35) which permits the definition of a metric tensor in the space-time manifold.

If we substitute eq. (6.4) into eq. (4.52) and take the normal field equations into account, we get the following tangential field equations:

$$(6.10)\qquad \tau_a = 0\,,$$

$$(6.11)\qquad \tau_i = \frac{1}{4\varkappa} F^{j}_{al} g^{lk} e_{ijkt} F^{a}_{rs}\omega^r\wedge\omega^s\wedge\omega^t + \frac{1}{2\varkappa}(L_r F^{j}_{al} + F^{c}_{ar} F^{j}_{cl}) g^{lk} e_{ijks}\omega^r\wedge\omega^s\wedge\omega^a\,.$$

Equation (6.11) has been simplified by means of eq. (2.10).

Equation (6.10) means that in the theory we are considering there is no spin angular momentum. If we assume that τ_i has the form (5.28), from eq. (6.11) we obtain condition (3.25), which permits the definition of the covariant derivatives in the space-time manifold, and the equation

$$\varkappa T_{ki} = R_{ki} - \tfrac{1}{2} g_{ki} R_{rs} g^{rs} , \tag{6.12}$$

where

$$R_{ki} = R^{s}_{ksi} = - F^{s}_{ak} F^{a}_{si} . \tag{6.13}$$

Equation (6.12) is just Einstein's equation of the gravitational field.

In order to get a complete understanding of the connection between the theory described in the present section and Einstein's theory, we should investigate the role played by the new fields F^{i}_{ak}. This is a delicate problem and we shall discuss it elsewhere. From eq. (6.11) we see that it is possible to describe also the gravitational field generated by energy-momentum distributions which have a nonlocal character. Starting from more complicated Lagrangian forms, one can build theories with weaker normal field equations, which can describe fields generated by sources of a still more general kind. For instance, one can build theories with nonvanishing torsion and nonvanishing spin angular momentum ([6]).

7. – The electromagnetic field.

The electromagnetic field can be treated as a matter field or, alternatively, as an additional geometric field. Here we want to develop the second approach, because the analogy with Maxwell's equations clarifies the structure of the geometric field equations.

In order to give a geometric meaning to electromagnetism, we have just to generalize the concept of frame of reference, assuming that, by giving a frame of reference, besides fixing the position of the origin and the directions of the axes, one fixes also the electromagnetic gauge at the origin. Then the space $\mathscr{S}$ becomes an eleven-dimensional manifold and also the vector space $\mathscr{T}$ acquires a new dimension, as it contains a new infinitesimal transformation, namely a gauge transformation of the first kind. This infinitesimal transformation corresponds to the new element A_{10} of the basis of the space $\mathscr{T}$. For the sake of typographic clarity we replace the index 10 by the simbol $\bullet$. In the present section the Greek indices take the values $0, \ldots, 10$, the Latin indices $a, b, \ldots, h$ take the values $4, \ldots, 9$ and the Latin indices $i, j, \ldots$ take the values $0, \ldots, 3$.

The whole treatment of sect. **2-5** holds in this more general case. One could also treat in a similar way a more complicated, possibly noncommutative, gauge theory. Due to the new dimension of the spaces $\mathscr{S}$ and $\mathscr{T}$, we have additional structure coefficients, which describe the electromagnetic field and a new differential 3-form $\tau_{\bullet}$ which describes the density and the flow of the electric charge.

A reasonable theory of electromagnetism can be obtained from the Lagrangian form (6.1) by extending the range of the Greek indices and adding the new term

$$(7.1)\qquad \lambda^{\mathrm{E}} = -\frac{1}{32\pi} F^{\bullet}_{ik} g^{ir} g^{ks} e_{rspq} F^{\bullet}_{\alpha\beta}\, \omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{p}\wedge\omega^{q} + \\ + \frac{3}{4\pi} F^{\bullet}_{ik} g^{ir} g^{ks} F^{\bullet}_{rs}\, \omega^{0}\wedge\omega^{1}\wedge\omega^{2}\wedge\omega^{3}\,.$$

Proceeding as in sect. **6**, we get other nonvanishing components of the quantity $G_{\varrho\gamma\delta}$, namely

$$(7.2)\qquad G_{\bullet pq} = -\frac{3}{4\pi} F^{\bullet}_{ik} g^{ir} g^{ks} e_{rspq}\,.$$

Besides eqs. (6.5), (6.8) and (6.9), we find the new normal field equations

$$(7.3)\qquad F^{r}_{\bullet\alpha} = 0\,,$$

$$(7.4)\qquad F^{\bullet}_{a\alpha} = 0\,,$$

$$(7.5)\qquad F^{\bullet}_{\bullet\alpha} = 0\,.$$

Finally, substituting eqs. (6.4) and (7.2) into eq. (4.49) or (4.52), taking the normal field equations into account and using also eq. (2.10), we obtain the tangential field equations

$$(7.6)\qquad \tau_{\bullet} = \mathbf{d}\left(\frac{1}{16\pi} F^{\bullet}_{ik} g^{ir} g^{ks} e_{rspq}\, \omega^{p}\wedge\omega^{q}\right),$$

$$(7.7)\qquad \tau_{a} = 0\,,$$

$$(7.8)\qquad \tau_{i} = \frac{1}{4\varkappa} F^{j}_{al} g^{lk} e_{ijkt} F^{a}_{rs}\, \omega^{r}\wedge\omega^{s}\wedge\omega^{t} + \frac{1}{2\varkappa}(\mathbf{L}_{r} F^{j}_{al} + F^{c}_{ar} F^{j}_{cl}) g^{lk} e_{ijks}\, \omega^{r}\wedge\omega^{s}\wedge\omega^{a} + \\ + \frac{1}{32\pi} F^{\bullet}_{pq} g^{pj} g^{qk} (e_{jkst} F^{\bullet}_{ir} - e_{jkir} F^{\bullet}_{st})\, \omega^{r}\wedge\omega^{s}\wedge\omega^{t} + \frac{1}{2\varkappa} F^{j}_{al} g^{lk} e_{ijkt} F^{a}_{\bullet s}\, \omega^{\bullet}\wedge\omega^{s}\wedge\omega^{t}\,.$$

From eq. (2.10), using the normal field equations, we obtain the formulae

$$(7.9)\qquad \mathbf{L}_{i} F^{\bullet}_{jk} + \mathbf{L}_{j} F^{\bullet}_{ki} + \mathbf{L}_{k} F^{\bullet}_{ij} = 0\,,$$

$$(7.10)\qquad \mathbf{L}_{a} F^{\bullet}_{ik} = F^{j}_{ai} F^{\bullet}_{jk} + F^{j}_{ak} F^{\bullet}_{ij}\,.$$

If we assume that $F^{\bullet}_{ik}$ is proportional to the electromagnetic field, eq. (7.9) is just the homogeneous Maxwell equation and eq. (7.10) is the consistency condition which ensures the tensor nature of the electromagnetic field.

From eq. (7.6), after some calculation, taking eq. (7.10) into account, we see that $\tau_\bullet$ can be written in the form (5.28) with

$$T^i_\bullet = \frac{1}{4\pi} L_l F^\bullet_{jk} g^{lj} g^{ik} . \tag{7.11}$$

This is just the inhomogeneous Maxwell equation if we identify $T^i_\bullet$ with the electric-current density and $-F^\bullet_{ik}$ with the electromagnetic field.

In eq. (7.8), the first two terms on the right-hand side are just those which appear in eq. (6.11). The third term can be written in the form

$$-\tfrac{1}{6}\hat{T}^k_i e_{krst}\omega^r\wedge\omega^s\wedge\omega^t , \tag{7.12}$$

where

$$\hat{T}^k_i = \frac{1}{4\pi}\left(-F^\bullet_{pq}g^{pk}g^{qr}F^\bullet_{ir} + \frac{1}{4}\,\delta^k_i F^\bullet_{pq}g^{pr}g^{ps}F^\bullet_{rs}\right) \tag{7.13}$$

is the energy-momentum tensor of the electromagnetic field. This term must appear because the forms τ_i, which concern the matter fields, do not contain the energy and the momentum of the electromagnetic field, which is considered as a geometric field.

If we assume that τ_i has the form (5.28), the last term in eq. (7.8) must vanish and this requirement gives rise to the equation

$$F^a_{\bullet s} = 0 . \tag{7.14}$$

We remark that, as a consequence of eqs. (7.3)-(7.5) and (7.14) the only non-vanishing structure coefficients which contain the index $\bullet$ are the electromagnetic field $-F^\bullet_{ik}$ and the coefficients $F^a_{\bullet b}$. From eq. (2.10) we have

$$F^b_{\bullet a}F^i_{bk} = L_\bullet F^i_{ak} \tag{7.15}$$

and we see that the problem of understanding the meaning of the quantities $F^a_{\bullet b}$ is connected with the problems concerning the fields F^i_{ak}.

8. – Final remarks.

In sect. 6 and 7 we have shown that the general formalism described in sect. 2-5 can be used to formulate the known theories of gravitation and electromagnetism. We have also obtained some indication on the possible modifications of these theories due to the presence of nonlocal terms in the sources of the fields.

The next step should be an investigation of all the possible Lagrangian forms of the geometric fields, in order to find the most satisfactory one. We have seen that the geometric fields have to satisfy the normal field equations which do not contain the field sources, and the tangential equations, which depend on the sources τ_ϱ. We have also seen that the tangential field equations impose some limitations to the form of the sources, namely the conservation equation (5.16) and also other conditions, for instance eq. (6.10) in the theory we have studied in sect. **6**.

We think that geometry should be determined by matter and not vice-versa. Then, in a satisfactory theory, the normal field equations, which limit the properties of geometry independently of the presence of matter, should be as weak as possible. Also the conditions imposed on the sources by the tangential field equations should be as weak as possible. Perhaps, these suggestions for the choice of a satisfactory Lagrangian could be considered as a generalized form of Mach's principle.

● RIASSUNTO

Si generalizza il formalismo della teoria classica dei campi sostituendo alla varietà spazio-temporale $\mathscr{M}$ la varietà $\mathscr{S}$ a dieci dimensioni costituita da tutti i sistemi di riferimento locali. La geometria della varietà $\mathscr{S}$ è determinata da dieci campi vettoriali corrispondenti a dieci trasformazioni infinitesime dei sistemi di riferimento, che sono definite operativamente. Si scrive il principio d'azione in termini di una forma differenziale del quarto ordine nello spazio $\mathscr{S}$ (forma lagrangiana). Le densità e le correnti sono rappresentate da forme differenziali del terzo ordine nello spazio $\mathscr{S}$. Dal principio d'azione si derivano le equazioni di campo e la relazione tra proprietà di simmetria e leggi di conservazione (teorema di Noether). Si riformulano in questo linguaggio la teoria di Einstein della gravitazione e la teoria di Maxwell dell'elettromagnetismo. Nel formalismo generale si possono formulare teorie in cui la carica, l'energia e la quantità di moto non possono essere localizzate nello spazio-tempo ed anche teorie in cui una varietà spazio-temporale non può essere definita esattamente in alcun modo utile.

Классическая теория поля в пространстве систем отсчета.

Резюме (*). — Формализм классической теории поля обобщается посредством замены пространственно-временного множества $\mathscr{M}$ десятимерным множеством $\mathscr{S}$ всех локальных систем отсчета. Геометрия множества $\mathscr{S}$ определяется с помощью десяти векторных полей, соответствующих десяти операторно заданных бесконечно малых преобразований систем отсчета. Принцип действия записывается в виде дифференциальной 4-формы в пространстве $\mathscr{S}$ (лагранжианная форма). Плотности и токи представляются с помощью дифференциальных 3-форм в $\mathscr{S}$. Из принципа действия выводятся уравнения поля и связь между сумметриями и законами сохранения (теорема Ноэтера). Заново формулируются теория гравитации Эйнштейна и теория электромагнетизма Максвелла. Общий формализм может быть также использован для формулировки теорий, в которых заряд, энергия и импульс не могут быть локализованы в пространстве-времени, и теорий, в которых пространственно-временное множество не может быть определено точно.

(*) *Переведено редакцией.*