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An Alternative Approach to the Proof of Unitarity for Gauge Theories (*) (**).

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Summary. — The proof of unitarity for gauge theories involving a Faddeev-Popov ghost is revisited. We assume the existence of a generator F of the Slavnov transformations. The subspace of the unphysical one-particle states is that spanned by F. Under the assumption that F is conserved and obeys the condition $F^2 = 0$, we prove that the unphysical states do not contribute to the unitarity equation restricted to the physical subspace.

In this paper we present an alternative approach to the proof of unitarity for gauge theories given by Becchi, Rouet and Stora (1).

In gauge theories one has to introduce unphysical modes and fields, in order to ensure manifest Lorentz covariance and renormalizability. Let G be the whole vector space of the asymptotic states. Usually, one tries to define a subspace $G' \subset G$ which has positive or positive semi-definite metric and allows a physical interpretation of the vectors (or of equivalence classes, like in the Gupta-Bleuler formulation of QED).

The problem of unitarity can be formulated in the following way. Let

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(1) C. Becchi, A. Rouet and R. Stora: *Gauge field models*, in *Lectures Given at the Erice Summer School* (1975). This paper gives the references of the previous works.

$\psi \in G'$ be any physical vector and S the scattering operator. In general, one expects that

$$S\psi \notin G' ,$$

i.e. S does not leave the physical subspace invariant. However, a physical interpretation is still possible when, for any φ, $\psi \in G'$, the S-matrix elements $(\varphi, S\psi)$ obey the unitarity equation

$$\sum_{n \in \text{complete set of } G'} (\varphi, S^+ n)(n, S\psi) = (\varphi, \psi) . \tag{1}$$

In non-Abelian gauge theories one usually introduces the Faddeev-Popov ghost field $\phi^i(x)$. This is a scalar field obeying Fermi statistics and trasforming according to the adjoint representation of the gauge group. This field enters the Lagrangian in a bilinear and quadrilinear way. Therefore, one can define a ghost charge Q, a conserved Hermitian scalar operator with the commutation relations

$$[Q, \phi^i] = -\phi^i , \qquad [Q, A^i_\mu] = 0 , \qquad [Q, \phi^{*i}] = \phi^{*i} , \tag{2}$$

where A^i_μ is the gauge field. Moreover, Q commutes with any matter field.

Let the field theory be invariant under Slavnov transformations ([1]). Then it is reasonable to assume that these transformations are generated by a translational invariant operator F, belonging to the one-dimensional representation of the gauge group. We assume then

$$[F, S] = 0 , \tag{3}$$

$$F|0\rangle = 0 , \tag{4}$$

$$[Q, F] = -F , \tag{5}$$

$$\begin{cases} \delta\phi^i = \delta\lambda\{F, \phi^i\} , \\ \delta\phi^{*i} = \delta\lambda\{F, \phi^{*i}\} , \\ \delta A^i_\mu = \delta\lambda[F, A^i_\mu] , \end{cases} \tag{6}$$

where the l.h.s. of eq. (6) indicates the Slavnov transforms of the fields, and $\delta\lambda$ is an anticommuting parameter. Moreover, we require that the field theory is such that

$$F^2 = 0 . \tag{7}$$

This assumption is a crucial one, although we did not succeed in finding a reasonable set of hypothesis that makes it also necessary. There are models in which unitarity is broken and (7) is not satisfied ([2]).

([2]) G. CURCI and R. FERRARI: *Phys. Lett.*, **63** B, 91 (1976).

We assume that G is a Fock space, given by the asymptotic states. Let $G^{(1)}$ be the one-particle state vector space. The physical subspace $G' \subset G$ can then be easily defined by splitting $G^{(1)}$ into the direct sum

$$G^{(1)} = G_p^{(1)} \oplus G_u^{(1)} , \tag{8}$$

where $G_p^{(1)}$ and $G_u^{(1)}$ are the physical and unphysical subspaces. Then G' is made up of vectors containing only physical modes. We characterize $G_p^{(1)}$ and $G_u^{(1)}$ by means of F, *i.e.* F is represented trivially in $G_p^{(1)}$, and $G_u^{(1)}$ gives an irreducible representation of F. Moreover, $G_p^{(1)}$ should have a positive definite metric. Then, for any $\psi \in G'$, we have

$$F\psi = 0 . \tag{9}$$

We denote by $|c\rangle$, $|\bar{c}\rangle$, $|a_i\rangle$, $i = 1, 2$, the one-particle states belonging to the sectors $Q = 1, -1, 0$ of $G_u^{(1)}$, respectively. We drop all the group representation indices and the momentum in the notations. We chose $|a_i\rangle$ in such a way that

$$\begin{cases} F|a_1\rangle = |\bar{c}\rangle , & F|\bar{c}\rangle = 0 , \\ F|c\rangle = |a_2\rangle , & F|a_2\rangle = 0 . \end{cases} \tag{10}$$

Since F is a Poincaré-invariant quantity and it is invariant under the group transformations, all these states must have the same mass and spin, and belong to the same representation of the gauge group. Condition (7) is satisfied and it is easy to see that (10) is the most general situation which realizes (7) with three sectors of charge $Q = 1, 0, -1$.

In terms of creation operators, we have

$$\begin{cases} [F, a_1^\dagger] = \bar{c}^+ , & \{F, \bar{c}^-\} = 0 , \\ [F, a_2^\dagger] = 0 , & \{F, c^-\} = a_2^\dagger . \end{cases} \tag{11}$$

From the canonical theory we have (we drop the momentum dependence and the group indices of creation and annihilation operators)

$$\{c, c^+\} = 1 , \qquad \{\bar{c}, \bar{c}^+\} = -1 \tag{12}$$

and, in general,

$$[a_i, a_j^\dagger] = \eta_{ij} , \tag{13}$$

where η is a nondegenerate numerical matrix. By using the Jacobi identity and eqs. (11)-(13), we get

$$\begin{cases} [F, \eta_{1i}^{-1} a_i] = 0 , & \{F, c\} = 0 , \\ [F, \eta_{2i}^{-1} a_i] = -c , & \{F, \bar{c}\} = -\eta_{1j}^{-1} a_j . \end{cases} \tag{14}$$

Some Lagrangians for non-Abelian gauge theories are not Hermitian and, therefore, the S-matrix is not unitary on the whole space G. In these cases, however, one can show that, to every order of the perturbative expansion for any φ, $\psi \in G'$,

$$(\varphi, S^+ \psi) = (\varphi, S^{-1} \psi) . \tag{15}$$

In fact, in these models the following relation is valid:

$$\mathscr{L}^+(x) = \mathscr{C}^{-1} \mathscr{L} \mathscr{C} , \tag{16}$$

where $\mathscr{C}$ is the Q-charge conjugation operator

$$\mathscr{C}: \phi^i \to \phi^{*i} , \qquad \mathscr{C}: \phi^{*i} \to - \phi^i , \qquad \mathscr{C}: A^i_\mu \to A^i_\mu .$$

Then eq. (15) follows from the invariance, under charge conjugation, of the physical states and from eq. (16); in fact,

$$\begin{aligned}\big(\varphi, T(\mathscr{L}(x_1) \dots \mathscr{L}(x_n))\, \psi\big)^* &= \big(\varphi, \overline{T}(\mathscr{L}^+(x_1) \dots \mathscr{L}^+(x_n))\, \varphi\big) = \\ &= \big(\mathscr{C}\psi, \overline{T}(\mathscr{L}(x_1) \dots \mathscr{L}(x_n))\mathscr{C}\varphi\big) = \big(\psi, \overline{T}(\mathscr{L}(x_1) \dots \mathscr{L}(x_n))\, \varphi\big) ,\end{aligned}$$

where $\overline{T}$ is the antitime-ordered product ([3]).

By using eq. (15), we can write eq. (1) in the form

$$\sum_{n \in \text{complete set of } G'} (\varphi, S^{-1} n)(n, S\psi) = (\varphi, \psi) . \tag{17}$$

We introduce the following operator product V. Let $AB \dots MN \dots$ be either creation or annihilation operators, then

$$V(AB \dots MN) = (-)^{\delta_P} A' B' \dots |0\rangle\langle 0| M' N' \dots , \tag{18}$$

where $A'B' \dots M'N' \dots$ are the operators $AB \dots MN \dots$ re-arranged in such a way that, on the left of the projector on the vacuum, there are only creation operators and, on the right, only annihilation operators. $(-)^{\delta_P}$ is the parity of the permutation of the anticommuting operators. V acts linearly on linear combinations of monomials and, moreover,

$$V(1) = |0\rangle\langle 0| .$$

([3]) N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields*, Chap. III (New York, N. Y., 1959).

The sum over a complete set of states in G can be then written

$$1 = V(\exp[\mathscr{A} + \mathscr{B}])\,, \tag{19}$$

where

$$\mathscr{A} = \sum(a_i^\dagger a_j \eta_{ij}^{-1} + c^+ c - \bar{c}^+ \bar{c}) \tag{20}$$

(the sum $\sum$ runs over the momenta and the internal indices) and $\mathscr{B}$ is the analogous quantity for the physical modes. Equation (17) will then be satisfied if

$$\mathscr{V}(\lambda) = \big(\varphi, S^{-1} V(\exp[\lambda\mathscr{A} + \mathscr{B}])\, S\psi\big) \tag{21}$$

is λ-independent for any $\varphi, \psi \in G'$. To show this, we consider the derivative with respect to λ:

$$\frac{\partial \mathscr{V}(\lambda)}{\partial \lambda} = \big(\varphi, S^{-1} \sum\{a_i^\dagger \mathscr{E}(\lambda) a_j n_{ij}^{-1} + c^+ \mathscr{E}(\lambda) c - \bar{c}^+ \mathscr{E}(\lambda)\bar{c}\} S\psi\big)\,, \tag{22}$$

where

$$\mathscr{E}(\lambda) \equiv V(\exp[\lambda\mathscr{A} + \mathscr{B}])\,. \tag{23}$$

By using eqs. (11) and (14), we can write eq. (22) in the form

$$\frac{\partial \mathscr{V}(\lambda)}{\partial \lambda} = \big(\varphi, S^{-1} \sum\{\{F, c^+\}\, \mathscr{E}(\lambda) a_j \eta_{2i}^{-1} - a_1^\dagger \mathscr{E}(\lambda)\{F, \bar{c}\} + {} \\ {} + c^+ \mathscr{E}(\lambda)[\eta_{2i}^{-1} a_i, F] - [F, a_1^\dagger]\mathscr{E}(\lambda)\bar{c}\}\psi\big)\,. \tag{24}$$

Now we use the fact that F commutes with S and annihilates the physical states (eqs. (3) and (9)):

$$\frac{\partial \mathscr{V}(\lambda)}{\partial \lambda} = \big(\varphi, S^{-1} \sum\{c^+[F, \mathscr{E}(\lambda)]\eta_{2i}^{-1} a_i + a_1^\dagger[F, \mathscr{E}(\lambda)]\bar{c}\}\, \psi\big)\,. \tag{25}$$

Finally, we show that

$$[F, \mathscr{E}(\lambda)] = 0\,. \tag{26}$$

This can be shown directly by using the commutation relations (11) and (14). More simply, one can see that F commutes by construction with the number of physical modes and with the number of unphysical modes. Then the coefficients of the power expansion in λ of the l.h.s. of eq. (26) coincide with the coefficients of the expansion in number of unphysical modes of $[F, \mathscr{E}(1)]$. But

$\mathscr{E}(1) = 1$ and, therefore, eq. (26) is obtained. Finally, we have

$$\frac{\partial \mathscr{V}(\lambda)}{\partial \lambda} = 0 \, .$$

The commutation relations (11) and (14) can be derived from the renormalized Slavnov transformations by taking the asymptotic limit, and from the equations of motion of the asymptotic fields.

In some Lagrangian models, dipole fields appear in the asymptotic limit [1], *e.g.* a field $\chi(x)$ which obeys the equation of motion

$$(\Box + m^2)^2 \chi = 0 \, . \tag{27}$$

Also in this case, one can define creation and annihilation operators and write commutation relations like those in eqs. (11) and (14). The field χ can be uniquely written in the form

$$\chi(x) = x^\mu \partial_\mu \chi_2(x) + \chi_1(x) \, , \tag{28}$$

where

$$(\Box + m^2)\chi_i = 0 \, , \qquad i = 1, 2 \, .$$

To the scalar fields χ_i now we can associate creation and annihilation operators a_i, $a_i^\dagger$. One-particle states can be also defined $|1\boldsymbol{p}\rangle$, $|2\boldsymbol{p}\rangle$. From eq. (28) and from $U(y)\chi(x)U^{-1}(y) = \chi(x+y)$ one can find how these states transform under space-time translations:

$$\begin{cases} U(y)|2\boldsymbol{p}\rangle = \exp[-ipy]\,|2\boldsymbol{p}\rangle \, , \\ U(y)|1\boldsymbol{p}\rangle = \exp[-ipy]\,(|1\boldsymbol{p}\rangle - ipy|2\boldsymbol{p}\rangle) \end{cases} \tag{29}$$

with $p^2 = m^2$.

The scalar product among these states is defined in terms of a matrix

$$\langle i\boldsymbol{p}'|j\boldsymbol{p}\rangle = \eta_{ij} 2p_0 \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}') \, . \tag{30}$$

The metric is translational invariant only if

$$\eta_{12} = \eta_{21} \, , \qquad \eta_{22} = 0 \, .$$

It can be easily seen that eq. (10) is compatible with translational invariance also when $|a_1\rangle$ and $|a_2\rangle$ represent the states associated to a dipole field.

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● RIASSUNTO

Si riesamina la prova dell'unitarietà per teorie di gauge che contengono un fantasma di Faddeev e Popov. Si assume l'esistenza di un generatore delle trasformazioni di Slavnov. Il sottospazio degli stati a una particella non fisica rappresenta in modo non banale F. Nell'ipotesi che F sia conservato e che obbedisca alla condizione $F^2 = 0$, si dimostra che gli stati non fisici non contribuiscono all'equazione di unitarietà ristretta al sottospazio dei vettori fisici.

Резюме не получено.