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Pseudoparticles and Conformal Symmetry.

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Summary. — The relation of conformal symmetry to the existence of zero energy-momentum (improved) solutions is investigated in four-dimensional space for a class of Lagrangian models. It is pointed out that in Minkowski space such solutions are either constant (or trivial) or the theory is conformally invariant. In Euclidean space such solutions may be constructed if solutions to the conformally invariant theories are known.

1. – Introduction.

Recently, a considerable amount of attention has been given to classical solutions to various field theories. Several authors ([1,2]) have considered conformally invariant field equations. In the approach of Fubini ([1]), importance is attached to solutions which break conformal invariance, but which still keep a lower $O_{3,2}$ symmetry. A simple example is the theory with one scalar field obeying the equation

$$\Box \Phi + 4g\Phi^3 = 0 . \tag{1.1}$$

([1]) S. FUBINI: *Nuovo Cimento*, **34** A, 521 (1976).
([2]) A. A. BELAVIN, A. M. POLYAKOV, A. S. SCHWARTZ and YU. S. TYUPHIN: *Phys. Lett.*, **59** B, 85 (1975).

Equation (1.1) allows for the solutions

$$\Phi = + \frac{2a}{\sqrt{2}g(x^2 + a^2)} . \tag{1.2}$$

Here a^2 is any real number. These solutions have the following properties:

1) localization in x for fixed t (they are also localized in four-dimensional space-time),

2) the improved energy-momentum tensor vanishes,

3) they are Lorentz invariant.

On the other hand, BELAVIN *et al.* (2) considered an O_4 non-Abelian Yang-Mills theory in Euclidean space, *i.e.* another example of a conformally invariant theory

$$\begin{cases} \partial_\nu F^{\mu\nu} = g[F^{\mu\nu}, A^\nu] , \\ F^{\mu\nu} \; = \partial^\mu A^\nu - \partial^\nu A^\mu + g[A^\mu, A^\nu] , \end{cases} \tag{1.3}$$

and discussed the solutions

$$\begin{cases} A^\mu = -\dfrac{2i}{g} \dfrac{\Sigma_{\mu\alpha} x^\alpha}{x^2 + a^2} , \\[2ex] F^{\mu\nu} = \dfrac{4i}{g} \dfrac{a^2 \Sigma^{\mu\nu}}{(x^2 + a^2)^2} . \end{cases} \tag{1.4}$$

Euclidean solutions (1.4) have the following properties:

I) localization in space-« time »,

II) the improved energy-momentum tensor vanishes,

III) invariance on the combined rotation of four-dimensional space and isospace.

The motivations of the authors of ref. (1,2) are different, but the solutions they consider have similar properties. With respect to the conformal group, these solutions are not invariant under a full group ($O_{5,1}$) but only under an O_5 subgroup (3,4).

(3) R. JACKIW and C. REBBI: *Phys. Rev. D*, **14**, 517 (1976).
(4) V. DE ALFARO, S. FUBINI and G. FURLAN: *A new classical solution of the Yang-Mills field equations*, Ref. TH 2232-CERN.

A specific property is the vanishing of the energy-momentum tensor (improved). Of course, the knowledge of a zero-energy class of solutions to a theory is interesting in itself. However, several authors [5,6] have pointed out another application; they attempted to construct the ground state for a Yang-Mills theory. They noted that in the Schrödinger picture the existence of imaginary-time zero-energy solutions indicates the occurrence of tunnelling among classical zero-energy configurations (in real time). Solution (1.4) interpolates between two different classical vacua for $x_4 = +\infty$ and $x_4 = -\infty$ (when gauge equivalence is suitably reinterpreted [5,6]). Again, as in the Schrödinger case, they conclude that tunnelling takes place. In fact, it is possible to form a nondegenerate set of states, each being a superposition of different classical vacua. For more details we refer the reader to the original papers [5,6] or extensive lectures by JACKIW [7]. Here we want to stress that the existence of solutions with vanishing Euclidean energy-momentum tensor enables one to conclude on the occurrence of the above effect.

Several questions may be posed:

I) To what extent is conformal invariance relevant to the existence of solutions with vanishing Euclidean energy-momentum tensor? In particular, are there other theories with broken conformal invariance which allow such solutions?

II) Are there other solutions of this type in a given theory?

In this paper we first consider scalar theories with the Lagrangian

$$L = \tfrac{1}{2}(\partial_\mu \Phi)^2 - V(\Phi) \tag{1.5}$$

and then gauge theories

$$L = -\tfrac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + \tfrac{1}{2}(D_\mu \Phi^p)(D_\mu \Phi^p) - V(\Phi^p \Phi^p) \tag{1.6}$$

with a compact gauge group G and a unitary representation $D(G)$ for the scalar fields Φ^p.

We shall find that conformal invariance is essential for the existence of the above solutions. More precisely, the only theories in Minkowski space that allow such solutions are the conformal invariant ones with $V = 0$ and $V = G(\Phi_p \Phi^p)^2$. In the case of scalar theories we are able to answer also the second

(5) R. JACKIW and C. REBBI: *Phys. Rev. Lett.*, **37**, 172 (1976).
(6) C. CALLAN, R. DASHEN and D. GROSS: *Phys. Lett.*, **63** B, 334 (1976).
(7) R. JACKIW: *Semi-classical analysis of quantum field theory*, in *Lectures at XVI Escuela Latino Americana, 1976, Caracas, Venezuela.*

question. The forms of all solutions for the free case are

$$\Phi = \text{const} \quad \text{and} \quad \Phi = \frac{\text{const}}{x^2}\,, \tag{1.7}$$

and solution (1.2), discussed by FUBINI, is unique for $V = G\Phi^4$.

In the Euclidean case the situation is different. It is possible to have such solutions for a general potential, but they are always of the type

$$(G_{\mu\nu},\, \Phi_c)\,,$$

where $G_{\mu\nu}$ is the solution of a pure Yang-Mills theory and Φ_c corresponds to $\Phi = 0$ if $V(0) = 0$ or to solutions of $\mathbf{\Phi}^2 = \mathbf{\Phi}_c^2$, $D_\mu \mathbf{\Phi} = 0$ with A_μ in D_μ being the same as that which generates $G_{\mu\nu}$. Also $V'(\Phi_c^2) = 0$. Thus, solutions of a general theory, if they exist, can be constructed from solutions of pure Yang-Mills theories which are conformally invariant. In this sense, conformal symmetry again plays a crucial role.

2. – Field theory with one scalar field.

$$L = \tfrac{1}{2}(\partial_\mu \Phi)^2 - V(\Phi)\,, \tag{2.1}$$

$$\Box \Phi + V'(\Phi) = 0\,. \tag{2.2}$$

The improved energy-momentum tensor is

$$\Theta^{\mu\nu} = \partial^\mu \Phi \partial^\nu \Phi - g^{\mu\nu}\{\tfrac{1}{2}(\partial^\alpha \Phi)^2 - V(\Phi)\} + \tfrac{1}{6}[g^{\mu\nu} - \partial^\mu \partial^\nu]\Phi^2\,. \tag{2.3}$$

We are looking for solutions satisfying

$$\Theta^{\mu\nu}(\Phi) = 0\,. \tag{2.4}$$

A necessary condition is

$$\operatorname{Tr} \Theta(\Phi) = 0\,, \tag{2.5}$$

$$\operatorname{Tr} \Theta = 4V(\Phi) - \Phi V'(\Phi) = 0\,. \tag{2.6}$$

There are two possibilities:

a) $\operatorname{Tr}\Theta = 0$ for all Φ and so (2.6) becomes a condition for the interaction which is satisfied only by

$$V(\Phi) = 0 \qquad \text{a free massless theory} \tag{2.7}$$

and

$$V(\Phi) = g\Phi^4 . \tag{2.8}$$

b) $\mathrm{Tr}\,\Theta \neq 0$ generally, but vanishes for some particular solutions of the equations of motion. Then, for a general type of interaction, eq. (2.6) becomes a condition for Φ. If we assume V to be analytical in Φ, eq. (2.6) is satisfied only by $\Phi = \text{const}$ (*i.e.* no space-time dependence otherwise V would be identically zero). Consequently,

$$\Phi_n = C_n , \qquad n = 1, 2, \dots , \tag{2.9}$$

may be solutions of eq. (2.6). By using the equation of motion

$$V'(\Phi_n) = 0 \qquad \text{and} \qquad V(\Phi_n) = 0 , \tag{2.10}$$

and as a consequence

$$\Theta^{\mu\nu}(\Phi_n) = 0 . \tag{2.11}$$

In conclusion, field configurations that are different from a constant and which give the vanishing energy-momentum tensor exist only in a free massless theory with $V(\Phi) = 0$ and $V(\Phi) = g\Phi^4$. These are also the only conformally invariant theories for the class of Lagrangians (2.1).

The next question is to find explicitly all such solutions that have the three properties mentioned in the introduction. The Lorentz-invariant solutions are of the form

$$\Phi(x) = \Phi(x^2) . \tag{2.12}$$

In this case, the energy-momentum is restricted to

$$\Theta^{\mu\nu} = \tfrac{1}{3}(\Phi\Phi'' - 2\Phi'^2)(x^2 g_{\mu\nu} - 4x_\mu x_\nu) \tag{2.13}$$

and the wave equation to

$$x^2\Phi'' + 2\Phi' + 4g\Phi^3 = 0 , \tag{2.13'}$$

where the derivatives are taken over x^2.

Condition (2.4) is reduced to the nonlinear differential equation

$$\Phi\Phi'' - 2\Phi'^2 = 0 . \tag{2.14}$$

Dividing eq. (2.14) by Φ'^2, one may write

$$\left(\frac{\Phi'}{\Phi^2}\right)' = -1 . \tag{2.15}$$

Thus, all solutions within proper functions (*i.e.* no δ-like terms) are

$$\Phi = A\,, \tag{2.16}$$

$$\Phi = \frac{B}{x^2 + a^2}\,, \tag{2.17}$$

where A, B and a^2 are arbitrary constants. We still have to require compatibility with wave equations.

2'1. *Free massless case.* – The constant a has to vanish and therefore

$$\Phi_1 = A\,, \qquad \Phi_2 = \frac{B}{x^2}\,. \tag{2.18}$$

A linear combination of Φ_1 and Φ_2 will again be a solution of the equations of motion, but in general with nonvanishing $\Theta^{\mu\nu}$. In fact

$$\Theta^{\mu\nu}\left(\Phi = A + \frac{B}{x^2}\right) = \frac{2}{3}\,\frac{AB}{(x^2)^3}\,, \tag{2.19}$$

and solutions with the same AB have identical $\Theta^{\mu\nu}$.

2'2. *Interaction case* ($V = g\Phi^4$). – Compatibility with the equations of motion requires

$$A = 0\,, \qquad B = \pm\frac{2a}{\sqrt{2g}}\,, \tag{2.20}$$

thus

$$\Phi_{1,2} = \pm\frac{2a}{\sqrt{2g}(x^2 + a^2)}\,, \tag{2.21}$$

which represents a one-parameter family of solutions considered in ref. (1). In fact, the above procedure shows that these solutions are the only ones within the class of proper functions with properties 1), 2) and 3).

3. – Gauge teories.

Consider a theory in which the Lagrangian has the form

$$L = -\tfrac{1}{4}\,G^a_{\mu\nu}G^a_{\mu\nu} + \tfrac{1}{2}\,(D_\mu\Phi^p)(D_\mu\Phi^p) - V(\Phi^2)\,, \tag{3.1}$$

where

$$D^p_\mu(\Phi) = \partial_\mu\Phi^p + ig(A^a_\mu T^a)^{pq}\Phi^q \tag{3.2}$$

and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gC^{abc} A^b_\mu A^c_\nu \,. \tag{3.3}$$

The compact gauge group G and the unitary representation $D(G)$ under which the scalar fields transform may be left arbitrary. The symmetric energy-momentum tensor (Belinfante tensor) reads

$$\Theta^{\mu\nu}(\text{Bel}) = - G^{a\mu}_{\ \ \nu} G^{a\nu}_{\ \ \lambda} + \tfrac{1}{4} g^{\mu\nu} G^a_{\lambda\beta} G^{a,\lambda\beta} + + D^{a\mu}(\Phi) D^{p\nu}(\Phi) - \tfrac{1}{2} g^{\mu\nu} D^p_\alpha(\Phi) D^{\alpha,p}(\Phi) + g^{\mu\nu} V \tag{3.4}$$

and the improved energy-momentum tensor is

$$\Theta^{\mu\nu} = \Theta^{\mu\nu}(\text{Bel}) - \tfrac{1}{6}(\partial^\mu\partial^\nu - g^{\mu\nu}) \Phi^p \Phi^p \,. \tag{3.5}$$

Again, we are looking for solutions to

$$\Theta^{\mu\nu} = 0 \,, \tag{3.6}$$

for which a necessary condition is

$$\operatorname{Tr}\Theta = 0 \,. \tag{3.7}$$

However, from (3.5)

$$\operatorname{Tr}\Theta = \operatorname{Tr}\Theta(\text{Bel}) + (\partial_\mu \Phi^p)(\partial_\mu \Phi^p) + \Phi^p \Box \Phi^p \,.$$

If we use the equations of motion, the trace is expressed by

$$\operatorname{Tr}\Theta = \operatorname{Tr}\Theta(\text{Bel}) + (D_\mu \Phi^p)(D_\mu \Phi^p) - 2\mathbf{\Phi}^2 V' \,.$$

From eq. (3.4)

$$\operatorname{Tr}\Theta(\text{Bel}) = - (D_\mu \Phi^p)(D_\mu \Phi^p) + 4V \,. \tag{3.8}$$

Finally,

$$\operatorname{Tr}\Theta = 4V(\mathbf{\Phi}^2) - 2\mathbf{\Phi}^2 V'(\mathbf{\Phi}^2) \,, \tag{3.9}$$

which is a very simple expression where the trace is given completely in terms of scalar fields, as expected.

Condition (3.7) for the vanishing of the trace now means that

$$2V(\mathbf{\Phi}^2) = \mathbf{\Phi}^2 V'(\mathbf{\Phi}^2) \,. \tag{3.10}$$

There are again two possibilities:

a) $\Theta^\mu_\mu \equiv 0$. Then, $V = 0$ and therefore $V = g(\mathbf{\Phi}^2)^2$, that is the case of conformally invariant theories.

b) $\Theta^\mu_\mu \neq 0$ generally, but vanishes for some particular solutions of the equations of motion. Assuming again V to be analytical in Φ^2, we conclude, as in sect. **2**, that the required solution must be of the form

$$\mathbf{\Phi}^2 = \text{const} \equiv \mathbf{\Phi}^2_c . \tag{3.11}$$

It is obvious that Φ_i may depend on the co-ordinates. For such solutions, the difference between the Belinfante tensor and the improved tensor vanishes

$$\Theta_{\mu\nu} = \Theta_{\mu\nu}(\mathrm{B}) . \tag{3.12}$$

Because of the vanishing of $\mathrm{Tr}\,\Theta$, $\mathrm{Tr}\,\Theta(\mathrm{B})$ also vanishes; so from (3.8) one obtains

$$(D_\mu \Phi^p D_\mu \Phi^p) = 4V(\Phi^2_c) = \text{const} . \tag{3.13}$$

3'1. *Localization.* – We require localization of ∂_μ and A_μ. More precisely, we require vanishing of $\partial_\mu \Phi$ and A_μ for fixed (but arbitrary) and large $\boldsymbol{x}$. Then, as the left-hand side of (3.13) vanishes at least somewhere in space-time, the constant $V(\Phi^2_c)$ must also vanish (*).

Finally, we can state that localized solutions to $\mathrm{Tr}\,\Theta = 0$ satisfy the following requirement:

$$\begin{cases} \mathbf{\Phi}^2 = \mathbf{\Phi}^2_c = \text{const} , \\ (D_\mu \Phi^p)(D_\mu \Phi^p) = 0 , \\ V(\Phi^2_c) = 0 , \qquad \mathbf{\Phi}^2_c V'(\Phi^2_c) = 0 . \end{cases} \tag{3.14}$$

The corresponding Θ^μ_ν is

$$\Theta_{\mu\nu} = -G^a_{\mu\lambda} G^a_{\nu\lambda} + \tfrac{1}{4} g_{\mu\nu} G^a_{\alpha\beta} G^a_{\alpha\beta} + D^p_\mu(\Phi) D^p_\nu(\Phi) . \tag{3.15}$$

3'2. *Minkowski space.* – If $\Theta^{\mu\nu} = 0$, Θ^{00} is

$$\Theta^{00} = \tfrac{1}{2}(G^a_{0i})^2 + \tfrac{1}{4}(G^a_{ij} G^a_{ij}) + D^a_0 D^a_0 = 0 ,$$

(*) This is the same requirement as that for finite-energy solutions. For such solutions, Φ tends asymptotically to a null point of the potential. (See, *e.g.*, ref. (8).)

(8) S. COLEMAN: *Classical lumps and their quantum descendants*, in *Lectures at the 1975 School «Ettore Majorana», Erice, Italy*.

and therefore

$$G^a_{0i} = 0 \,, \qquad G^p_{ij} = 0 \,, \qquad D^p_0 = 0 \,.$$

Thus, combining these expressions with (3.14), we find that the vanishing of Θ implies

$$G^a_{\mu\nu} = 0 \qquad \text{and} \qquad D^a_\mu(\Phi) = 0 \,. \tag{3.16}$$

Equations (3.16) have only trivial solutions:

$$\Phi^p = C\delta^{pn} \qquad \text{and} \qquad A^a_\mu = 0 \qquad\qquad \text{with } \mathbf{\Phi}^2 = \mathbf{\Phi}^2_c .$$

The only candidates for which nontrivial solutions may exist are the theories with $V=0$ and $V=\lambda\Phi^4$.

Example 1:

$$V = 0 \,, \qquad D(G) \text{ is an adjoint representation.}$$

A solution is

$$A^a_\mu = 0 = \Phi^a = \varphi(x^2)\begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix}, \qquad \varphi(x^2) = \frac{C}{x^2}\,, \qquad \varphi(x^2) = \text{const}\,,$$

$$\Theta^{\mu\nu} = 0 \,.$$

Example 2 (*):

$$V = \lambda(\Phi^2)^2 \,, \qquad D(G) \text{ is an adjoint representation.}$$

$$A^a_\mu = 0 \,, \qquad \Phi^a = \varphi(x^2)\begin{pmatrix}0\\ \vdots\\ i\end{pmatrix}, \qquad \varphi(x^2) = \pm\frac{2a}{\sqrt{2g}(a^2+x^2)} \,.$$

Again, $\Theta^{\mu\nu} = 0$.

3'3. *Euclidean space.* – In the Euclidean space, the situation is slightly different. Because of the metric, conclusion (3.16) is no longer allowed. However, condition (3.14)

$$(D_\mu \Phi^p)(D_\mu \Phi^p) = 0$$

now implies

$$D_\mu \Phi^p = 0 \,, \tag{3.17}$$

(*) An analogous solution to an Abelian Yang-Mills theory was discussed by FURLAN in context with Fubini's program [9].

[9] G. FURLAN: private communication.

where it was essential that G was compact. For solutions with vanishing trace, the tensor is

$$\Theta_{\mu\nu} = - G^a_{\mu\lambda} G^a_{\nu\lambda} + \tfrac{1}{4} g^{\mu\nu} G^a_{\alpha\beta} G^a_{\alpha\beta} . \tag{3.18}$$

Thus any solution to a pure Yang-Mills theory with $\Theta = 0$ will also be a solution to our problem. More precisely, solutions with vanishing energy-momentum tensor are of the type

$$(G^a_{\mu\nu}, \Phi^p) ,$$

where $G^a_{\mu\nu}$ is a solution to a pure Yang-Mills theory (no Higgs fields).

For Φ_p there are at most two classes of solutions. One class are solutions with $\Phi = 0$ if $V(0) = 0$. The other class are solutions to $\mathbf{\Phi}^2 = \mathbf{\Phi}^2_c$, $D_\mu \mathbf{\Phi} = 0$, with A_μ in D_μ being a solution to a pure Yang-Mills theory.

Example:

$G = SO_3$; A^a_μ and Φ^a are triplets. Choose gauge such that $\Phi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Phi_c$;

Φ_c is, of course, constant due to (3.14). Then

$$D_\mu \Phi_p = g \Phi_c A^a_\mu C_{pa3} , \qquad A^a_\mu = \delta^a_3 A_\mu(x) , \tag{3.19}$$

$$G^a_{\mu\nu} = \delta^a_3 F_{\mu\nu} , \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{3.20}$$

Consequently, solutions are

$$(G^a_{\mu\nu}, \Phi^p) ,$$

where $G^a_{\mu\nu}$ are given by (3.20) and $F_{\mu\nu}$ is a solution of a pure Yang-Mills theory

$$\partial_\mu F_{\mu\nu} = 0 ,$$

$$\Phi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Phi_{c'} , \qquad \Theta_c \text{ is a solution of } V'(\Phi^2_c) = 0 .$$

Another trivial possibility is to choose $\Phi = 0$. Then, from any solution G to the interacting Yang-Mills theory we can construct a solution to the theory with general V

$$(G_{\mu\nu}, \Phi) , \qquad \Phi = 0 .$$

We may conclude that solutions with vanishing Euclidean energy-momentum tensor (improved) in conformally noninvariant theories can be obtained from the knowledge of solutions of invariant theories. The existence of the latter was shown in ref. (2). At least the class with $\Phi = 0$ is nonempty.

* * *

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● RIASSUNTO

Si discute la connessione fra simmetria conforme e l'esistenza, per una classe di modelli Lagrangiani, di soluzioni per cui il tensore «migliorato» energia-impulso sia nullo. Si nota che, nel caso dello spazio di Minkowski a quattro dimensioni, tali soluzioni sono costanti (o banali) oppure il modello è conformemente invariante. Nello spazio Euclideo tali soluzioni possono essere costruite una volta note le soluzioni dei modelli conformemente invarianti.

Псевдочастицы и конформная симметрия.

Резюме (*). — Исследуется связь конформной симметрии с существованием решений с нулевым значением энергии-импульса в четырехмерном пространстве для класса моделей с Лагранжианами. Отмечается, что в пространстве Минковского такие решения либо являются постоянными (или тривиальными), либо теория является конформно инвариантной. В эвклидовом пространстве такие решения могут быть сконструированы, если известны решения в конформно инвариантных теориях.

(*) *Переведено редакцией.*