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Quantum Mechanics of Two Coupled Oscillators.

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Summary. — From the point of view of quantum mechanics, the problem of two coupled oscillators with two different coupling parameters is considered. By using the accurate definition of Dirac operators the wave function in both coherent state and number state (Schrödinger picture) are obtained. The Green's function and the expectation value of the energy are calculated; the transition amplitude between the coherent states when the coupling parameters are different and equal, as well as the eigenstates, are given. The constants of the motion for such a system have been also considered.

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1. – Introduction.

In the last few decades the most relevant problem to the field of quantum optics, is the problem of frequency converter and parametric amplifier, where two electromagnetic fields are coupled [1-11]. In fact, this problem has played a central role in several physical phenomena of interest, such as coherent Raman and Brillouin scattering, spontaneous and stimulated emission of radiation, super-radiance, etc. In the mean time some authors [12-14] have taken the problem further by considering three coupled electromagnetic fields, which leads to more complicated non-linear parametric interactions. The frequency converter, parametric amplifier and the photon-photon process of Raman or Brillouin scattering have been treated extensively by direct use of the coupled equations of motion. The treatment has been extended to include the statistical properties such as the expectation values and fluctuations, as well as the density matrix and the quasi-probability distribution functions (P-representation, W-Wigner, and Q-function). Obviously, the analysis is drastically simplified if we first diagonalize the Hamiltonian at exact resonance and only one coupling parameter is used. In the present paper we shall deal quantum-mechanically with two more general coupled oscillators, where the Hamiltonian with two different coupling parameters will be diagonalized at off resonance.

The Hamiltonian we shall consider is given by

$$H = H_1 + H_2 , \tag{1.1a}$$

$$H_1 = \hbar\omega_1 a^\dagger a + \hbar\omega_2 b^\dagger b , \tag{1.1b}$$

$$H_2 = \hbar\lambda_1 (a^\dagger b + ab^\dagger) + \hbar\lambda_2 (a^\dagger b^\dagger + ab) , \tag{1.1c}$$

where ω_1 and ω_2 are the fields frequencies, while λ_1 is the coupling constant that incorporates the non-linear susceptibility of the crystal and the driving field amplitude, and λ_2 is the coupling constant measuring the strength of the mixing of the two modes. It would be interesting to point out that the Hamiltonian (1.1) can be obtained from the quantization of the cavity modes, by taking the magnetic permeability and the electric permittivity to vary at the same time, see for example ref. [2]. Here we may say that the Hamiltonian given by eq. (1.1*c*) has been used by the authors of ref. [15] to describe the simultaneous non-degenerate parametric amplification and mixing of two modes via a rotation of their polarization, where the Schrödinger-cat wave function is calculated. Also in ref. [16] the evolution operator of this Hamiltonian is used as a squeeze operator to calculate the Glauber second-order correlation function, and to discuss some other statistical properties, which is of interest to the field of quantum optics. In the following section of the present paper we shall introduce the accurate definition for the Dirac operator to diagonalize the Hamiltonian (1.1) and then to calculate the wave function in the coherent state as well as in the number state. Section **3** is devoted to give the exact expression for the Green's function, and to calculate the partition function. In sect. **4** we shall concentrate with the calculation of the transition amplitude in both coherent and number states. Finally, in sect. **5** we have constructed a linear and a quadratic invariant for the Hamiltonian (1.1). In sect. **6** we give our conclusion.

2. – The coherent and the number states wave functions.

In this section we shall calculate the wave function in both coherent and number states, by using the accurate definition for the Dirac operators, that is to diagonalize the Hamiltonian given by eq. (1.1). To reach this goal, let us define the following operators:

$$a = (2\omega_1\hbar)^{-1/2} (\omega_1 q_1 + ip_1) , \tag{2.1a}$$

$$b = (2\omega_2\hbar)^{-1/2} (\omega_2 q_2 + ip_2) . \tag{2.1b}$$

Therefore, if we substitute eqs. (2.1*a*), (2.1*b*) into eq. (1.1), we have

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(\omega_1^2 q_1^2 + \omega_2^2 q_2^2) + \mu_1 q_1 q_2 + \mu_2 p_1 p_2 , \tag{2.2}$$

where

$$\mu_1 = (\lambda_1 + \lambda_2)\sqrt{\omega_1\omega_2} , \tag{2.3a}$$

(2.3b) $$\mu_2 = (\lambda_1 - \lambda_2)/\sqrt{\omega_1 \omega_2}\,.$$

To diagonalize the above Hamiltonian, we shall use the following operators:

(2.4a) $$A(t) = [2\Omega_+ \hbar\sqrt{\gamma_1\gamma_2}]^{-1/2}[k_x(\sqrt{\gamma_2}\cos\theta q_1(t) + \sqrt{\gamma_1}\sin\theta q_2(t)) + \\ + iJ_x(\sqrt{\gamma_1}\cos\theta p_1(t) + \sqrt{\gamma_2}\sin\theta p_2(t))]\,,$$

(2.4b) $$B(t) = [2\Omega_- \hbar\sqrt{\gamma_1\gamma_2}]^{-1/2}[k_y(\sqrt{\gamma_1}\cos\theta q_2(t) - \sqrt{\gamma_2}\sin\theta q_1(t)) + \\ + iJ_y(\sqrt{\gamma_2}\cos\theta p_2(t) - \sqrt{\gamma_1}\sin\theta p_1(t))]\,.$$

The following abbreviations have been used in the above equations:

(2.5a) $$J_x = (\gamma_1\gamma_2)^{-1/4}[\gamma_2\cos^2\theta + \gamma_1\sin^2\theta + \mu_2\sqrt{\gamma_1\gamma_2}\sin 2\theta]^{1/2}\,,$$

(2.5b) $$J_y = (\gamma_1\gamma_2)^{-1/4}[\gamma_1\cos^2\theta + \gamma_2\sin^2\theta - \mu_2\sqrt{\gamma_1\gamma_2}\sin 2\theta]^{1/2}\,,$$

(2.5c) $$k_x = (\gamma_1\gamma_2)^{-1/4}[\gamma_1\omega_1^2\cos^2\theta + \gamma_2\omega_2^2\sin^2\theta + \mu_1\sqrt{\gamma_1\gamma_2}\sin 2\theta]^{1/2}\,,$$

(2.5d) $$k_y = (\gamma_1\gamma_2)^{-1/4}[\gamma_2\omega_2^2\cos^2\theta + \gamma_1\omega_1^2\sin^2\theta - \mu_1\sqrt{\gamma_1\gamma_2}\sin 2\theta]^{1/2}\,,$$

(2.5e) $$\gamma_1 = \sqrt{\frac{\omega_2}{\omega_1}}[\lambda_1(\omega_1 + \omega_2) + \lambda_2(\omega_1 - \omega_2)]\,,$$

(2.5f) $$\gamma_2 = \sqrt{\frac{\omega_1}{\omega_2}}[\lambda_1(\omega_1 + \omega_2) - \lambda_2(\omega_1 - \omega_2)]\,,$$

(2.5g) $$\theta = \frac{1}{2}\,\mathrm{tg}^{-1}\left(\frac{2\sqrt{\gamma_1\gamma_2}}{\omega_1^2 - \omega_2^2}\right)$$

and

(2.6a) $$\Omega_+^2 = k_1^2\cos^2\theta + k_2^2\sin^2\theta + \sqrt{\gamma_1\gamma_2}\sin 2\theta\,,$$

(2.6b) $$\Omega_-^2 = k_2^2\cos^2\theta + k_1^2\sin^2\theta - \sqrt{\gamma_1\gamma_2}\sin 2\theta\,,$$

where

(2.6c) $$k_i^2 = \omega_i^2 + (\lambda_1^2 + \lambda_2^2)\,, \qquad i = 1, 2\,.$$

Now, if we use eqs. (2.4a), (2.4b) together with eq. (2.2) we find

(2.7) $$H = \hbar\Omega_+\left(A^\dagger A + \frac{1}{2}\right) + \hbar\Omega_-\left(B^\dagger B + \frac{1}{2}\right),$$

where A and B satisfy the commutation relation

(2.8) $$[A, A^\dagger] = [B, B^\dagger] = 1\,.$$

In order to calculate the wave function in the coherent states, let us define the state $|\alpha, \beta\rangle$ as follows:

$$|\alpha, \beta\rangle = \exp\left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2)\right]\sum_{m,n=0}^{\infty}\frac{\alpha^n\beta^m}{\sqrt{n!\,m!}}|n, m\rangle. \tag{2.9}$$

From eqs. (2.4) and (2.9) we have

$$A(t)|\alpha, \beta\rangle = \alpha(t)|\alpha, \beta\rangle, \tag{2.10a}$$

$$B(t)|\alpha, \beta\rangle = \beta(t)|\alpha, \beta\rangle. \tag{2.10b}$$

Therefore, if we use eqs. (2.4) together with eqs. (2.10) we get

$$J_x\left[\sqrt{\gamma_1}\cos\theta\frac{\partial}{\partial q_1} + \sqrt{\gamma_2}\sin\theta\frac{\partial}{\partial q_2}\right]\psi_{\alpha\beta} = \left[\alpha(t)\sqrt{\frac{2\Omega_+}{\hbar}}\sqrt[4]{\gamma_1\gamma_2} - k_x(\sqrt{\gamma_2}\cos\theta q_1 + \sqrt{\gamma_1}\sin\theta q_2)\right]\psi_{\alpha\beta}, \tag{2.11a}$$

$$J_y\left[\sqrt{\gamma_2}\cos\theta\frac{\partial}{\partial q_2} - \sqrt{\gamma_1}\sin\theta\frac{\partial}{\partial q_1}\right]\psi_{\alpha\beta} = \left[\beta(t)\sqrt{\frac{2\Omega_-}{\hbar}}\sqrt[4]{\gamma_1\gamma_2} - k_y(\sqrt{\gamma_1}\cos\theta q_2 - \sqrt{\gamma_2}\sin\theta q_1)\right]\psi_{\alpha\beta}. \tag{2.11b}$$

From eqs. (2.11a), (2.11b) we obtain the wave function in the following form:

$$\begin{aligned}\psi_{\alpha\beta}(q_1, q_2, t) = N\exp&\left[-\frac{1}{2\hbar}\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}[(k_x/J_x)\cos^2\theta + (k_y/J_y)\sin^2\theta]q_1^2\right]\cdot\\ \cdot\exp&\left[-\frac{1}{2\hbar}\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}[(k_y/J_y)\cos^2\theta + (k_x/J_x)\sin^2\theta]q_2^2\right]\cdot\\ \cdot\exp&\left[\frac{1}{2\hbar}[(k_y/J_y) - (k_x/J_x)]\sin 2\theta q_1 q_2\right]\cdot\\ \cdot\exp&\left[\left[\alpha(t)\sqrt{2\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}\frac{k_x/J_x}{\hbar}}\cos\theta - \beta(t)\sqrt{2\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}\frac{k_y/J_y}{\hbar}}\sin\theta\right]q_1\right]\cdot\\ \cdot\exp&\left[\left[\beta(t)\sqrt{2\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}\frac{k_y/J_y}{\hbar}}\cos\theta + \alpha(t)\sqrt{2\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}\frac{k_x/J_x}{\hbar}}\sin\theta\right]q_2\right],\end{aligned} \tag{2.12}$$

where N is the normalizing constant, given by

$$N = [\pi\hbar J_x J_y]^{-1/2}(\Omega_+\Omega_-)^{1/4}\exp\left[-\frac{1}{2}[\alpha^2(t) + \beta^2(t) + |\alpha|^2 + |\beta|^2]\right] \tag{2.13}$$

and

$$\alpha(t) = \alpha(0) \exp[-i\Omega_+ t], \qquad \beta(t) = \beta(0) \exp[-i\Omega_- t]. \tag{2.14}$$

To calculate the wave function in the number state (Schrödinger wave function), we have to use eqs. (2.9) and (2.12) and we find that

$$\psi_{nm}(q_1, q_2, t) = \left[\frac{\Omega_+ \Omega_-}{(\hbar\pi)^2}\right]^{1/4} (J_x J_y 2^{n+m} n!\, m!]^{-1/2} \cdot \tag{2.15}$$

$$\cdot H_n\left(\left(\frac{k_x}{\hbar J_x}\right)^{1/2}\left[\left(\frac{\gamma_2}{\gamma_1}\right)^{1/4} \cos\theta q_1 + \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4} \sin\theta q_2\right]\right) \cdot$$

$$\cdot H_m\left(\left(\frac{k_y}{\hbar J_y}\right)^{1/2}\left[\left(\frac{\gamma_1}{\gamma_2}\right)^{1/4} \cos\theta q_2 - \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4} \sin\theta q_1\right]\right) \cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\sqrt{\frac{\gamma_2}{\gamma_1}}\, [(k_x/J_x)\cos^2\theta + (k_y/J_y)\sin^2\theta]\, q_1^2\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\sqrt{\frac{\gamma_1}{\gamma_2}}\, [(k_y/J_y)\cos^2\theta + (k_x/J_x)\sin^2\theta]\, q_2^2\right] \cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}[(k_x/J_x) - (k_y/J_y)] \sin 2\theta q_1 q_2\right] \exp\left[-i\left[\Omega_+\left(n + \frac{1}{2}\right)t + \Omega_-\left(m + \frac{1}{2}\right)t\right]\right].$$

In the following we shall employ the result obtained in this section to calculate the Bloch density matrix, and then to calculate the expectation value of the energy.

3. – The Bloch density matrix and the expectation value of the energy.

In the present section we shall make use of the coherent-state wave function given by eq. (2.12) to calculate the Bloch density matrix. However, we shall first start by calculating the Green's function. The Green's function is connected with the coherent-states wave function by the equation

$$G(q_1, q_2, q_1', q_2', t) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \psi_{\alpha\beta}(q_1, q_2, t)\, \psi^*_{\alpha\beta}(q_1', q_2', 0)\, \mathrm{d}^2\alpha\, \mathrm{d}^2\beta. \tag{3.1}$$

Therefore, by inserting eq. (2.12) and the corresponding complex conjugate at $t = 0$ into eq. (3.1) and by evaluating the integral one finds

$$G(q_1, q_2, q_1', q_2', t) = (2\hbar\pi J_x J_y)^{-1} [\Omega_+ \Omega_- / \sin\Omega_+ t \sin\Omega_- t]^{1/2} \cdot \tag{3.2}$$

$$\cdot \exp\left[\frac{i}{2\hbar}\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}\left(\frac{k_x}{J_x}\operatorname{ctg}\Omega_+ t \cos^2\theta + \frac{k_y}{J_y}\operatorname{ctg}\Omega_- t \sin^2\theta\right)(q_1^2 + q_1'^2)\right] \cdot$$

$$\cdot \exp\left[\frac{i}{2\hbar}\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}\left(\frac{k_y}{J_y}\operatorname{ctg}\Omega_- t\cos^2\theta + \frac{k_x}{J_x}\operatorname{ctg}\Omega_+ t\sin^2\theta\right)(q_2^2 + q_2'^2)\right]\cdot$$

$$\cdot \exp\left[\frac{i}{2\hbar}\left(\frac{k_x}{J_x}\operatorname{ctg}\Omega_+ t - \frac{k_y}{J_y}\operatorname{ctg}\Omega_- t\right)\sin 2\theta(q_1 q_2 + q_1' q_2')\right]\cdot$$

$$\cdot \exp\left[-\frac{i}{\hbar}\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}\left(\frac{k_x}{J_x}\operatorname{cosec}\Omega_+ t\cos^2\theta + \frac{k_y}{J_y}\operatorname{cosec}\Omega_- t\sin^2\theta\right)q_1 q_1'\right]\cdot$$

$$\cdot \exp\left[-\frac{i}{\hbar}\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}\left(\frac{k_y}{J_y}\operatorname{cosec}\Omega_- t\cos^2\theta + \frac{k_x}{J_x}\operatorname{cosec}\Omega_+ t\sin^2\theta\right)q_2 q_2'\right]\cdot$$

$$\cdot \exp\left[-\frac{i}{2\hbar}\left(\frac{k_x}{J_x}\operatorname{cosec}\Omega_+ t - \frac{k_y}{J_y}\operatorname{cosec}\Omega_- t\right)\sin 2\theta(q_1 q_2' + q_2 q_1')\right].$$

Since the Hamiltonian (1.1) is a constant of the motion, therefore the system is in equilibrium, and then the analytic continuation $t \to -i\gamma\hbar$ in the Green's function (3.2) leads immediately to the Bloch density matrix. In this case we have the following expression:

$$(3.3)\qquad C(q_1, q_2, q_1', q_2', \gamma) = (2\hbar\pi J_x J_y)^{-1}[\Omega_+\Omega_-/\sin\Omega_+ t\sin\Omega_- t]^{1/2}\cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\left(\frac{\gamma_2}{\gamma_1}\right)^{1/2}\left\{\frac{k_x}{J_x}[(q_1^2 + q_1'^2)\operatorname{ctgh}(\gamma\hbar\Omega_+) - 2q_1 q_1'\operatorname{cosech}(\gamma\hbar\Omega_+)]\cos^2\theta + \right.\right.$$

$$\left.\left. + \frac{k_y}{J_y}[(q_1^2 + q_1'^2)\operatorname{ctgh}(\gamma\hbar\Omega_-) - 2q_1 q_1'\operatorname{cosech}(\gamma\hbar\Omega_-)]\sin^2\theta\right\}\right]\cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\left(\frac{\gamma_1}{\gamma_2}\right)^{1/2}\left\{\frac{k_y}{J_y}[(q_2^2 + q_2'^2)\operatorname{ctgh}(\gamma\hbar\Omega_-) - 2q_2 q_2'\operatorname{cosech}(\gamma\hbar\Omega\ \ )]\cos^2\theta + \right.\right.$$

$$\left.\left. + \frac{k_x}{J_x}[(q_2^2 + q_2'^2)\operatorname{ctgh}(\gamma\hbar\Omega_+) - 2q_2 q_2'\operatorname{cosech}(\gamma\hbar\Omega_+)]\sin^2\theta\right\}\right]\cdot$$

$$\cdot \exp\left[-\frac{1}{2\hbar}\left[\frac{k_x}{J_x}\{(q_1 q_2 + q_1' q_2')\operatorname{ctgh}(\gamma\hbar\Omega_+) - (q_1 q_2' + q_2 q_1')\operatorname{cosech}(\gamma\hbar\Omega_+)\} - \right.\right.$$

$$\left.\left. - \frac{k_y}{J_y}\{(q_1 q_2 + q_1' q_2')\operatorname{ctgh}(\gamma\hbar\Omega_-) - (q_1 q_2' + q_2 q_1')\operatorname{cosech}(\gamma\hbar\Omega_-)\}\right]\sin 2\theta\right].$$

As a special case if we take the coupling parameter $\lambda_2 \to 0$, we obtain eq. (4.1) of ref. [10].

To calculate the expectation value of the energy, we have to calculate the partition function which is given by

$$Q(\gamma) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(q_1, q_2, \gamma)\, dq_1 dq_2 , \tag{3.4}$$

where C is the density matrix. From eqs. (3.3) and (3.4) one7 finds

$$Q(\gamma) = \frac{1}{4}\,\mathrm{cosech}\left(\frac{1}{2}\gamma\hbar\Omega_+\right)\mathrm{cosech}\left(\frac{1}{2}\gamma\hbar\Omega_-\right). \tag{3.5}$$

The expectation value of the energy can be obtained by using the equation

$$\langle E\rangle = -\frac{\partial}{\partial\gamma}\ln[Q(\gamma)]. \tag{3.6}$$

Thus

$$\left\langle\frac{E}{\hbar}\right\rangle = \frac{1}{2}\left[\Omega_+\,\mathrm{ctgh}\left(\frac{\gamma\hbar\Omega_+}{2}\right) + \Omega_-\,\mathrm{ctgh}\left(\frac{\gamma\hbar\Omega_-}{2}\right)\right]. \tag{3.7}$$

4. – The transition amplitude.

Suppose we take the coupling parameters λ_1 and λ_2 to be equal in eq. (2.2), therefore we shall find the parameter $\mu_2 \to 0$, and hence the energy for the system when $\lambda_1 = \lambda_2$ will be different from the energy when $\lambda_1 \neq \lambda_2$. In this case we may turn our attention to calculate the transition amplitude between the states when $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$, in both the coherent and the number states.

To do so, we have first to obtain the wave function in the coherent state for the Hamiltonian (2.2) when $\lambda_1 = \lambda_2 = \lambda$. We find the following expression:

$$\psi_{\bar\alpha\bar\beta}(q_1, q_2, t) = \left[\frac{k_+ k_-}{(\hbar\pi)^2}\right]^{1/4} \exp\left[-\frac{1}{2}[\bar\alpha^2(t) + \bar\beta^2(t) + |\bar\alpha|^2 + |\bar\beta|^2]\right]\cdot$$
$$\cdot\exp\left[-\frac{1}{2\hbar}[k_+(\cos\delta q_1 + \sin\delta q_2)^2 + k_-(\cos\delta q_2 - \sin\delta q_1)^2]\right]\cdot$$
$$\cdot\exp\left[\bar\alpha(t)\sqrt{\frac{2k_+}{\hbar}}(q_1\cos\delta + q_2\sin\delta)\right]\exp\left[\bar\beta(t)\sqrt{\frac{2k_-}{\hbar}}(q_2\cos\delta - q_1\sin\delta)\right], \tag{4.1}$$

where

$$k_+^2 = \omega_1^2\cos^2\delta + \omega_2^2\sin^2\delta + 2\lambda\sqrt{\omega_1\omega_2}\sin 2\delta , \tag{4.2a}$$

$$k_-^2 = \omega_2^2\cos^2\delta + \omega_1^2\sin^2\delta - 2\lambda\sqrt{\omega_1\omega_2}\sin 2\delta , \tag{4.2b}$$

(4.2c)
$$\delta = \frac{1}{2}\,\mathrm{tg}^{-1}\left(\frac{4\lambda\sqrt{\omega_1\omega_2}}{\omega_1^2-\omega_2^2}\right)$$

and

(4.2d)
$$\bar{\alpha}(t) = \bar{\alpha}(0)\exp[-ik_+ t]\,, \qquad \bar{\beta}(t) = \bar{\beta}(0)\exp[-ik_- t]\,.$$

The amplitude connecting two coherent states is given by

(4.3)
$$\langle\bar{\alpha}, \bar{\beta}|\beta, \alpha\rangle = \int_{-\infty}^{\infty} \psi^*_{\bar{\alpha}\bar{\beta}}(q_1, q_2, t)\,\psi_{\alpha\beta}(q_1, q_2, t)\,\mathrm{d}q_1\,\mathrm{d}q_2\,.$$

By inserting eqs. (2.13) and the complex conjugate of eq. (4.1) into eq. (4.3) and then evaluating the integral, one finds

(4.4)
$$\langle\bar{\alpha}, \bar{\beta}|\alpha, \beta\rangle = \frac{2}{\sqrt{l'}}(k_+k_-\Omega_+\Omega_-)^{1/2}\cdot$$
$$\cdot\exp\left[-\frac{1}{2}[\bar{\alpha}^{*2}(t) + \bar{\beta}^{*2}(t) + \alpha^2(t) + \beta^2(t) + |\bar{\alpha}|^2 + |\bar{\beta}|^2 + |\alpha|^2 + |\beta|^2]\right]\cdot$$
$$\cdot\exp\left[\frac{k_x}{l'}[\beta(t)\sqrt{k_y} - \sqrt{J_y}(\bar{\alpha}^*(t)\sqrt{k_+}f_2 - \bar{\beta}^*(t)\sqrt{k_-}f_1)]^2\right]\cdot$$
$$\cdot\exp\left[\frac{k_y}{l'}[\alpha(t)\sqrt{k_x} + \sqrt{J_x}(\bar{\alpha}^*(t)\sqrt{k_+}f_3 + \bar{\beta}^*(t)\sqrt{k_-}f_4)]^2\right]\cdot$$
$$\cdot\exp\left[\frac{k_-}{l'}(\bar{\alpha}^*(t)\sqrt{k_+J_xJ_y} + \alpha(t)\sqrt{k_xJ_y}f_1 - \beta(t)\sqrt{k_yJ_x}f_4)^2\right]\cdot$$
$$\cdot\exp\left[\frac{k_+}{l'}(\bar{\beta}^*(t)\sqrt{k_-J_xJ_y} + \alpha(t)\sqrt{J_yk_x}f_2 + \beta(t)\sqrt{k_yJ_x}f_3)^2\right],$$

where

(4.5)
$$l' = [k_+k_-J_xJ_y + k_xJ_y(k_+f_2^2 + k_-f_1^2) + k_yJ_x(k_+f_3^2 + k_-f_4^2) + k_xk_y]$$

and

(4.6a)
$$f_1 = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}\cos\theta\cos\delta + \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}\sin\theta\sin\delta\,,$$

(4.6b)
$$f_2 = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}\sin\theta\cos\delta - \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}\cos\theta\sin\delta\,,$$

(4.6c)
$$f_3 = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}\cos\theta\cos\delta + \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}\sin\theta\sin\delta\,,$$

(4.6d) $$f_4 = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4} \sin\theta \cos\delta - \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4} \cos\theta \sin\delta .$$

Now we shall calculate the transition amplitude $A_{nm\bar{n}\bar{m}}$ from the state $|n, m\rangle$ to the state $|\bar{n}, \bar{m}\rangle$, where $|n, m\rangle$ and $|\bar{n}, \bar{m}\rangle$ are the eigenstates for the cases $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$, respectively. The transition amplitude can be calculated from the following equation:

(4.7) $$A_{nm\bar{n}\bar{m}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi^*_{\bar{n}\bar{m}}(q_1, q_2, t)\,\psi_{nm}(q_1, q_2, t)\,\mathrm{d}q_1\,\mathrm{d}q_2 ,$$

where $\psi_{\bar{n}\bar{m}}(q_1, q_2, t)$ is the wave function for eq. (2.2) when $\lambda_1 = \lambda_2$ which takes the form

(4.8) $$\psi_{nm}(q_1, q_2, t) = \left[\frac{k_+ k_-}{(\hbar\pi)^2}\right]^{1/4} [2^{\bar{n}+\bar{m}}\bar{n}!\,\bar{m}!]^{-1/2} H_{\bar{n}}\left[\sqrt{\frac{k_+}{\hbar}}(\cos\delta q_1 + \sin\delta q_2)\right]\cdot$$
$$\cdot H_{\bar{m}}\left[\sqrt{\frac{k_-}{\hbar}}(\cos\delta q_2 - \sin\delta q_1)\right] \exp\left[-\frac{1}{2\hbar}(k_+\cos^2\delta + k_-\sin^2\delta)\,q_1^2\right]\cdot$$
$$\cdot\exp\left[-\frac{1}{2\hbar}(k_-\cos^2\delta + k_+\sin^2\delta)\,q_2^2\right]\exp\left[-\frac{1}{2\hbar}(k_+ - k_-)\sin 2\delta q_1 q_2\right]\cdot$$
$$\cdot\exp\left[-i\left[k_+\left(\bar{n}+\frac{1}{2}\right)t + k_-\left(\bar{m}+\frac{1}{2}\right)t\right]\right].$$

By inserting eq. (2.14) and the complex conjugate of eq. (4.8) into eq. (4.7) we have

(4.9) $$A_{nm\bar{n}\bar{m}} = (\hbar\pi)^{-1}(k_+ k_- \Omega_+ \Omega_-)^{1/4}[J_x J_y 2^{n+m+\bar{n}+\bar{m}} n!\,m!\,\bar{n}!\,\bar{m}!]^{-1/2}\cdot$$
$$\cdot\exp\left[i\left[k_+\left(\bar{n}+\frac{1}{2}\right) + k_-\left(\bar{m}+\frac{1}{2}\right) - \Omega_+\left(n+\frac{1}{2}\right) - \Omega_-\left(m+\frac{1}{2}\right)\right]t\right]\cdot$$
$$\cdot\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\, H_{\bar{n}}\left[\sqrt{\frac{k_+}{\hbar}}\,x\right] H_{\bar{m}}\left[\sqrt{\frac{k_-}{\hbar}}\,y\right]\exp\left[-\frac{1}{2\hbar}(k_+x^2 + k_-y^2)\right]\cdot$$
$$\cdot H_n\left[\sqrt{\frac{k_x}{\hbar J_x}}(xf_1 + yf_2)\right] H_m\left[\sqrt{\frac{k_y}{\hbar J_y}}(yf_3 - xf_4)\right]\cdot$$
$$\cdot\exp\left[-\frac{1}{2\hbar}\left[\frac{k_x}{J_x}(xf_1 + yf_2)^2 + \frac{k_y}{J_y}(yf_3 - xf_4)^2\right]\right],$$

where f_1, f_2, f_3 and f_4 are given by eqs. (4.6).

To calculate the integral in eq. (4.9) we shall introduce the generating function

$$(4.10)\quad A_{s_1s_2s_3s_4} = \int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}y \exp\left[-\frac{1}{2\hbar}\left[\left(k_+ + \frac{k_x}{J_x} f_1^2 + \frac{k_y}{J_y} f_4^2\right)x^2 + \right.\right.$$

$$\left.\left. + \left(k_- + \frac{k_x}{J_x} f_2^2 + \frac{k_y}{J_y} f_3^2\right) y^2 + 2\left(\frac{k_x}{J_x} f_1 f_2 - \frac{k_y}{J_y} f_3 f_4\right) xy\right]\right] \cdot$$

$$\cdot \exp[-(s_1^2 + s_2^2 + s_3^2 + s_4^2)] \exp\left[2x\left[s_1\left(\frac{k_+}{\hbar}\right)^{1/2} + s_3\left(\frac{k_x}{\hbar J_x}\right)^{1/2} f_1 - s_4\left(\frac{k_y}{\hbar J_y}\right)^{1/2} f_4\right]\right] \cdot$$

$$\cdot \exp\left[2y\left[s_2\left(\frac{k_-}{\hbar}\right)^{1/2} + s_3\left(\frac{k_x}{\hbar J_x}\right)^{1/2} f_2 + s_4\left(\frac{k_y}{\hbar J_y}\right)^{1/2} f_3\right]\right].$$

Therefore, the transition amplitude is given by

$$(4.11)\quad A_{nm\bar{n}\bar{m}} = \frac{2}{\sqrt{l'}} (k_+ k_- \Omega_+ \Omega_-)^{1/4} [2^{n+m+\bar{n}+\bar{m}} n!\, m!\, \bar{n}!\, \bar{m}!]^{-1/2} \cdot$$

$$\cdot \exp\left[i\left[k_+\left(\bar{n} + \frac{1}{2}\right)t + k_-\left(\bar{m} + \frac{1}{2}\right)t - \Omega_+\left(n + \frac{1}{2}\right)t - \Omega_-\left(m + \frac{1}{2}\right)t\right]\right] \cdot$$

$$\cdot \frac{\partial^{n+\bar{m}+n+m}}{\partial s_1^{\bar{n}}\, \partial s_2^{\bar{m}}\, \partial s_3^{n}\, \partial s_4^{m}} \exp[-(s_1^2 + s_2^2 + s_3^2 + s_4^2)] \cdot$$

$$\cdot \exp\left[\frac{2}{l'}[s_2\sqrt{k_+ k_- J_x J_y} + s_3 \bar{f}_2 + s_4 \bar{f}_3]^2\right] \cdot \exp\left[\frac{2}{l'}[s_1\sqrt{k_+ k_- J_x J_y} + s_3 \bar{f}_1 - s_4 \bar{f}_4]^2\right] \cdot$$

$$\cdot \exp\left[\frac{2}{l'}[s_3\sqrt{k_x k_y} + s_1 \bar{f}_3 - s_2 \bar{f}_4]^2\right] \exp\left[\frac{2}{l'}[s_4\sqrt{k_x k_y} + s_2 \bar{f}_1 - s_1 \bar{f}_2]^2\right]\Bigg|_{s_1=s_2=s_3=s_4=0},$$

where

$$(4.12a)\quad \bar{f}_1 = \sqrt{k_- k_x J_y}\, f_1 ,$$

$$(4.12b)\quad \bar{f}_2 = \sqrt{k_+ k_x J_y}\, f_2 ,$$

$$(4.12c)\quad \bar{f}_3 = \sqrt{k_+ k_y J_x}\, f_3 ,$$

$$(4.12d)\quad \bar{f}_4 = \sqrt{k_- k_y J_x}\, f_4 ,$$

and f_1, f_2, f_3 and f_4 are given by eqs. (4.6).

After some minor algebra, eq. (4.11) can be expressed in the following form:

$$A_{nm\bar{n}\bar{m}} = \frac{2}{\sqrt{l'}}(k_+ k_- \Omega_+ \Omega_-)^{1/4}[n!\, m!\, \bar{n}!\, \bar{m}!]^{1/2} \cdot \tag{4.13}$$

$$\cdot \exp\left[i\left[k_+\left(\bar{n}+\frac{1}{2}\right)t + k_-\left(\bar{m}+\frac{1}{2}\right)t - \Omega_+\left(n+\frac{1}{2}\right)t - \Omega_-\left(m+\frac{1}{2}\right)t\right]\right]\cdot$$

$$\cdot \sum_{k=0}^{\bar{n}} \sum_{j=0}^{\bar{m}} \sum_{p=0}^{n} \sum_{r=0}^{p} [r!(p-r)!\,(\bar{n}-k-p+r)!\,(\bar{m}-j-r)!]^{-1}\, 4^{[\bar{n}+m-j-k]}\cdot$$

$$\cdot (-)^{n+r-k-p}(F_{13}/F_{42})^{p-r}(F_{24}/F_{31})^{r}(F_{31})^{\bar{m}-j}(F_{42})^{\bar{n}-k}\cdot$$

$$\cdot (A_+/A_-)^{(\bar{n}-\bar{m})/4}(B_+/B_-)^{(n-m)/4}(i^{\tilde{l}+\tilde{r}}\tilde{l}!\,\tilde{r}!)^{-1}(A_+A_- - F^2)^{(\bar{n}+\bar{m})/4}\cdot$$

$$\cdot (B_+B_- - G^2)^{(n+m)/4} P^{(1/2)|\bar{n}-\bar{m}|}_{(\bar{n}+\bar{m})/2}\left[\frac{iF}{\sqrt{A_+A_- - F^2}}\right] P^{(1/2)|n-m|}_{(n+m)/2}\left[\frac{iG}{\sqrt{B_+B_- - G^2}}\right],$$

where P stands for an associated Legendre function of the first kind. Note that in the above expression $\bar{m} \geqslant r + j$, and $\bar{n} \geqslant p + k - r$, also $n - m$ and $\bar{n} - \bar{m}$ are even, while $A_{nm\bar{n}\bar{m}}$ is equal to zero otherwise. The following abbreviations have been used:

$$F_{13} = \frac{1}{l'}[\sqrt{k_+ k_- J_x J_y}\,\bar{f}_1 + \sqrt{k_x k_y}\,\bar{f}_3], \tag{4.14a}$$

$$F_{31} = \frac{1}{l'}[\sqrt{k_+ k_- J_x J_y}\,\bar{f}_3 + \sqrt{k_x k_y}\,\bar{f}_1], \tag{4.14b}$$

$$F_{24} = \frac{1}{l'}[\sqrt{k_+ k_- J_x J_y}\,\bar{f}_2 + \sqrt{k_x k_y}\,\bar{f}_4], \tag{4.14c}$$

$$F_{42} = \frac{1}{l'}[\sqrt{k_+ k_- J_x J_y}\,\bar{f}_4 + \sqrt{k_x k_y}\,\bar{f}_2], \tag{4.14d}$$

$$F = \frac{2}{l'}(\bar{f}_3\bar{f}_4 - \bar{f}_1\bar{f}_2), \tag{4.14e}$$

$$G = \frac{2}{l'}(\bar{f}_2\bar{f}_3 - \bar{f}_1\bar{f}_4), \tag{4.14f}$$

while

$$A_\pm = \frac{1}{l'}[(k_+ k_- J_x J_y - k_x k_y) \pm (\bar{f}_2^2 + \bar{f}_3^2 - \bar{f}_1^2 - \bar{f}_4^2)], \tag{4.15a}$$

(4.15b) $$B_\pm = \frac{1}{l'}[(k_x k_y - k_+ k_- J_x J_y) \pm (\bar{f}_1^2 + \bar{f}_2^2 - \bar{f}_3^2 - \bar{f}_4^2)],$$

and

(4.16a) $$\tilde{l} = \frac{1}{2}[(n+m) + |n-m|],$$

(4.16b) $$\tilde{r} = \frac{1}{2}[(\bar{n}+\bar{m}) + |\bar{n}-\bar{m}|].$$

5. – The constants of the motion.

As has been stated in earlier work [17], if one can find explicitly time-dependent invariants, then the eigenvalues and eigenstates of these invariants would be helpful for solving some explicit quantum-mechanical problems. Therefore, we shall concentrate in this section on discussing the constants of the motion for the system given by eq. (2.2). Let us start by seeking linear invariants in the form [18, 19]

(5.1) $$I = \sum_{i=1}^{2}[\alpha_i(t)p_i + \beta_i(t)q_i].$$

To construct a linear invariant we need

(5.2) $$\dot{I} = \frac{\partial I}{\partial t} + \sum_{i=1}^{2}\frac{\partial I}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial I}{\partial p_i}\frac{\partial H}{\partial q_i} = 0, \qquad i = 1, 2.$$

From eqs. (2.2) and (5.1) together with eq. (5.2) we have

(5.3a) $$\frac{d\alpha_1}{dt} = \omega_1^2\beta_1 + \mu_1\beta_2,$$

(5.3b) $$\frac{d\alpha_2}{dt} = \omega_2^2\alpha_2 + \mu_1\alpha_1,$$

(5.3c) $$\frac{d\beta_1}{dt} = -\alpha_1 - \mu_2\alpha_2,$$

(5.3d) $$\frac{d\beta_2}{dt} = -\alpha_2 - \mu_2\alpha_1.$$

After some minor algebra we find

(5.4a) $$\alpha_1(t) = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}[A_1\sin(\Omega_+ t + \phi_1)\cos\theta - B_1\sin(\Omega_- t + \phi_2)\sin\theta],$$

(5.4b) $$\alpha_2(t) = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4} [B_1 \sin(\Omega_- t + \phi_2)\cos\theta + A_1 \sin(\Omega_+ t + \phi_1)\sin\theta],$$

(5.4c) $$\beta_1(t) = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}\left[A_1 \frac{k_x}{J_x}\cos(\Omega_+ t + \phi_1)\cos\theta - B_1 \frac{k_y}{J_y}\cos(\Omega_- t + \phi_2)\sin\theta\right],$$

(5.4d) $$\beta_2(t) = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}\left[B_1 \frac{k_y}{J_y}\cos(\Omega_- t + \phi_2)\cos\theta + A_1 \frac{k_x}{J_x}\cos(\Omega_+ t + \phi_1)\sin\theta\right],$$

A_1, B_1, ϕ_1 and ϕ_2 are arbitrary constants, while γ_1, γ_2, θ and Ω_+ and Ω_- are given by eqs. (2.5) and (2.6), respectively.

Now we turn our attention to construct a quadratic invariant, therefore, we shall introduce the following transformation:

(5.5a) $$q_1 = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}(x\cos\theta - y\sin\theta),$$

(5.5b) $$q_2 = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}(y\cos\theta + x\sin\theta),$$

(5.5c) $$p_1 = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/4}(p_x\cos\theta - p_y\sin\theta),$$

(5.5d) $$p_2 = \left(\frac{\gamma_1}{\gamma_2}\right)^{1/4}(p_y\cos\theta + p_x\sin\theta).$$

Then the Hamiltonian (2.2) will take the form

(5.6) $$H \to H' = \frac{1}{2}(J_x^2 P_x^2 + k_x^2 x^2) + \frac{1}{2}(J_y P_y^2 + k_y^2 y^2),$$

where

(5.7) $$[x, P_x] = i\hbar = [y, P_y].$$

Since we assume that the relation (5.7) holds, therefore eq. (5.6) is simply identified as the linear superposition of two independent oscillators

(5.8a) $$H_x = \frac{1}{2}(J_x^2 P_x^2 + k_x^2 x^2),$$

(5.8b) $$H_y = \frac{1}{2}(J_y^2 P_y^2 + k_y^2 y^2).$$

Now, we are in the position to construct two pairs of invariants I_x and I_y satisfying eq. (5.2), thus

(5.9a) $$I_x = \alpha_x P_x^2 + \beta_x x^2 + \gamma_x [x, P_x]_+ ,$$

(5.9*b*) $$I_y = \alpha_y P_y^2 + \beta_y y^2 + \gamma_y [y, P_y]_+ ,$$

where α_j, β_j and γ_j, $j = x, y$ are real functions of time, then by substituting eqs. (5.8) and (5.9) into eq. (5.2) and after some manipulation, we have

(5.10*a*) $$I_x^{(q)} = C_1^{1/2} J_x^{-2} \left[\frac{x^2}{\sigma_+^2} + (\dot{\sigma}_+ x - J_x^2 \sigma_+ P_x)^2 \right],$$

(5.10*b*) $$I_y^{(q)} = C_2^{1/2} J_y^{-2} \left[\frac{y^2}{\sigma_-^2} + (\dot{\sigma}_- y - J_y^2 \sigma_- P_y)^2 \right],$$

(5.10*c*) $$I_x^{(p)} = C_1^{1/2} k_x^{-2} \left[\frac{P_x^2}{\rho_+^2} + (\dot{\rho}_+ P_x + \rho_+ k_x^2 x)^2 \right],$$

(5.10*d*) $$I_y^{(p)} = C_2^{1/2} k_y^{-2} \left[\frac{P_y^2}{\rho_-^2} + (\dot{\rho}_- P_y + \rho_- k_y^2 y)^2 \right],$$

where

(5.11*a*) $$\rho_\pm(t) = \gamma_\pm |\Omega_\pm|^{-1/2} [\cosh \delta_\pm + \mu_\pm \sinh \delta_\pm \sin(2\Omega_\pm t + \phi_\pm)]^{1/2} ,$$

(5.11*b*) $$\sigma_\pm(t) = \tilde{\gamma}_\pm |\Omega_\pm|^{-1/2} [\cosh \tilde{\delta}_\pm + \tilde{\mu}_\pm \sinh \tilde{\delta}_\pm \sin(2\Omega_\pm t + \eta_\pm)]^{1/2} ,$$

$\gamma_\pm$, $\tilde{\gamma}_\pm$, $\delta_\pm$, $\tilde{\delta}_\pm$, $\mu_\pm$, $\tilde{\mu}_\pm$, C_1 and C_2 are independent constants, while $\phi_\pm$ and $\eta_\pm$ are real phases.

By making use of the inverse transformation of eqs. (5.5) and substituting the result into eqs. (5.10), we obtain the quadratic invariants for the Hamiltonian (2.2).

6. – Conclusion.

In the present paper we have considered the problem of two-mode coupled oscillators with two different coupling parameters, which is the result of the quantization of the cavity modes, by taking forth the magnetic permeability and the electric permittivity to vary at the same time. The problem has been considered in a purely quantum-mechanical way, for example we have managed to calculate the wave function in both coherent state and Schrödinger picture, by using the accurate definition of Dirac operators (2.4) where the Hamiltonian (2.2) can be diagonalized. The Green's function has been calculated by employing the coherent-state wave function and then the average value of the energy is obtained.

From the earlier work [20], it is well known that the transition amplitude between two different eigenvalues can be regarded as a general coherent-state wave function, which has wide applications in quantum optics, and, since the energy for the system given by eq. (2.2) will be different if we take the coupling parameters λ_1 and λ_2 to be equal, therefore we extended our discussion to include the transition amplitude in both coherent state and number state. Finally, explicit formulae are obtained for the linear invariant, and a construction for two equivalent families of quadratic invariants for each mode is given. Elsewhere the statistical properties at exact resonance for the Hamiltonian (1.1) are discussed in detail [21].

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