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A Dispersion Relation for Open Spiral Galaxies

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Abstract. The Lin-Shu dispersion relation is applicable in the (asymptotic) case of tight spirals (large wave number k_R). Here we reconsider the various steps leading to the Lin-Shu dispersion relation in higher approximation, under the assumption that the wave number k_R is not large $[(k_R r) = O(1)]$, and derive a new dispersion relation. This is valid for open spiral waves and bars. We prove that this dispersion relation is the appropriate limit of the nonlinear self-consistency condition in the case where the linear theory is applicable.

Key words: galaxies—density waves—dispersion relation

1. Introduction

The basic dispersion relation of Lin and Shu (1964) for spiral density waves was derived on the asymptotic assumption of a large wave number k_R, *i.e.* of a small pitch angle (tight spirals). However, this assumption is not applicable to open spirals, or to bars.

As we will see below in the response calculations the wave number k_R enters through the combination $kr\varepsilon$, where

$$\epsilon = \langle \dot{r}^2 \rangle^{\frac{1}{2}} / \kappa r \tag{1}$$

is the epicyclic small parameter, with $\langle \dot{r}^2 \rangle^{1/2}$ the dispersion of velocities and κ the epicyclic frequency. In the Lin-Shu dispersion relation the quantity $[k_R r \epsilon]$ is about 1 at co-rotation. Thus it is smaller than 1 in the whole 'long wave' branch of the dispersion relation from the inner Lindblad resonance up to co-rotation, as well as in the extension of the 'short wave' branch beyond co-rotation up to the outer Lindblad resonance.

The various steps and assumptions used in deriving the Lin-Shu dispersion relation were analysed in the Maryland Notes on the *Dynamics of Spiral Structure*

(Contopoulos 1972). There, however, only a first order epicyclic theory in ϵ was used.

In order to find a dispersion relation appropriate for relatively small k_R ($k_R r = O(1)$, or smaller), we will need a second order epicyclic theory.

In §2 we derive the response density in the case of a flat (two-dimensional) galaxy. Then, using also Poisson's equation, we derive, in §3, the new dispersion relation. We show that this dispersion relation agrees with the self-consistency conditions derived for nonlinear waves (Contopoulos 1979) in the limit where the linear theory can be applied.

2. The response density

We write the potential of the spiral galaxy V in the form

$$V = V_0 + V_1 \tag{2}$$

where $V_0 = V_0(r)$ is the axisymmetric background and

$$V_1 = V_1^*(r) \exp\,[i(\omega t - 2\theta)], \tag{3}$$

Represents a two-armed spiral mode with eigenvalue $\varpi = 2\Omega_s$, where Ω_s is the angular velocity of the spiral pattern. We write

$$V_1^*(r) = \exp\,[i\Phi(r)]. \tag{4}$$

where Φ is complex; the derivative of Φ is the complex wave number

$$\Phi' = k = k_R + ik_I. \tag{5}$$

(Accents mean derivatives with respect to r). The imaginary part K_I is related to the amplitude A of the potential by the relation

$$k_I = -d\ln A/dr. \tag{6}$$

Thus k_R is the wave number used by Lin and Shu. In our case k_R is of $0(r^{-1})$ and it may even be zero (bar).

The corresponding density response is given by Shu (1968)

$$r\sigma_1^*(r) = \iint d\dot{r}\, dJ_0 \left\{ \frac{\partial f}{\partial E} V_1^*(r) - \frac{\left(2\frac{\partial f_0}{\partial J_0} + \omega\frac{\partial f_0}{\partial E_0}\right)}{2\sin(\omega\tau_0 - 2\theta_0)} \right.$$

$$\left. \times \int_{-\tau_0}^{\tau_0} V_1^*(\tilde{r}) \exp\,[i(\omega\tilde{\tau} - 2\tilde{\theta})]\, d\tilde{\tau} \right\}, \tag{7}$$

where f_0 is the axisymmetric distribution function, given as a function of the energy

$$E_0 = \tfrac{1}{2}(\dot{r}^2 + J_0^2/r^2) + V_0(r), \tag{8}$$

and the angular momentum

$$J_0 = (r_0^3 V_0')^{1/2} = r_0^2 \Omega_0. \tag{9}$$

Here τ_0 is the half-period of the unperturbed orbit, θ_0 the angle between the pericentron and apocentron of an epicyclic orbit going through a given point (r, θ) at time τ_0. The quantity r_0 is defined by equation (9).

A second order epicyclic theory gives (Contopoulos 1975)

$$\tilde{r} = s_0 + s_1 \cos\theta_1 + s_2 \cos 2\theta_1, \tag{10}$$

and $$\dot{\tilde{r}} = -\frac{\pi}{\tau_0}(s_1 \sin\theta_1 + 2s_2 \sin 2\theta_1), \tag{11}$$

where $$\theta_1 = \frac{\pi}{\tau_0}(\tilde{\tau} - \tau_1) = \gamma - \gamma_1, \tag{12}$$

with $$\gamma = \tilde{\tau}\,\frac{\pi}{\tau_0},\ \gamma_1 = \tau_1 \frac{\pi}{\tau_0}, \tag{13}$$

τ_1 being the time of the apocentron passage (where θ_1=0). This orbit goes through $\tilde{r}$=r, with radial velocity $\dot{\tilde{r}} = r$ (defined by equation (8) if E_0 is given) at time $\tilde{\tau} = \tau_0$, *i.e.* for $\theta_1 = \pi - \gamma_1$.

We have further

$$s_2/s_1^2 = \frac{1}{12\kappa_0^2}\left(\frac{d\kappa_0^2}{dr_0} - \frac{3\kappa_0^2}{r_0}\right), \tag{14}$$

where κ_0 is the 'epicyclic frequency'

$$\kappa_0 = \left(\frac{3V_0'}{r_0} + V_0''\right)^{1/2} \tag{15}$$

and $s_0 = r_0 - 3s_2$. (16)

We can see that s_1/r_0 is of $O(\epsilon)$, while $s_2/r_0 = O(\epsilon^2)$.

The value of τ_0 is given by

$$\pi/\tau_0 = \kappa_0(1 + a_0 s_1^2), \tag{17}$$

But the expression of a_0 will not be needed.

Using these relations, and omitting terms of $O(\epsilon^3)$ we find

$$\tilde{r} - r = R_1 + R_2, \tag{18}$$

where $R_1 = (r_0 - r)(1 + \cos\gamma) - \frac{\dot{r}}{\kappa_0}\sin\gamma,$ (19)

and $$R_2 = \frac{s_2}{s_1^2}\left\{-2\left[(r_0 - r)^2 + \frac{2\dot{r}^2}{\kappa_0^2}\right]\cos\gamma - 4(r_0 - r)\frac{\dot{r}}{\kappa_0}\sin\gamma\right.$$
$$\left. + \left[(r_0 - r)^2 - \frac{\dot{r}^2}{\kappa_0^2}\right]\cos 2\gamma - 2(r_0 - r)\frac{\dot{r}}{\kappa_0}\sin 2\gamma - 3\left[(r_0 - r)^2 + \frac{\dot{r}^2}{\kappa_0^2}\right]\right\}. \tag{20}$$

We have also

$$\dot{\tilde{r}} = \dot{R}_1 + \dot{R}_2, \tag{21}$$

where the dot means differentiation with respect to $\tilde{\tau}$. Hence, in this approximation,

$$s_1^2 = (r_0 - r)^2 + \frac{\dot{r}^2}{\kappa_0^2}, \tag{22}$$

therefore $(r_0 - r)/r$ and $\dot{r}/\kappa r$ are of $0(\epsilon)$.

The dispersion of velocities $\langle \dot{r}^2 \rangle^{1/2}$ cannot be smaller than a minimum value $(0{\cdot}2857)^{1/2}\kappa/k_T$, necessary for axisymmetric stability (Toomre 1964), where

$$k_T = \kappa^2/2\pi G\,\sigma_0, \tag{23}$$

And σ_0 is the surface density. Thus we write

$$\langle \dot{r}^2 \rangle^{1/2} = Q(0{\cdot}2857)^{1/2}\,\kappa/k_T, \tag{24}$$

Where $Q \geqslant 1$. In realistic models Q may be of order 2. For $Q = 1$ (marginal stability) the small parameter ε in our galaxy varies between 0·26 at r = 4kpc and 0·10 at r = 20kpc, while for Q = 2 it varies between 0·5 and 0·2.

In equation (7) we need also the expression (derived from the formulae of Appendix A of contopoulos 1975):

$$\omega\tilde{\tau} - 2\tilde{\theta} = (\omega\tau_0 - 2\theta_0)\frac{\gamma}{\pi} + \Lambda_1 + \Lambda_2, \tag{25}$$

where $\theta_0 = \tau_0\,\Omega_0\left(1 + \frac{\kappa_0'}{r_0\kappa_0}\,s_1^2\right),$ (26)

$$\Lambda_1 = \frac{4\Omega_0}{r_0\,\kappa_0}\left[\frac{\dot{r}}{\kappa_0}(1 + \cos\gamma) + (r_0 - r)\sin\gamma\right], \tag{27}$$

and $$\Lambda_2 = \frac{8\Omega_0 s_2}{r_0\kappa_0 s_1^2}\left\{2(r_0-r)\frac{\dot r}{\kappa_0}(1+\cos\gamma)-\left[(r_0-r)^2+\frac{2\dot r^2}{\kappa_0^2}\right]\sin\gamma\right\}$$

$$+\frac{4\bar s_2}{s_1^2}\left\{2(r_0-r)\frac{\dot r}{\kappa_0}\sin^2\gamma-\left[(r_0-r)^2-\frac{\dot r^2}{\kappa_0^2}\right]\sin\gamma\cos\gamma\right\}, \tag{28}$$

where $\bar s_2 = O(\epsilon^2)$. As we will see the terms containing $\bar s_2$ do not contribute in the response integral, thus its value is not given.

We write further

$$\omega\tau_0 - 2\theta_0 = \nu_0\pi = \pi(1-a_0 s_1^2)\left(\bar\nu_0 - \frac{2\Omega_0\kappa_0'}{r_0\kappa_0^2}s_1^2\right), \tag{29}$$

where $$\bar\nu_0 = (\omega - 2\Omega_0)/\kappa_0. \tag{30}$$

In the last integral of equation (7) we have also

$$V_1^*(\tilde r) = V_1^*(r)\exp\left[i\Phi(\tilde r)-i\Phi(r)\right] = V_1^*(r)\exp\left\{i\left[k(R_1+R_2+\ldots)\right.\right.$$

$$\left.\left.+\tfrac{1}{2}k'(R_1+R_2\ldots)^2+\ldots\right]\right\}. \tag{31}$$

Thus equation (7) is written

$$r\,\sigma_1^*(r) = V_1^*(r)\int_{-\infty}^{\infty}d\dot r\int_0^{\infty}dr_0\frac{dJ_0}{dr_0}\left[\frac{\partial f_0}{\partial E_0}-\frac{\left(2\dfrac{\partial f_0}{\partial J_0}+\omega\dfrac{\partial f_0}{\partial E_0}\right)}{2\sin(\nu_0\pi)}\frac{\tau_0}{\pi}\right.$$

$$\left.\times\int_{-\pi}^{\pi}d\gamma\exp\left\{i\left[\nu_0\gamma+\Lambda_1+\Lambda_2+k(R_1+R_2)+\tfrac{1}{2}k'R_1^2+\ldots\right]\right\}\right], \tag{32}$$

where k, k' … are calculated at r.

We introduce not the unperturbed distribution function (Shu 1970)

$$f_0 = \frac{2\Omega_0\sigma_0(r_0)}{\kappa_0 2\pi\langle\dot r^2\rangle_0}\exp\left[-\frac{(E_0-E_{00})}{\langle\dot r^2\rangle_0}\right], \tag{33}$$

where E_0, J_0 are given by equations (8) and (9) and

$$E_{00} = \frac{1}{2}\frac{J_0^2}{r_0^2}+V_0(r_0). \tag{34}$$

Thus $$E_0 - E_{00} = \frac{1}{2}\left[\dot r^2+\kappa_0^2(r-r_0)^2+\frac{4\kappa_0^2 s_2}{s_1^2}(r-r_0)^3+\ldots\right]. \tag{35}$$

After some operations we find

$$2\frac{\partial f_0}{\partial J_0}+\omega\frac{\partial f_0}{\partial E_0}=-\frac{f_0}{\langle\dot r^2\rangle_0}\left(\bar\nu_0\,\kappa_0-T\langle\dot r^2\rangle_0\right), \tag{36}$$

where $$T=\frac{4\Omega_0}{\kappa_0^2 r_0}\left[\frac{d\ln}{dr_0}\left(\frac{\Omega_0\,\sigma_0\,(r_0)}{\kappa_0\,\langle\dot r^2\rangle_0}\right)+\left(\frac{E_0-E_{00}}{\langle\dot r^2\rangle_0}\right)\left(\frac{d\ln\langle\dot r^2\rangle_0}{dr_0}\right)\right]. \tag{37}$$

Further

$$\bar\nu_0\,\kappa_0\,\frac{\tau_0}{\pi}=\nu_0+\frac{2\Omega_0\kappa_0'}{r_0\,\kappa_0^2}\,s_1^2. \tag{38}$$

Using these values in equation (32) together with the expressions

$$dJ_0/dr_0=\kappa_0^2\,r_0/2\Omega_0, \tag{39}$$

and $$\partial f_0/\partial E_0=-f_0/\langle\dot r^2\rangle_0, \tag{40}$$

we find $$r\sigma_1^*(r)=V_1^*(r)\int_{-\infty}^{\infty}d\dot r\int_0^{\infty}dr_0\,\frac{\kappa_0 r_0\sigma_0(\tau_0)}{2\pi\langle\dot r^2\rangle_0^2}\exp\left[-(E_0-E_{00})/\langle\dot r^2\rangle_0\right]$$

$$\times\left[-1+\frac{1}{2\sin(\nu_0\pi)}\left(\nu_0+\frac{2\Omega_0\,\kappa_0'}{r_0\kappa_0^2}\,s_1^2-\frac{T\langle\dot r^2\rangle_0}{\kappa_0}\right)\int_{-\pi}^{\pi}d\gamma\right.$$

$$\left.\times\exp\left\{i[\nu_0\gamma+\Lambda_1+\Lambda_2+k(R_1+R_2)+\tfrac{1}{2}k'R_1^2]\right\}\right]. \tag{41}$$

This is the basic ‘response equation’. From this equation we can easily derive the Lin-Shu formula (1974) under the following assumptions and simplifications:

(i) We assume that $|kr|$ is large, of $0(\epsilon^{-1})$.
(ii) We assume that we are not close to the Lindblad resonances or the particle resonance, *i.e.* $\sin(\nu_0\pi)$ is not of $0(\epsilon^2)$ or smaller.
(iii) We write $\xi=r_0-r$ and integrate ξ from $-\infty$ (instead of $-r$) to ∞.
(iv) We omit all terms except those of the lowest order.

Then we find the well-known formula

$$\sigma_1^*=-\frac{V_1^*\,\sigma_0}{\langle\dot r^2\rangle}\left(1-\frac{\nu\pi}{\sin(\nu\pi)}\,G_\nu(\chi_R)\right) \tag{42}$$

where $$G_\nu(\chi_R)=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\gamma\cos(\nu\gamma)\exp\left[-\chi_R(1+\cos\gamma)\right] \tag{43}$$

and $\quad X_R = k_R^2 \langle \dot{r}^2 \rangle / \kappa^2.$ (44)

The various quantities like ν, κ are calculated at r, while the corresponding quantities ν_0, K_0 are calculated at r_0. [An exception to this rule is the unperturbed surface density σ_0, which means $\sigma_0(r)$, while the surface density at r_0 is written explicity $\sigma_0(r_0)$]. In the next approximation in ε we use also in equation (41) the terms kR_2 and $\frac{1}{2}k'R_1^2$ and the cubic term in $(E_0 - E_{00})$. After several operations we find

$$\sigma_1^* = -\frac{V_1^* \sigma_0}{\langle \dot{r}^2 \rangle} \left\{ 1 - \frac{\nu\pi}{\sin(\nu\pi)} \left[G_\nu(X) - \frac{iX}{kr} \frac{dG_\nu}{dX} \frac{d \ln}{d \ln r} \left(\frac{dG_\nu}{dX} \frac{\sigma_0 k_R r \nu \pi}{\kappa^2 \sin(\nu\pi)} \right) \right] \right\}, \tag{45}$$

where k is now complex. This formula is essentially that of Shu (1970) if we set $k = k_R + ik_I$ and omit terms of $O(k_I^2)$.

In fact, if k is complex, we can write

$$X = k^2 \langle \dot{r}^2 \rangle / \kappa^2 = X_R(1 + i2k_I/k_R),$$

omitting terms of $O(k_I^2)$, and

$$G_\nu(X) = G_\nu(X_R) + \frac{i2k_I r}{k_R r} X_R \frac{d G_\nu}{dX}. \tag{46}$$

But $2k_I r = -d \ln A^2 / d \ln r$, thus the response density (45) becomes

$$\sigma_1^* = -\frac{V_1^* \sigma_0}{\langle \dot{r}^2 \rangle} \left\{ 1 - \frac{\nu\pi}{\sin(\nu\pi)} \left[G_\nu(X_R) - \frac{iX_R}{k_R r} \frac{d G_\nu}{dX} \right.\right.$$
$$\left.\left. \times \frac{d \ln}{d \ln r} \left(\frac{d G_\nu}{dX} \frac{\sigma_0 k_R r \nu \pi A^2}{\kappa^2 \sin(\nu\pi)} \right) \right] \right\}. \tag{47}$$

We can check that this form is exactly that of Shu (1970).

In deriving eq. (45) we notice that the epicyclic terms of $O(\epsilon^2)$, namely the terms containing s_2/s_1^2 do not contribute in the response density. The same can be seen in Shu's (1970) derivation of the dispersion relation.

We come now to the case where $kr = O(1)$ *i.e.* kr is not large. In this case we start again with equation (41), assuming now $\mathbf{\Lambda}_1$ and kR_1 of $O(\epsilon)$ and $\mathbf{\Lambda}_2$, kR_2, and $\frac{1}{2}\, k'R_1^2$ of $O(\epsilon^2)$.

As regards $\sin(\nu_0\pi)$ it can be away from zero (non-resonant case) or close to zero (resonant cases). We consider here the non-resonant case and leave a special resonant case ($\nu_0 \simeq 0$) for the Appendix.

We develop the last exponential of (41) and sort out the terms of various orders. The terms of orders $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ in the second member of (41) give

$$\frac{V_1^* \sigma_0}{\langle \dot{r}^2 \rangle} \left[-1 + \frac{\nu}{2\sin(\nu\pi)} \int_{-\pi}^{\pi} d\gamma \, \exp(i\nu\gamma) \right] \tag{48}$$

and this is equal to zero.

In the next higher order, $O(1)$, we find

$$r\sigma_1^*(r) = V_1^*(r)\int_{-\infty}^{\infty} d\dot{r} \int_0^{\infty} dr_0\, \frac{r_0\kappa_0\sigma_0(r_0)}{2\pi\,\langle\dot{r}^2\rangle_0^{\frac{3}{2}}} \exp\left\{-\frac{1}{2\langle\dot{r}^2\rangle_0}\left[\dot{r}^2+\kappa_0^2(r_0-r)^2\right.\right.$$

$$\left.\left.-\frac{4\kappa_0^2\, s_2}{s_1^2}\,(r_0-r)^3\right]+O_2\right\}\left[-1+\frac{\nu_0}{2\sin(\nu_0\pi)}\left\{1-\frac{T_1\langle\dot{r}^2\rangle_0}{\kappa_0\,\nu_0}\right.\right.$$

$$\left.+\frac{1}{\kappa_0\,\nu_0}\left(\frac{2\,\Omega_0\,\kappa_0'}{r_0\,\kappa_0^3}-T_2\right)[\dot{r}^2+\kappa_0^2\,(r_0-r)^2]\right\}\int_{-\pi}^{\pi} d\gamma\, \exp\,(i\nu_0\,\gamma)$$

$$\left.\times\left\{1+i(\Lambda_1+kR_1)+i(\Lambda_2+kR_2)-\tfrac{1}{2}(\Lambda_1+kR_1)^2+\frac{ik'}{2}R_1^2\right\}\right] \qquad (49)$$

where $$T_1=\frac{4\Omega}{\kappa^2 r}\,\frac{d\ln}{dr}\left(\frac{\Omega\kappa}{\sigma_0}\right),\; T_2=\frac{4\Omega}{\kappa^2 r}\,\frac{d\ln}{dr}\,(\sigma_0/\kappa), \qquad (50)$$

and O_2 is a term containing the factor $(r_0-r)^4/\langle\dot{r}^2\rangle_0$.

If we use the values (19), (20), (27) and (28) for R_1, R_2, Λ_1, Λ_2 and perform the integration with respect to $\dot{r}$, we find

$$r\sigma_1^*(r)=V_1^*(r)\int_0^{\infty} dr_0\, Z_0 \exp\left[-\frac{\kappa_0^2\,(r_0-r)^2}{2\langle\dot{r}^2\rangle_0}+\frac{2\kappa_0^2 s_2\,(r_0-r)^3}{s_1^2\,\langle\dot{r}^2\rangle_0}+O_2\right]$$

$$\times\left[-1+\frac{\nu_0}{2\sin(\nu_0\pi)}\left\{1-\frac{T_1\langle\dot{r}^2\rangle_0}{\kappa_0\nu_0}+\frac{1}{\kappa\nu}\left(\frac{2\Omega_0\kappa_0'}{r_0\kappa_0^3}-T_2\right)[\langle\dot{r}^2\rangle_0\right.\right.$$

$$\left.+\kappa_0^2\,(r_0-r)^2]\right\}\int_{-\pi}^{\pi} d\gamma\,\exp\,(i\nu_0\gamma)\left\{1-\frac{i2\,\bar{s}_1}{s_1}\,(r_0-r)\sin\gamma\right.$$

$$+ik\,(r_0-r)\,(1+\cos\gamma)+\frac{i4\,\bar{s}_1}{s_1^3}\,s_2\,\sin\,\gamma\left[(r_0-r)^2+\frac{2\langle\dot{r}^2\rangle_0}{\kappa_0^2}\right]$$

$$-\frac{i4\,\bar{s}_2}{s_1^2}\left[(r_0-r)^2-\frac{\langle\dot{r}^2\rangle_0}{\kappa_0^2}\right]\sin\gamma\cos\gamma$$

$$-\frac{ik\,2\,s_2}{s_1^2}\left[(r_0-r)^2+\frac{2\,\langle\dot{r}^2\rangle_0}{\kappa_0^2}\right]\cos\gamma+\frac{iks_2}{s_1^2}\left[(r_0-r)^2\right.$$

$$\left.-\frac{\langle\dot{r}^2\rangle_0}{\kappa_0^2}\right]\cos 2\gamma-\frac{ik\,3\,s_2}{s_1^2}\left[(r_0-r)^2+\frac{\langle\dot{r}^2\rangle_0}{2\,\kappa_0^2}\right]-\frac{\langle\dot{r}^2\rangle_0}{2\kappa_0^2}\left[\frac{2\,\bar{s}_1}{s_1}\,(1+\cos\gamma)\right.$$

$$+ k \sin \gamma \Big]^2 - \frac{(r_0 - r)^2}{2} \Big[- \frac{2\,\bar{s}_1}{s_1} \sin \gamma + k(1 + \cos \gamma) \Big]^2$$

$$+ \frac{ik'}{2} \Big[\frac{\langle \dot{r}^2 \rangle_0}{\kappa_0^2} \sin^2 \gamma + (r_0 - r)^2 \, (1 + \cos \gamma)^2 \Big] \Big\} \Big], \tag{51}$$

where $\bar{s}_1 = -2\,\Omega_0 s_1 / \, r_0 \kappa_0,$ (52)

and $Z_0 = \dfrac{r_0 \kappa_0 \sigma_0 (r_0)}{(2\pi \langle \dot{r}^2 \rangle^3)^{1/2}}.$ (53)

We set now $r_0 - r = \xi$ and expand every quantity around $\xi = 0$. Thus

$$\exp \left[- \frac{\kappa^2 (r_0 - r)^2}{2 \langle \dot{r}^2 \rangle} + \frac{2\kappa_0^2 s_2 (r_0 - r)^3}{s_1^2 \langle \dot{r}^2 \rangle_0} + O_2 \right]$$

$$= \exp \left(\frac{\kappa^2 \xi^2}{2 \langle \dot{r}^2 \rangle} \right) \left\{ 1 + \frac{\kappa^2}{\langle \dot{r}^2 \rangle} \left[\frac{2\, s_2}{s_1^2} - \frac{d \ln}{2\,dr} \left(\frac{\kappa^2}{\langle r^2 \rangle} \right) \right] \xi^3 + O_2 \right\}, \tag{54}$$

where the last O_2 contains terms with $\xi^4/\langle \dot{r}^2 \rangle$ and $\xi/\langle \dot{r}^2 \rangle^2$.
Also

$$Z_0 = Z \left(1 + \frac{Z'}{Z} \xi + \frac{Z''}{2Z} \, \xi^2 \right), \tag{55}$$

with $Z = \dfrac{r \kappa \sigma_0 \,(r)}{(2\pi \langle \dot{r}^2 \rangle^3)^{1/2}},$ (56)

$$\frac{\nu_0}{\sin (\nu_0 \pi)} = \frac{\nu}{\sin (\nu \pi)} \left[1 + \frac{d \ln}{dr} \left(\frac{\nu}{\sin (\nu \pi)} \right) \xi + \frac{\sin (\nu \pi)}{2\nu} \frac{d^2}{dr^2} \left(\frac{\nu}{\sin (\nu \pi)} \right) \xi^2 \right] \tag{57}$$

and $\exp \; (i \nu_0 \gamma) = \exp \, (i \nu \gamma) \, (1 + i \gamma \nu' \xi + \dfrac{i \gamma \nu''}{2} \xi^2 - \dfrac{\gamma^2 \nu'^2 \xi^2}{2}.$ (58)

Further $\bar{s}_1/s_1$, calculated previously at r_0, is replaced by

$$\bar{s}_1/s_1 + \frac{d}{dr} (\bar{s}_1/s_1) \, \xi$$

where now $\bar{s}_1/s_1$ is calculated at $r(\xi = 0)$.

If we integrate now over ξ from $-\infty$ to $+\infty$ (replacing the lower limit by $-\infty$ as done by Lin and Shu) we see that the terms of $O(\epsilon^{-2})$ are zero. The terms of $O(\epsilon^{-1})$ are odd in ξ, thus they are also zero. After some operations we find

$$\sigma_1^* = \frac{V_1^* \, \sigma_0 \, \nu}{r \kappa^2 \sin (\nu \pi)} \int_{-\pi}^{\pi} d\gamma \exp \, (i \nu \gamma) \left\{ \left[\frac{d \ln}{dr} \left(\frac{\nu}{\sin (\nu \pi)} \right) + i \gamma \nu' - \frac{i 2 \bar{s}_1}{s_1} \sin \gamma \right. \right.$$

$$+ ik(1+\cos\gamma)\Big]\left[\frac{Z'}{Z} + \frac{6s_2}{s_1^2} - \frac{3}{2}\frac{d\ln}{dr}\left(\frac{\kappa^2}{\langle \dot{r}^2\rangle}\right)\right] + \frac{d\ln}{dr}\left(\frac{\nu}{\sin(\nu\pi)}\right)$$

$$\times\left[i\gamma\nu' - \frac{i2\bar{s}_1}{s_1}\sin\gamma + ik(1+\cos\gamma)\right] + i\gamma\nu'\left[-i\frac{2\bar{s}_1}{s_1}\sin\gamma + ik(1+\cos\gamma)\right]$$

$$+ \frac{\sin(\nu\pi)}{2\nu}\frac{d^2}{dr^2}\left(\frac{\nu}{\sin(\nu\pi)}\right) + \frac{1}{\nu}\left[\frac{4\,\Omega\,\kappa'}{r\kappa^2} - \kappa\,(T_1 + 2T_2)\right] + \frac{i\gamma\nu''}{2} - \frac{\gamma^2\,\nu'^2}{2}$$

$$- i\,\frac{2d}{dr}\left(\frac{\bar{s}_1}{s_1}\right)\sin\gamma + \frac{i12\bar{s}_1 s_2}{s_1^3}\sin\gamma - \frac{ik6s_2}{s_1^2}(1+\cos\gamma) - \left[\left(\frac{2\bar{s}_1}{s_1}\right)^2 + k^2\right](1+\cos\gamma)$$

$$+ ik'(1+\cos\gamma)\Big\}\,. \tag{59}$$

Using now the values of Z, s_2/s_1^2, $\bar{s}_1/s_1$ and T_1, given by (56), (14), (52) and (50), we finally find

$$\sigma_1^* = \frac{V_1^*}{2\pi Gr}\left\{\frac{d}{dr}\left[\frac{r}{k_T(1-\nu^2)}\left(i\,k - \frac{4\,\Omega\,\nu}{\kappa r}\right)\right] - \frac{r}{k_T(1-\nu^2)}\left[k^2 + \left(\frac{4\Omega}{\kappa r}\right)^2\right]\right.$$

$$\left. + \frac{4\Omega}{k_T\,\kappa\nu}\,\frac{d}{dr}\left(\frac{\Omega}{k_T}\right)\right\}. \tag{60}$$

We notice that this response is by two orders in ε σ smaller than the response of Lin and Shu (42). However the present formula is valid for $k_R\, r$ not large, namely for $k_R\, r = O(1)$.

3. Dispersion relation

We find the dispersion relation if we set the response density given by equation (60), equal to the imposed density derived by solving Poisson's equation.

Kalnajs (1971) solved Poisson's equation in the case of a flat galaxy, by finding a relation between the Mellin transforms of the 'reduced potential' $r^{1/2}\, V_1$ and the 'reduced surface density' $r^{1/2}\,\sigma_1$. In the case of a potential of the form

$$V_1 = \exp\left[(i\alpha - \tfrac{1}{2})\ln r + i(\omega t - 2\theta)\right], \tag{61}$$

the corresponding density is

$$\sigma_1 = -\frac{1}{2\pi GK(\alpha, 2)}\exp\left[\left(i\alpha - \frac{3}{2}\right)\ln r + i(\omega t - 2\theta)\right], \tag{62}$$

where, approximately

$$K(\alpha, r) \simeq (\alpha^2 + 4)^{-\frac{1}{2}}. \tag{63}$$

The approximation is very good for $|\alpha|$ large, while the greatest error occurs for $\alpha = 0$ and is less than 3 per cent.

As the form of most spiral galaxies is close to a logarithmic spiral we can use a formula of the form (61) and derive

$$kr = \alpha + \frac{i}{2}. \tag{64}$$

In Kalnajs' calculations α is considered real.

However one can extend these formulae to complex α provided that the imaginary part of α is small (Contopoulos 1980).

We write the solution of Poisson's equation in the form

$$\sigma_1^* = -\frac{V_1^*}{2\pi G r}\left[\left(kr - \frac{i}{2}\right)^2 + 4\right]^{1/2} \tag{65}$$

with k complex.

Using (65), (60) and (23) we find

$$\left[\left(kr - \frac{i}{2}\right)^2 + 4\right]^{1/2} + \frac{d}{dr}\left[\frac{r}{k_T(1-\nu^2)}\left(ik - \frac{4\Omega\nu}{\kappa r}\right)\right] - \frac{r}{k_T(1-\nu^2)}\left[k^2 + \left(\frac{4\Omega}{\kappa r}\right)^2\right] - \frac{4\Omega}{k_T\kappa\nu}\frac{d\ln}{dr}\left(\frac{\Omega}{k_T}\right) = 0. \tag{66}$$

This is the required dispersion relation. It is valid away from the main resonances of the galaxy (where $\nu^2 = 1$ or $\nu = 0$), provided $k_R\, r$ is of $0(1)$, and $(k_T\, r - ½)^2 < (k_R\, r)^2 + 4$.

If we disregard the quantity $i/2$ in the first term of (66) the dispersion relation becomes

$$\operatorname{sgn}(k_R)\, kr\left(1 + \frac{4}{k^2r^2}\right)^{1/2} + \frac{d}{dr}\left[\frac{r}{k_T(1-\nu^2)}\left(ik - \frac{4\Omega\nu}{\kappa r}\right)\right] - \frac{r}{k_T(1-\nu^2)}\left[k^2 + \left(\frac{4\Omega}{\kappa r}\right)^2\right] - \frac{4\Omega}{k_T\kappa\nu}\frac{d\ln}{dr}\left(\frac{\Omega}{k_T}\right) = 0. \tag{67}$$

In this form the dispersion relation was given without proof by Contopoulos (1973)

A resonant form of the dispersion relation valid near the particle resonance is given in the Appendix.

It is of interest now to compare our dispersion relation (66) or (67) with the classical Lin-Shu dispersion relation

$$\frac{|k|}{k_T\chi}\left[1 - \frac{\nu\pi}{\sin(\nu\pi)}\, G_\nu(X)\right] = 1, \tag{68}$$

which is derived by solving Poisson's equation for large real k, in which case (65) gives $\sigma_1^* = -V_1^* |k|/2\pi G$, and inserting this value in (42).

We notice that

$$\chi = (kr)^2 \epsilon^2. \tag{69}$$

In our case, where $kr = O(1)$, we have $\chi = O(\epsilon^2)$. Thus we can expand the exponential in (43) and write $\exp[-\chi(1 + \cos\gamma)] = 1 - \chi - \chi\cos\gamma$.
Then we find, up to terms of $O(\epsilon^2)$,

$$1 - \frac{\nu\pi}{\sin(\nu\pi)} G_\nu(X) = \frac{\chi}{1-\nu^2}, \tag{70}$$

and the dispersion relation becomes

$$|k| = k_T(1-\nu^2). \tag{71}$$

This dispersion relation contains only a few of the terms of our dispersion relation (66). On the other hand if in (66) we consider kr real and large the most important terms give again the relation (71).

We conclude that the Lin-Shu dispersion relation for kr not large tends to our dispersion relation (66) for large kr. This establishes a continuity between the two formulae but indicates clearly that the dispersion relation (66) is the correct one if kr is of $O(1)$, or smaller, and not large of $O(\epsilon^{-1})$.

We will prove now that the dispersion relation (66) can be derived as the limiting case of the nonlinear self-consistency equations derived by Contopoulos (1979).

If we are not very near resonance the self-consistency equations (80) and (81) of Contopoulos (1979) are written, in the present symbolism,

$$\frac{\sigma_0 \sin q_+ x}{k_R A}\left\{\left(-k_I + \frac{4\Omega}{\kappa r}\right)\left[\frac{1}{r}\left(\frac{2\Omega}{\Omega-\Omega_s} - 1\right) - \frac{\sigma_0'}{\sigma_0} - \frac{x'}{x}\right] + (k_R + q_+')k_R\right\} = \frac{1}{2\pi Gr}\mathrm{Re}\left[\left(kr - \frac{i}{2}\right)^2 + 4\right]^{1/2}, \tag{72}$$

and

$$\frac{\sigma_0 \sin q_+ x}{k_R A}\left\{k_R\left[\frac{1}{r}\left(\frac{2\Omega}{\Omega-\Omega_s} - 1\right) - \frac{\sigma_0'}{\sigma_0} - \frac{x'}{x}\right] - \left(k_R + q_+\right) \times \left(-k_I + \frac{4\Omega}{kr}\right)\right\} = \frac{1}{2\pi Gr}\mathrm{Im}\left[\left(kr - \frac{i}{2}\right)^2 + 4\right]^{\frac{1}{2}}, \tag{73}$$

where x is the deviation of a periodic orbit from a circle, and

$$\cot q_+ = \frac{1}{k_R}\left(-k_I + \frac{4\Omega}{\kappa r}\right). \tag{74}$$

Combining (72) and (73) in one complex equation we find

$$\left[\left(kr-\frac{i}{2}\right)^2+4\right]^{1/2}=\frac{\kappa^2 r\sin q_+ x}{k_T k_R A}\left\{\left(ik+\frac{4\Omega}{\kappa r}\right)\left[\frac{2\Omega}{r(\Omega-\Omega_s)}\right.\right.$$

$$\left.\left.-\frac{1}{r}-\frac{\sigma_0'}{\sigma_0}-\frac{x'}{x}+\left(k_R+q_+'\right)\left(k-\frac{i4\Omega}{kr}\right)\right\}. \quad (75)$$

In the linear approximation the deviation x is (Contopoulos 1979)

$$x=\frac{Ak}{2\kappa^2(1+\nu)\sin q_+}. \quad (76)$$

If we insert this value of x in (75) we find, after some operations

$$\left[\left(kr-\frac{i}{2}\right)^2+4\right]^{\frac{1}{2}}=\frac{r}{2k_T(1+\nu)}\left\{\left(ik+\frac{4\Omega}{\kappa r}\right)\left[\frac{2\Omega}{r(\Omega-\Omega_s)}-\frac{1}{r}-\frac{\sigma_0'}{\sigma_0}\right.\right.$$

$$\left.\left.+\frac{2\kappa'}{\kappa}+\frac{\nu'}{1+\nu}-ik\right]-\left(ik+\frac{4\Omega}{\kappa r}\right)'\right\}. \quad (77)$$

In the general neighbourhood of the inner Lindblad resonance (for which our theory was mainly developed) we have

$$1+\nu=\eta \text{ (small)}. \quad (78)$$

If we keep only terms of $0(\eta^{-2})$ and $0(\eta^{-1})$ in the second member of (77), we can write $\Omega-\Omega_s=-\tfrac{1}{2}\kappa\nu$, hence

$$\left[\left(kr-\frac{i}{2}\right)^2+4\right]^{1/2}=\frac{r}{2k_T\,\eta}\left\{\left(ik+\frac{4\Omega}{kr}\right)\left(-\frac{1}{r}-\frac{\sigma_0'}{\sigma_0}+\frac{2\kappa'}{\kappa}+\frac{\nu'}{\eta}\right)\right.$$

$$\left.+k^2+\left(\frac{4\Omega}{kr}\right)^2-\left(ik+\frac{4\Omega}{\kappa r}\right)'\right\}. \quad (79)$$

Exactly the same formula is derived from the dispersion relation (66) in the same approximation, *i.e.* including only terms of $0(\eta^{-2})$ and $0(\eta^{-1})$. This provides a verification both of the dispersion relation (66) and of the self-consistency conditions of our previous paper.

This coincidence provides also the necessary continuity between the non-linear theory and the linear theory. Very close to the resonance the value of x given by (76) increases very much and tends to infinity as $\nu\to 1$. This is not possible, and the nonlinear theory provides the correct value of x there. However, further away from resonance the linear theory is sufficient and one can use the linearised self-consistency conditions or the dispersion relation (66). We must remember here that the nonlinear theory of Contopoulos (1979) was developed for kr not large,

while the corresponding theory for large *kr* (tight spirals) is much more complicated (Mertzanides 1976).
We finish this section by writing the dispersion relation in the case of a bar ($k_R = 0$):

$$\left[4-\left(k_I r-\frac{1}{2}\right)^2\right]^{\frac{1}{2}}-\frac{d}{dr}\left[\frac{r}{k_T(1-\nu^2)}\left(k_T+\frac{4\,\Omega\,\nu}{\kappa\,r}\right)\right]$$

$$-\frac{r}{k_T(1-\nu^2)}\left[-k_I^2+\left(\frac{4\,\Omega}{kr}\right)^2\right]-\frac{4\,\Omega}{k_T\,\kappa\nu}\frac{d\ln}{dr}\left(\frac{\Omega}{k_T}\right)=0. \qquad (80)$$

The first term represents approximately the function $[K(\alpha, 2)]^{-1}$ of Kalnajs (1971) and is valid if

$$\left|k_I\,r-\frac{i}{2}\right|<2.$$

A better representation of $[K(\alpha, 2)]^{-1}$ is the expansion

$$2\left[1-\frac{(k_I\,r-\frac{1}{2})^2}{8}-\frac{(k_I\,r-\frac{1}{2})^4}{128}\right]$$

Which is approximately valid for $|k\,r - ½| \lesssim 2{\cdot}2$.

4. Conclusions

We summarise here the main conclusions of the present paper.

(i) We found a dispersion relation which is valid for relatively open spirals and bars.

However, it is expected that this dispersion relation is approximately valid also for smaller pitch angles, of the order of 20°. The evidence is derived from the fact that the linear theoretical formula (76) above, gives good numerical results for pitch angles even less than 20° (Contopoulos 1979, Appendix C).

(ii) Our dispersion relation has a common limit with the Lin-Shu (1964) dispersion relation. Namely the Lin-Shu dispersion relation for *kr* not large coincides with our dispersion relation for large *kr*.

(iii) Our dispersion relation is the limit of the nonlinear self-consistency conditions (Contopoulos 1979) when the linear approximation is valid, namely not very near resonances.

(iv) The dispersion relation is in complex form, therefore it contains two equations that have to be satisfied simultaneously. These equations correspond to the self-consistency conditions, requiring that the phase and amplitude of the response density should coincide with the phase and amplitude of the imposed density.

In the case of a bar we have only one equation, that refers to the amplitude of the bar.

(v) The complex second order dispersion relation of Shu (1970) depends only on the first order epicyclic orbits and not on the second order epicyclic terms.

On the other hand in our dispersion relation we have also contributions from the epicyclic orbits of $O(\epsilon^2)$.

(vi) In the Appendix we give a dispersion relation valid near the particle resonance.

Numerical applications of the new dispersion relations will be given in another paper.

Appendix

We will apply now the basic response equation (41) to the neighbourhood of the particle resonance. In this case the result (60) is not valid because it contains ν in the denominator which tends to zero as r tends to the particle resonance r_*. The Lin-Shu formula (42) gives the impression that no singularity occurs at $\nu=0$ because $\nu\pi/\sin(\nu\pi) \rightarrow 1$ as $\nu \rightarrow 0$. However, the singularity appears in the amplitude A if we solve Shu's (1970) complex dispersion relation.

A more careful examination of the basic response equation (41) shows that if ν_0 is small, of $O(\epsilon^2)$, then we cannot disregard the other terms of $O(\epsilon^2)$, besides ν_0, in the coefficient of the last integral. This coefficient contains the denominator $\sin(\nu_0 \pi)$, therefore it tends to infinity as $\nu_0 \rightarrow 0$.

One way to avoid the singularity is to consider ν_0 complex

$$\nu_0 = \nu_R + i\nu_I, \tag{Al}$$

where $\nu_R = 2(\Omega\ \tau_0 - \theta_0)/\pi$ and

$$\nu_I = \frac{\tau_0}{\pi}\omega_I \simeq \frac{\omega_I}{\kappa} < 0, \tag{A2}$$

i.e. we assume that the eigenvalue ω is complex

$$\omega = 2\Omega_s + i\omega_I, \tag{A3}$$

containing a small negative imaginary part ($\omega_I < 0$), corresponding to a slightly growing wave. As we will see this method is valid also if $\omega_I \rightarrow 0$, *i.e.* if a neutral wave is considered as the limit of a growing wave.

Near the resonance ν_0 can be replaced by

$$\nu_0 \simeq \nu'_*(\xi - \xi_*) + i\nu_I, \tag{A4}$$

where $\xi = r_0 - r,\ \xi_* = r_* - r.$ (A5)

This approach was used in the case of the Lindblad resonances by Mark (1971). If ν_I is small the major contribution to the integral over $\xi = r_0 - r$ comes from the region near $\xi - \xi_*$.

In equation (51) the lowest order terms in the expression for σ^*_1 that become infinite as $\nu_0 \to 0$ are

$$\frac{V_1^*}{r}\int_0^\infty dr_0\, Z_0 \exp\left[-\frac{\kappa_0^2 (r_0-r)^2}{2\langle \dot r^2\rangle_0}\right]\frac{1}{2\sin(\nu_0\pi)}\left\{-\frac{T_1\langle \dot r^2\rangle_0}{\kappa_0}+\frac{1}{\kappa_0}\left(\frac{2\Omega_0\kappa_0'}{r_0\kappa_0^3}-T_2\right)\right.$$

$$\left.\times\left[\langle \dot r^2\rangle_0+\kappa_0^2(r_0-r)^2\right]\right\}\int_{-\pi}^{\pi} d\gamma \exp(i\nu_0\gamma). \tag{A6}$$

These terms are of $0(\epsilon^2)$. Introducing the above expression (A4) and omitting higher order terms we find

$$\frac{V_1^* Z}{\kappa r\, \nu_*'}\int_{-\infty}^{\infty}\frac{d\xi}{(\xi-\xi_p)}\exp\left(-\frac{\kappa^2\xi^2}{2\langle \dot r^2\rangle}\right)\left[\left(-T_1+\frac{2\Omega\kappa'}{r\kappa^3}-T_2\right)\langle \dot r^2\rangle\right.$$

$$\left.+\kappa^2\left(\frac{2\Omega\kappa'}{r\kappa^3}-T_2\right)\xi^2\right], \tag{A7}$$

where $\xi_p = \xi_* - i\nu_I/\nu_*'$. (A8)

The integral extends from $-\infty$ (instead of $-r$) to $+\infty$, as in the case of Lin and Shu.

Using the w function (Abramowitz and Stegun 1965)

$$w(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\exp(-t^2)\,dt}{z-t}, \tag{A9}$$

we find $\frac{V_1^* Z}{\kappa r \nu_*'}\left\{ i\pi w(u_p)\left[\left(-T_1+\frac{2\Omega\kappa'}{r\kappa^3}-T_2\right)\langle \dot r^2\rangle+\left(\frac{2\Omega\kappa'}{r\kappa^3}-T_2\right)\kappa^2\xi_p^2\right]\right.$

$$\left.+\kappa\xi_p(2\pi\langle \dot r^2\rangle)^{\frac{1}{2}}\left(\frac{2\Omega\kappa'}{r\kappa^3}-T_2\right)\right\} \tag{A10}$$

where $u_p = \kappa\xi_p/(2\langle \dot r^2\rangle)^{\frac{1}{2}}$. (A11)

Assuming that $\nu_I \to 0$ we have for a neutral wave, considered as a limit of growing waves, $u_p = u_*$, where

$$u_* = \kappa\xi_*/(2\langle \dot r^2\rangle)^{\frac{1}{2}}. \tag{A12}$$

Thus the last term of the response density (60) must be replaced by

$$\frac{V_1^*\sigma_0 2\Omega}{\kappa^2 r\nu_*'}\left\{\frac{i\pi w(u_*)}{(2\pi\langle \dot r^2\rangle)^{1/2}}\left[\frac{d\ln}{dr}\left(\frac{\kappa}{\Omega^2}\right)+\frac{\kappa^2\xi_*^2}{\langle r^2\rangle}\frac{d\ln}{dr}\left(\frac{k_T^2}{\kappa}\right)\right]+\left[\frac{\kappa\xi_*}{\langle \dot r^2\rangle}\frac{d\ln}{dr}\left(\frac{k_T}{\kappa}\right)\right]\right\}, \tag{A13}$$

and the final form of the dispersion relation near the particle resonance is†

$$\left[\left(kr-\frac{i}{2}\right)^2+4\right]^{1/2}+\frac{d}{dr}\left[\frac{r}{k_T(1-\nu^2)}\left(ik-\frac{4\Omega\nu}{\kappa r}\right)\right]-\frac{r}{k_T(1-\nu^2)}$$
$$\times\left[k^2+\left(\frac{4\Omega}{\kappa r}\right)^2\right]+\frac{2\Omega}{k_T\nu'_*}\left\{\frac{i\pi w(u_*)}{(2\pi\langle\dot{r}^2\rangle)^{1/2}}\left[\frac{d\ln}{dr}\left(\frac{\kappa}{\Omega^2}\right)+\frac{\kappa^2\,\xi_*^2}{\langle\dot{r}^2\rangle}\frac{d\ln}{dr}\left(\frac{k_T^2}{\kappa}\right)\right]\right.$$
$$\left.+\frac{\kappa\xi_*}{\langle\dot{r}^2\rangle}\frac{d\ln}{dr}\left(\frac{k_T^2}{\kappa}\right)\right\}. \qquad \text{(A14)}$$

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†This dispersion relation was given without proof by Contopoulos (1973) but with an erratum: A factor κ should be put in front of $w\,(u_*)$ there.