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Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform. – I

F. CALOGERO

Istituto di Fisica dell'Università - 00185 Roma, Italia
Istituto Nazionale di Fisica Nucleare - Sezione di Roma

A. DEGASPERIS

Istituto di Fisica dell'Università - 00185 Roma, Italia
Istituto di Fisica dell'Università - Lecce
Istituto Nazionale di Fisica Nucleare - Sezione di Roma

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Summary. — This paper is the first of a series based on a general method to discover and investigate nonlinear partial differential equations solvable via the inverse spectral transform technique. The results of this paper are those that obtain applying this method to the generalized Zakharov-Shabat linear problem. We give a class of nonlinear evolution equations solvable by the inverse spectral transform, that is more general than that introduced by Ablowitz, Kaup, Newell and Segur because it includes equations involving more than one space variable and containing coefficients that are not constant. We also report a very general class of Bäcklund transformations that includes all such transformations previously considered and clarifies their significance. And we produce, for a somewhat less general class of nonlinear evolution equations (involving only one space variable), a remarkable functional equation that relates the solution at time t to the same solution at time t'. This paper is focussed on a general presentation of the approach and the proof of the main results (some of which had been previously reported without proof). Although the analysis of special equations and special solutions is deferred to subsequent papers of this series, there are here also a few results of this kind, including the explicit display of the exact nonsoliton solution of the sine-Gordon equation corresponding to a double pole of the associated spectral parameter.

1. – Introduction.

Most physical problems are represented mathematically by partial differential equations. In many cases these equations are linear, or at least a linearized version is adequate to account for the main features of the physical process. In many other cases the physical phenomenon is described by nonlinear partial differential equations (NLPDE's), with the nonlinearity playing a nonnegligible rôle ([1]).

The theory of linear partial differential equations has traditionally constituted the backbone of mathematical physics; this has mainly occurred because no theoretical approach to NLPDE's existed whose generality and power were comparable to the methods available in the linear case (such as, for instance, the Fourier-transform technique to solve linear partial differential equations with constant coefficients, an instance arising in innumerable physical applications, as reflected in the importance of the Fourier transform in physics and, more generally, in applied mathematics).

A few years ago a technique to solve a NLPDE has been invented ([2]). The subsequent demonstration of the applicability of this technique (suitably generalized) to large classes of NLPDE's ([3]) constitutes a major development in mathematical physics, or, more generally, in applied mathematics. This is underscored by the recognition ([3g]) that the new technique may be viewed as an extension of the Fourier-transform method, to which it does indeed reduce in the linear (or linearized) case. Moreover, the remarkable properties of those

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([1]) Throughout this paper whenever we mention NLPDE's we include also the possibility that these be integro-differential equations (that may, or may not, reduce to pure partial differential equations, possibly by an appropriate redefinition of the dependent variable).

([2]) C. S. GARDNER, J. M. GREENE, M. D. KRUSKAL and R. M. MIURA: *Phys. Rev. Lett.*, **19**, 1095 (1967); *Comm. Pure Appl. Math.*, **27**, 97 (1974).

([3]) Out of the extensive literature on this topic we list here only the most significant contributions, selected on the basis of their review nature, their landmark character or their technical closeness to the approach of this paper: *a*) A. C. SCOTT, F. Y. F. CHU and D. W. McLAUGHLIN: *Proc. IEEE*, **61**, 1443 (1973); *b*) G. B. WHITHAM: *Linear and Nonlinear Waves* (New York, N.Y., 1974); *c*) J. MOSER, Editor: *Dynamical Systems, Theory and Applications* (Berlin, 1974) (see in particular the papers by M. KRUSKAL and by H. FLASCHKA and A. C. NEWELL); *d*) P. D. LAX: *Comm. Pure Appl. Math.*, **21**, 467 (1968); *e*) V. E. ZAKHAROV and L. D. FADDEEV: *Func. Anal. Appl.*, **5**, 280 (1971); *f*) V. E. ZAKHAROV and A. B. SHABAT: *Sov. Phys. JETP*, **34**, 62 (1972); *g*) M. J. ABLOWITZ, D. J. KAUP, A. C. NEWELL and H. SEGUR: *Stud. Appl. Math.*, **53**, 249 (1974), hereafter referred to as AKNS; *h*) F. CALOGERO: *Lett. Nuovo Cimento*, **14**, 443 (1965); *i*) T. KOTERA and K. SAWADA: *Journ. Phys. Soc. Japan*, **39**, 501 (1975). Presumably another useful reference, that we have however not yet been able to consult, is *Nonlinear Wave Motion*, edited by A. C. NEWELL, *Lectures in Applied Math.*, **15** (Providence, R. I., 1974).

NLPDE's to which the novel technique is applicable open perspectives that are highly interesting also from a purely mathematical point of view (4).

It is convenient in this discussion to focus upon *evolution equations*, *i.e.* NLPDE's describing the evolution in time of a field q (that may depend on several « space » variables x, y, z ..., besides the time t, and that might also be a multicomponent quantity, *i.e.* a vector or a matrix; see below). The main idea of the development mentioned above consists in the association, with every « solvable » NLPDE, of a linear operator of (generalized) Sturm-Liouville type, whose spectral parameters (defined more precisely below) evolve simply in time while q evolves (generally in a quite complicated way) according to the NLPDE. The field q at time t can then be evaluated from its values at time t_0 by first determining, at time t_0, the spectral parameters of the associated operator, letting them evolve to the time t, and finally recovering the field q at time t from the corresponding spectral parameters. The first and third steps of this procedure correspond to the « direct » and « inverse » spectral problems for the linear operator associated with the original NLPDE.

When this method was first introduced (2), the NLPDE was the celebrated KdV equation (5), and the associated linear problem was the one-dimensional scattering and bound-state Schrödinger problem, with q playing the rôle of the potential. In this case the spectral parameters are the scattering and bound-state data (reflection coefficient, bound-state energies and normalization constants—see below), whose determination from the potential corresponds to the solution of the direct Schrödinger scattering and bound-state problem, and that in their turn determine the potential via the solution of the inverse scattering problem (6). From this last step the procedure has been named « inverse scattering method »; in their landmark contribution AKNS (3g) emphasized the relationship of this technique to the Fourier-transform method for solving linear partial differential equations, and introduced therefore the name « inverse scattering transform ». We prefer to use here a name—inverse spectral transform (IST)—that reflects more accurately the nature and generality of the method (and moreover preserves the acronym, IST, introduced by AKNS).

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(4) We list again only a few contributions, particularly significant in the context of this paper: H. D. WAHLQUIST and F. B. ESTABROOK: *a*) *Phys. Rev. Lett.*, **31**, 1386 (1973); *b*) *Journ. Math. Phys.*, **16**, 1 (1975); *c*) D. W. MCLAUGHLIN and A. C. SCOTT: *Journ. Math. Phys.*, **14**, 1817 (1973); *d*) G. L. LAMB jr.: *Journ. Math. Phys.*, **15**, 2157 (1974); *e*) F. CALOGERO: *Lett. Nuovo Cimento*, **14**, 537 (1975); see also the papers of ref. (3).

(5) D. J. KORTEWEG and G. DEVRIES: *Phil. Mag.*, **39**, 422 (1895).

(6) I. M. GEL'FAND and B. M. LEVITAN: *Amer. Math. Soc. Transl.*, **1**, 253 (1955); Z. S. AGRANOVICH and V. A. MARCHENKO: *The Inverse Problem of Scattering Theory* (translated from the Russian by B. D. SECKLER) (New York, N. Y., 1963); I. KAY and H. E. MOSES: *Nuovo Cimento*, **2**, 917 (1955); **3**, 66, 276 (1956); *Journ. Appl. Phys.*, **27**, 1503 (1956); I. KAY: *Comm. Pure Appl. Math.*, **13**, 371 (1960).

The applicability of the IST to solve a NLPDE depends upon the discovery of an associated linear problem that allows the three steps described above to be performed. The main development of the last few years has therefore focussed on this issue; generally the starting point of the analysis is the linear problem, and the NLPDE (or rather the class of NLPDE's) associated with it are then uncovered. Three parallel, and occasionally osculating, techniques have mainly emerged.

The first originates from the remark by LAX ([3d]) that, if the time-dependent linear operator L satisfies the operator equation

$$L_t = [L, M] \tag{1.1}$$

with M some other operator, the time evolution of its spectral parameters is particularly simple. Taking this point of departure, it has been possible to obtain a whole class of NLPDE's that are solvable by the IST ([3d,3c,7]). This approach has moreover been particularly fruitful in the context of the discrete problem, leading to the discovery of a number of exactly integrable many-body systems ([8]), and to a deeper understanding ([9]) of some such models whose solvability had been previously demonstrated by other means ([10]).

The main merit of the Lax approach is its all-encompassing nature, connected with its operator-theoretic standpoint. Its main drawback, as a tool to enlarge the class of solvable NLPDE's, is its reliance on a starting point, eq. (1), that is not very suited to a systematic approach.

The second, and related, technique is due to the Clarkson school ([11]), and its more complete exposition is in the AKNS paper ([3g]). Its starting point is an appropriate Sturm-Liouville problem, such as the Zakharov-Shabat system ([3f]) (rather, a generalized version of it; see below) or the one-dimensional Schrödinger equation, whose direct and inverse spectral problems are well in hand. A large class of NLPDE's, solvable by the IST associated with such problems, is then generated by a systematic procedure, related to the Lax formula and based on a convenient ansatz for the time dependence of the wave

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([7]) M. WADATI and T. KAMIJO: *Prog. Theor. Phys.*, **52**, 397 (1974).

([8]) Paper by J. MOSER in ref. ([3c]); J. MOSER: *Adv. Math.*, **16**, 197 (1975); F. CALOGERO, C. MARCHIORO and O. RAGNISCO: *Lett. Nuovo Cimento*, **13**, 383 (1975); F. CALOGERO: *Lett. Nuovo Cimento*, **13**, 411 (1975); M. ADLER: preprint (*A new integrable system and a conjecture by Calogero*, to be published).

([9]) H. FLASCHKA: *Phys. Rev. B*, **9**, 1924 (1974); *Prog. Theor. Phys.*, **51**, 703 (1974) (see also the paper by the same author in ref. ([3c])); S. V. MANAKOV: *Sov. Phys. JETP*, **67**, 543 (1974).

([10]) M. TODA: *Journ. Phys. Soc. Japan*, **23**, 501 (1967); *Phys. Rep.* (1974); M. HENON: *Phys. Rev. B*, **9**, 1921 (1974); F. CALOGERO: *Journ. Math. Phys.*, **12**, 419 (1971).

([11]) M. J. ABLOWITZ, D. J. KAUP, A. C. NEWELL and H. SEGUR: *Phys. Rev. Lett.*, **31**, 125 (1973).

functions of the linear problem. In compact form, these NLPDE's may be written as [12]

(1.2)
$$\begin{pmatrix} r_t(x,t) \\ -q_t(x,t) \end{pmatrix} + 2A(L_-)\begin{pmatrix} r(x,t) \\ q(x,t) \end{pmatrix} = 0$$

with

(1.3)
$$L_- = \frac{1}{2i}\begin{pmatrix} \frac{\partial}{\partial x} - 2rI_-q & 2rI_-r \\ -2qI_-q & -\frac{\partial}{\partial x} + 2qI_-r \end{pmatrix},$$

where we have introduced for short the integral operator

(1.4)
$$I_- = \int_{-\infty}^{x} d\xi \cdot ,$$

or

(1.5)
$$q_t + B(L_s)\,q_x = 0$$

with

(1.6)
$$L_s = -\frac{1}{4}\frac{\partial^2}{\partial x^2} - q + \frac{1}{2}\,q_x\int_x^{+\infty} d\xi \cdot .$$

The first formula, (1.2) corresponds to the generalized Zakharov-Shabat linear problem; it yields a system of coupled nonlinear evolution equations for the two fields $q(x,t)$ and $r(x,t)$, that may reduce to a single equation for a single field in special cases, such as, for instance, that treated by ZAKHAROV and SHABAT ([3f]), characterized by $r = -q^*$. The second formula, eq. (1.5), corresponds to the one-dimensional Schrödinger problem.

The generality of the class of equations yielded by the AKNS approach is demonstrated by the arbitrariness of the functions A and B in eqs. (1.2) and (1.5) (they are only required to be ratios of entire functions); and the direct connection of these functions with the dispersion relation characterizing the linearized version of the NLPDE's (1.2) and (1.5) is of major importance, displaying, as emphasized by AKNS, the analogy of the IST treatment of NLPDE's to the Fourier-transform technique to solve linear partial differential equations with constant coefficients. Moreover, by choosing simple polynomial, or ra-

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([12]) Throughout this paper we occasionally differ, to streamline our presentation, from the notation used previously.

tional, expressions for the functions A and B, AKNS were able to reobtain all the previously known solvable NLPDE's, including the celebrated KdV ([2,5]), nonlinear Schrödinger ([3f]) and sine-Gordon ([13]) equations. A limitation of the AKNS approach, that may however be in the process of being overcome ([14]), is its restriction to NLPDE's involving only one space variable.

A third approach ([3h]) employs generalized Wronskian-type equations, relating the wave functions and the spectral parameters of (generalized) Sturm-Liouville problems, to derive a large class of NLPDE's that can be solved by the IST. This technique is quite straightforward, and since it relies essentially only upon integrations by parts, it might be applicable also in the context of multidimensional Sturm-Liouville problems. In this paper its potentiality in the context of one-dimensional Sturm-Liouville problems is displayed; the linear problem taken as starting point of the analysis is the generalized Zakharov-Shabat problem. The following paper of this series will deal similarly with the multichannel Schrödinger problem.

The results yielded by this technique in the context of the one-dimensional Schrödinger equation have been already published ([3h,4c,15]) as well as some of the results described below and in the following paper of this series (but without proofs) ([16]). An important advantage of this approach is its deliverance of NLPDE's, solvable via the IST, that may involve more than one space variable and contain coefficients that are not constant; note that these results obtain even though the linear problem related by the IST to the « solvable » NLPDE's refers only to one variable. When the approach is strictly limited to problems involving only one space variable, it reproduces essentially the same results as the AKNS method, in the context in which that technique has been used, namely when the linear problems taken as starting points are the (single channel) Schrödinger or the generalized Zakharov-Shabat problems; it is also applicable in more general contexts, such as the multichannel Schrödinger case, in which case it yields novel classes of NLPDE's ([16,17]).

([13]) The literature on the sine-Gordon equation and its applications is large (see, *e.g.*, ref. ([3a,3g])); its complete solution was first given by M. J. ABLOWITZ, D. J. KAUP, A. C. NEWELL and H. SEGUR: *Phys. Rev. Lett.*, **30**, 1262 (1973); and by L. D. FADDEEV and L. A. TAKHTAJAN: Commun. JINR Dubna, E2-7998 (1974). See also D. J. KAUP: *Stud. Appl. Math.*, **54**, 165 (1975).

([14]) M. J. ABLOWITZ and R. HABERMAN: *Phys. Rev. Lett.* (in press).

([15]) F. CALOGERO: *Nuovo Cimento*, **29** B, 509 (1975). See also the paper by the same author in the forthcoming Festschrift in honor of V. BARGMANN, edited by B. SIMON and A. S. WIGHTMAN.

([16]) F. CALOGERO and A. DEGASPERIS: *a*) *Phys. Rev. Lett* (submitted to); *b*) *Lett. Nuovo Cimento*, **15**, 65 (1976).

([17]) Indeed, even in the case with one space variable only, only a more limited class of NLPDE's than the one reported here (and in ref. ([16b])) had been identified as solvable by the IST associated with the multichannel Schrödinger equation ([7]).

The NLPDE's that are shown (below and in the following paper of this series) to be solvable by the IST may be written as

$$(1.7)\qquad \begin{pmatrix} r_t(x, y, t) \\ -q_t(x, y, t) \end{pmatrix} + \gamma(L, y, t)\begin{pmatrix} r(x, y, t) \\ q(x, y, t) \end{pmatrix} + \nu(L, y; t)\frac{\partial}{\partial y}\begin{pmatrix} r(x, y, t) \\ -q(x, y, t) \end{pmatrix} = 0\,,$$

where L is the integro-differential operator

$$(1.8)\qquad L = \frac{1}{2i}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + 2\begin{pmatrix} rIq & -rIr \\ qIq & -qIr \end{pmatrix}\right]$$

with

$$(1.9)\qquad I \equiv \int_x^{+\infty} \mathrm{d}\xi\, \cdot\,,$$

and

$$(1.10)\qquad Q_t(x, y, t) = 2\beta_0(\underline{L}_s, y, t)Q_x(x, y, t) + \alpha_n(\underline{L}_s, y, t)[\sigma_n, Q(x, y, t)] + \\ + \beta_n(\underline{L}_s, y, t)\,\underline{G}\sigma_n + \gamma(\underline{L}_s, y, t)\frac{\partial}{\partial y}Q(x, y, t)\,,$$

where $\underline{L}_s$ and $\underline{G}$ are the integro-differential operators defined by

$$(1.11)\qquad \underline{L}_s F(x) = F_{xx}(x) - 2\{Q(x), F(x)\} + \underline{G}\int_x^{+\infty} \mathrm{d}\xi\, F(\xi)\,,$$

$$(1.12)\qquad \underline{G}F(x) = \{Q_x(x), F(x)\} + \left[Q(x), \int_x^{+\infty} \mathrm{d}\xi\, [Q(\xi), F(\xi)]\right].$$

The NLPDE (1.7) involves the 2 fields r and q; in special cases (see below) it reduces to a NLPDE for a single field. It is solvable by the IST related to the generalized Zakharov-Shabat linear problem; the 2 fields r and q depend generally on the variables x, y and t (with y an M-dimensional vector); solvability means here the possibility to evaluate, employing linear techniques only, $q(x, y, t)$ and $r(x, y, t)$ from given $\bar{q}(x, y) = q(x, y, t_0)$ and $\bar{r}(x, y) = r(x, y, t_0)$. The functions $\gamma(z, y, t)$ and $\nu(z, y, t)$ are only required to be ratios of entire functions of z (with the same singularities for finite z, if any; see below). If the y-dependence in (1.7) disappears, this NLPDE reduces essentially to that treated previously by AKNS, eq. (1.2) ([18]); it should be noted that in

([18]) The difference between the integral operators I_- and I, eqs. (1.4) and (1.9), compensates exactly the sign differences between the definitions of L_- and L, eqs. (1.3) and (1.8); see the appendix.

this case, if $\gamma(z, t)$ is a polynomial in z, the NLPDE (1.7) (or, equivalently, (1.2)) contains no integrals (*i.e.* it is a NLPDE in the strict sense, not an integro-differential equation), in spite of the presence of the integral operator (see below).

The NLPDE (1.10) is an $N\times N$ matrix equation; it is solvable by the IST related to the N-channel Schrödinger problem. The $N\times N$ matrix Q depends generally on the variables x, $\boldsymbol{y}$ and t (again with $\boldsymbol{y}$ an M-dimensional vector); the constant matrices σ_n provide, together with the unit matrix, an orthogonal basis for $N\times N$ matrices (in the 2×2 case, they may be identified with the Pauli matrices), so that the index n runs from 1 to N^2-1 (and is, by convention, summed upon when repeated); $[A, B]=AB-BA$ and $\{A, B\}=AB+BA$; the operators $\underline{L}_s$ and $\underline{G}$ transform $N\times N$ matrices into $N\times N$ matrices according to (1.11) and (1.12), where $F(x)$ stands for a generic $N\times N$ matrix. Solvability means again the possibility to evaluate $Q(x, \boldsymbol{y}, t)$ from a given $\bar{Q}(x, \boldsymbol{y})\equiv Q(x, \boldsymbol{y}, t_0)$ by linear techniques only; special (« soliton ») solutions of (1.10) can be displayed explicitly. The functions $\beta_0(z, \boldsymbol{y}, t)$, $\alpha_n(z, \boldsymbol{y}, t)$, $\beta_n(z, \boldsymbol{y}, t)$ and $\boldsymbol{\gamma}(z, \boldsymbol{y}, t)$ are only required to be ratios of entire functions in z (with the same singularities, if any).

The NLPDE's (1.7) and (1.10) are clearly rather general; particularly interesting are the cases corresponding to the simplest choices (constants, or ratios of polynomials of very low degree) for the arbitrary functions that enter them; this is particularly so in the case of the more novel NLPDE (1.10) ([16,17]). This analysis is, however, postponed to a subsequent paper of this series.

It should also be mentioned that in this paper (and in the following one of this series) we consider only problems in which the unknown fields (r and q in the case of eq. (1.7), Q in the case of eq. (1.10)) are defined, as functions of the x-variable, over the whole real axis and vanish asymptotically (as $x\to\pm\infty$). Adoption of the approach employed here also in the context of problems with different boundary conditions is an appealing possibility that remains to be explored.

The generality of the NLPDE's reported above and the simplicity with which the solvability by the IST technique can be established (see below) witness to the convenience of the approach based on generalized Wronskian relations ([3h,15]). But the power of this technique is not limited to the generation of solvable NLPDE's; indeed, its main merit is rather to provide a convenient tool to investigate the solutions of these NLPDE's, and in particular to obtain explicit equations (Bäcklund transformations) that relate different solutions, and even, in some cases, explicit equations that relate a solution at one time to the *same* solution at another time (a remarkable result that might be viewed as an extension to NLPDE's of the resolvent formula for linear equations). These results have been first given for the class of NLPDE's (KdV and generalizations) that are solvable by the IST associated with the one-channel Schrödinger equation ([4e]); it was thereby possible to reobtain, explain and extend the beautiful results (Bäcklund transformations and nonlinear superposition principle) previously given, for the KdV equation, by WAHLQUIST and ESTABROOK ([4a]), and to discover a remarkable functional equation relating the solutions of the NLPDE's at time t and $t + \Delta t$ with Δt finite ([4e]). Similar results, but in the more general contexts of the NLPDE's written above, were also reported (without proofs) ([16]); they are proved and discussed below and in the following paper of this series. Special cases of these results reproduce all previously known Bäcklund transformations ([19]); the results given here are however much more general than those given heretofore, because they apply directly to large classes of NLPDE's and because they yield large classes of Bäcklund transformations, not just those previously known, providing moreover an illuminating explanation of their origin and significance ([20]). In particular the permutability of Bäcklund transformations is generally demonstrated, and its significance displayed, together with the beautiful results (nonlinear superposition principles) that follow from it in the context of the various NLPDE's ([4]).

Quite novel and most remarkable is the functional equation relating the same solution of one of these NLPDE's at different times. Even in the very simplest cases this equation is far from trivial; in some such case it degenerates into remarkable operator identities, that may be considered nonlinear generalizations of the well-known linear operator formula

$$f(z + a) = \exp\left[a \frac{\mathrm{d}}{\mathrm{d}z}\right] f(z) . \tag{1.13}$$

Before ending this introduction we would like to call attention to the paper by KOTERA and SAWADA ([3i]), whose approach is in some respects similar to that employed here ([21]). We would also like to mention that there exists another general approach, originated by ZAKHAROV and FADDEEV ([3e]), to the problem of solvable NLPDE's, that views them as integrable Hamiltonian systems; although very important from a philosophical point of view (and also in many

([19]) Results for Bäcklund transformations have been obtained and discussed, for some special equations (KdV, modified KdV, nonlinear Schrödinger, sine-Gordon), by AKNS and by many others; see, for instance, M. WADATI, H. SANUKI and K. KONNO: *Prog. Theor. Phys.*, **53**, 419 (1975), the papers of ref. ([4]) and some of the papers of ref. ([3]).

([20]) After this paper was partially drafted (and the two papers of ref. ([16]) had been submitted for publication) we received a preprint by H. FLASCHKA and D. W. McLAUGHLIN (*Some comments on Bäcklund transformations, canonical transformations and the inverse scattering method,* to be published) that takes a point of view similar to that of this paper, and reports some results that coincide with special cases of those given here.

([21]) A preprint by D. J. KAUP: *The closure of the squared Zakharov-Shabat eigenstates* (to appear in *Journ. Math. Anal. Appl.*) also takes a somewhat similar point of view to that of this paper.

applications, such as the quantization problem), this standpoint does not however appear particularly appropriate to enlarge the class of solvable NLPDE's. Its relation to the results of this paper is an interesting point ([20]), that deserves further investigation.

The organization of this paper is clearly indicated by the titles of the following sections and subsections, so that we need not describe it here.

2. – Notation and preliminaries.

2'1. *Basic notation.* – We use generally (but not exclusively) upper-case characters for 2×2 matrices, and lower-case characters for 2-component vectors (or rather spinors); an exception to this convention is the use of the usual Pauli matrices

$$\sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.1}$$

The 2 eigenstates of σ_3 are indicated with the notation

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.1.2}$$

The conventional notation for commutators and anticommutators used throughout is $[A, B] = AB - BA$, $\{A, B\} = AB + BA$. A^T is the transpose of the matrix A, $(u, v) = u_1 v_1 + u_2 v_2$ if $u = u_1 \chi_+ + u_2 \chi_-$, $v = v_1 \chi_+ + v_2 \chi_-$.

2'2. *The direct problem.* – The generalized Zakharov-Shabat problem ([3f,3g]) is characterized by the differential equation

$$\psi_x + ik\sigma_3 \psi = [q_1 \sigma_1 + iq_2 \sigma_2]\, \psi\,. \tag{2.2.1}$$

In this equation the two scalars q_1 and q_2 depend on the real variable x, while the spinor ψ depends on x and k; all these quantities may also depend parametrically on other variables. The subscript x indicates of course partial differentiation, a notation that is used throughout this paper. Note that the problem (2.2.1) could be easily reformulated in the form $H\psi = k\psi$, the linear operator H being however generally not Hermitian (unless q_2 is imaginary and q_1 is real). The connection of (2.2.1) with the usual notation ([3g]) obtains if we set

$$q_1 = \tfrac{1}{2}(q + r)\,, \qquad q_2 = \tfrac{1}{2}(q - r)\,; \tag{2.2.2}$$

in the following we use either the variables (r, q) or (q_1, q_2), whichever allows us to write in more compact form the various formulae, with the understanding that they are always related to each other by (2.2.2). We shall also use occasionally, to make the notation more compact, the spinor

$$v = r\chi_+ + q\chi_- , \tag{2.2.3}$$

and related symbols that will be defined below whenever appropriate. Note that the components of v are r, q, not q_1, q_2.

The two functions q and r are assumed to vanish asymptotically exponentially or faster, *i.e.* we assume that, for some positive ε,

$$\lim_{x\to\pm\infty} \left[\exp[\varepsilon|x|]\, v(x)\right] = 0 . \tag{2.2.4}$$

This is a stronger condition than it is actually needed for the validity of most of the following results; but it is adequate to cover all interesting cases, so we assume its validity for the sake of simplicity. In some cases (that will be specified below) we shall assume even stronger conditions.

The continuum part of the spectrum associated with (2.2.1) in the Zakharov-Shabat problem is characterized by the asymptotic boundary conditions

$$\Psi(x, k) \xrightarrow[x\to+\infty]{} \exp[-ikx\sigma_3] + \tfrac{1}{2}\alpha^{(-)}(k)(\sigma_1 + i\sigma_2)\exp[-ikx] + \tfrac{1}{2}\alpha^{(+)}(k)(\sigma_1 - i\sigma_2)\exp[ikx] , \tag{2.2.5a}$$

$$\Psi(x, k) \xrightarrow[x\to-\infty]{} \tfrac{1}{2}\beta^{(+)}(k)(1 + \sigma_3)\exp[-ikx] + \tfrac{1}{2}\beta^{(-)}(k)(1 - \sigma_3)\exp[ikx] . \tag{2.2.5b}$$

Note that, for the sake of notational compactness, we consider here (and below) a matrix solution of (2.2.1), that is of course built out of two spinor solutions (used as columns of the matrix). The functions $\alpha^{(\pm)}$ and $\beta^{(\pm)}$ depend of course parametrically on other variables, if q and r (and therefore also Ψ) do.

It should be emphasized that, although the differential equations (2.2.1) could be transformed into those characterizing a two-channel Schrödinger problem, the boundary conditions (2.2.5) of the Zakharov-Shabat problem differ from those of the corresponding Schrödinger problem [22].

The discrete part of the spectrum consists of a finite number of eigenvalues $k_n^{(\pm)}$, whose corresponding eigenfunctions may be normalized as follows:

$$\int_{-\infty}^{+\infty} dx\, \left(\psi^{(\pm)}(x), \sigma_1 \psi^{(\pm)}(x)\right) = 1 . \tag{2.2.6}$$

[22] This point is ignored in ref. [7], where the interested reader may find the explicit connection between the differential equations of the two problems.

It is easily seen that a necessary condition for this to happen is that these functions be characterized by either one of the two asymptotic behaviours described by the formula

(2.2.7a) $$\psi^{(\pm)}(x) \xrightarrow[x\to+\infty]{} \gamma^{(\pm)} \exp[\pm ik^{(\pm)}x]\,\chi_{\mp}\,,$$

(2.2.7b) $$\psi^{(\pm)}(x) \xrightarrow[x\to-\infty]{} \delta^{(\pm)} \exp[\mp ik^{(\pm)}x]\,\chi_{\pm}\,,$$

with

(2.2.8) $$\pm\,\mathrm{Im}\,[k^{(\pm)}] > 0\,.$$

Here, and often in the following, we have, for notational simplicity, not labelled explicitly the different eigenvalues (as well as the eigenspinors $\psi^{(\pm)}$ and the quantities $\gamma^{(\pm)}$, $\delta^{(\pm)}$) with the subscript n. The distinction between discrete eigenvalues (and the corresponding eigenfunctions, etc.) with superscripts « plus » or « minus » is hereafter characterized by (2.2.8) (we ignore, for simplicity, the possibility of real eigenvalues).

The order of the neglected terms in these asymptotic formulae is given by the expression

(2.2.9) $$\chi_+ O\left[\exp[-ikx]\int\limits_x^{\pm\infty} d\xi\, q(\xi)\exp[2ik\xi]\right] + \\ + \chi_- O\left[\exp[ikx]\int\limits_x^{\pm\infty} d\xi\, r(\xi)\exp[-2ik\xi]\right],$$

the $\pm$ sign corresponding of course to the limits $x \to \pm\infty$. From this formula, the known analytic properties of $\alpha^{(\pm)}(k)$ and $\beta^{(\pm)}(k)$ ([3/.3g]) and a comparison of eqs. (2.2.5) and (2.2.7), one concludes that, corresponding to the values $k_n^{(+)}$ respectively $k_n^{(-)}$, the functions $\alpha^{(+)}$, $\beta^{(+)}$ respectively $\alpha^{(-)}$, $\beta^{(-)}$ have a pole, and

(2.2.10) $$i\gamma^{(\pm)}\delta^{(\pm)} = \varrho^{(\pm)} = \operatorname*{res}_{k^{(\pm)}} \alpha^{(\pm)}(k) = \lim_{k\to k^{(\pm)}} \{[k-k^{(\pm)}]\,\alpha^{(\pm)}(k)\}\,,$$

provided the values of $k^{(+)}$ respectively $k^{(-)}$ are such that, when substituted in (2.2.9), they yield an asymptotically vanishing contribution. Note that this is guaranteed to happen if r and q vanish asymptotically faster than exponentially. It should however be cautioned that eq. (2.2.10), as well as eq. (2.2.6), are applicable only in the case of single poles, to which we restrict, for simplicity, our considerations in this paper (except in a special case in subsect. 4.4 below).

Clearly if r and q are given functions of x, the quantities $\alpha^{(\pm)}(k)$, $\beta^{(\pm)}(k)$ and the parameters $k_n^{(\pm)}$, $\gamma_n^{(\pm)}$ and $\delta_n^{(\pm)}$ of the discrete part of the spectrum (if any) are uniquely determined, through eqs. (2.2.1), (2.2.5) and (2.2.7). Such a determination constitutes the « direct » (generalized Zakharov-Shabat) problem.

2'3. *The inverse problem.* – The corresponding «inverse» problem consists in the evaluation of r and q from the spectral parameters, defined above by the asymptotic formulae (2.2.5) and (2.2.7). We report here the Marchenko-type equation that solves this problem, referring to AKNS for its derivation and a discussion of its peculiarities, related to the non-Hermitian nature of (2.2.1). The input data that are sufficient to determine r and q are the functions $\alpha^{(\pm)}(k)$, the eigenvalues $k_n^{(\pm)}$ and the corresponding quantities $\gamma_n^{(\pm)}\delta_n^{(\pm)}$; in this paper we shall generally assume that the parameters of the discrete spectrum, $k_n^{(\pm)}$ and $\gamma_n^{(\pm)}\delta_n^{(\pm)}$, are obtainable by analytic continuation from those characterizing the continuum spectrum, $\alpha^{(\pm)}(k)$, being respectively the positions of the poles and (up to a constant; see eq. (2.2.10)) the residues of $\alpha^{(\pm)}(k)$. The relationship of this restriction to the asymptotic behaviour of r and q has been clarified above [23].

The Marchenko-type equations read

$$m^{(\pm)}(z) = \mp i \sum_n \varrho_n^{(\pm)} \exp[\pm ik_n^{(\pm)}z] + (2\pi)^{-1}\int_{-\infty}^{+\infty} dk\, \alpha^{(\pm)}(k) \exp[\pm ikz]\,, \tag{2.3.1}$$

$$M(z) = \tfrac{1}{2} m^{(-)}(z)(1+\sigma_3) + \tfrac{1}{2} m^{(+)}(z)(1-\sigma_3)\,, \tag{2.3.2}$$

$$K(x, x') + M(x+x') + \int_x^{+\infty} d\xi\, K(x,\xi)\,\sigma_1\, M(\xi + x') = 0\,, \qquad x' \geqslant x, \tag{2.3.3}$$

$$\begin{cases} q(x) = -2K_{11}(x,x)\,, \qquad r(x) = -2K_{22}(x,x)\,, \\ \displaystyle\int_x^{+\infty} d\xi\, q(\xi)\, r(\xi) = 2K_{12}(x,x) = 2K_{21}(x,x)\,. \end{cases} \tag{2.3.4}$$

We reiterate that these equations (in particular, eq. (2.3.1)) refer to the case with simple poles only, and that the quantity $\varrho_n^{(\pm)}$ is defined by eq. (2.2.10) (with the last equality being a consequence of the assumption mentioned above, that shall be used in the following without further warning).

2'4. *Transformation properties.* – It is finally convenient to report 4 transformations of the fields r and q, whose corresponding effect on the quantities $\alpha^{(\pm)}(k)$ is simple and can be easily evinced from eqs. (2.2.1) and (2.2.5). We do not report the effect of these transformations on the parameters of the discrete spectrum, since they can be directly read from the properties of $\alpha^{(\pm)}(k)$. Note

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[23] The significance of such a restriction is well understood in the context of the usual Schrödinger scattering problem; see, for instance, F. CALOGERO and J. R. COX: *Nuovo Cimento*, **55** A, 786 (1968). In the present context the limitation is not a serious one, but it deserves a separate discussion, in view of its relevance for soliton solutions (that are, however, already included in the present treatment; see below).

that these transformations can be multiplied (*i.e.* applied sequentially), that their square is unity and that they commute with one another (so that the 4 transformations given below yield in fact 16 different transformations). These transformations follow:

(2.4.1a) $r'(x) = -r(x)\,,\quad q'(x) = -q(x)\,,\quad q_j'(x) = -q_j(x)\,,\qquad j=1,2\,,$

(2.4.1b) $\alpha^{(\pm)\prime}(k) = -\alpha^{(\pm)}(k)\,,\quad \beta^{(\pm)\prime}(k) = \beta^{(\pm)}(k)\,;$

(2.4.2a) $r'(x) = r^*(x)\,,\quad q'(x) = q^*(x)\,,\quad q_j'(x) = q_j^*(x)\,,\qquad j=1,2\,,$

(2.4.2b) $\alpha^{(\pm)\prime}(k) = \alpha^{(\pm)*}(-k^*)\,,\quad \beta^{(\pm)\prime}(k) = \beta^{(\pm)*}(-k^*)\,;$

(2.4.3a) $r'(x) = q(x)\,,\quad q'(x) = r(x)\,,\quad q_1'(x) = q_1(x)\,,\quad q_2'(x) = -q_2(x)\,,$

(2.4.3b) $\alpha^{(\pm)\prime}(k) = \alpha^{(\mp)}(-k)\,,\quad \beta^{(\pm)\prime}(k) = \beta^{(\mp)}(-k)\,;$

(2.4.4a) $r'(x) = r(-x+a),\ q'(x) = q(-x+a),\ q_j'(x) = q_j(-x+a),\quad j=1,2\,,$

(2.4.4b) $\alpha^{(\pm)\prime}(k) = -\alpha^{(\pm)}(-k)\exp(\mp 2ika)\,,\quad \beta^{(\pm)\prime}(k) = \beta^{(\pm)}(-k)\,.$

The third of these transformations, eqs. (2.4.3), is particularly interesting, since, in contrast to the others, it interchanges the two fields r and q; it may be combined with the other 3, to yield altogether 4 transformations that share this property, and that are therefore suitable to ascertain which properties of $\alpha^{(\pm)}(k)$ correspond to special subcases of eq. (2.2.1), containing only one field (or, in the language of the inverse problem, what properties must the input functions $\alpha^{(\pm)}(k)$ have in order to generate two fields r and q simply related, *i.e.* essentially only a single independent field).

3. – Generalized Wronskian relation and derivation of the basic formulae.

3·1. *Generalized Wronskian relation.* – The starting point of our analysis is the generalized Wronskian relation

(3.1.1)
$$[\Psi'^T(x,k)F(x)\Psi(x,k)]|_{x_1}^{x_2} = \int_{x_1}^{x_2} dx\,\Psi'^T(x,k)\Big[-ik\{\sigma_3, F(x)\} + \frac{1}{2}S_1(x)\{\sigma_1, F(x)\} - \frac{i}{2}S_2(x)[\sigma_2, F(x)] + \frac{1}{2}D_1(x)[\sigma_1, F(x)] - \frac{i}{2}D_2(x)\{\sigma_2, F(x)\} + F_x(x)\Big]\Psi(x,k)\,.$$

Here, and always below,

(3.1.2)
$$S_j(x) = q_j'(x) + q_j(x)\,,\qquad D_j(x) = q_j'(x) - q_j(x)\,,\qquad j=1,2\,,$$

where q_j and q_j' are two different pairs of fields and Ψ, Ψ' are two matrix solutions of the corresponding eqs. (2.2.1). $F(x)$ is an essentially arbitrary (at least once differentiable) 2×2 matrix. The validity of eq. (3.1.1) is a straightforward consequence of (2.2.1).

3'2. *Application to the continuum spectrum.* – We now consider eq. (3.1.1) for $x_1=-\infty$, $x_2=+\infty$, inserting two solutions Ψ and Ψ' of (2.2.1) characterized by the boundary conditions appropriate to the continuum spectrum, eqs. (2.2.5), and assuming moreover that the matrix $F(x)$ satisfies the asymptotic conditions

$$F(+\infty)=0\,, \tag{3.2.1}$$

$$F(-\infty)=F_1(-\infty)\,\sigma_1+F_2(-\infty)\,i\sigma_2\,, \tag{3.2.2}$$

the second of which obviously implies $\{F(-\infty), \sigma_3\}=0$. This yields

$$\begin{aligned}[F_1(-\infty)+F_2(-\infty)\,\sigma_3]\,B+ik\int_{-\infty}^{+\infty}\mathrm{d}x\,\Psi'^T(x,k)\{\sigma_3, F(x)\}\Psi(x,k) = \\ =\tfrac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}x\,\Psi'^T(x,k)[S_1(x)\{\sigma_1, F(x)\}-iS_2(x)[\sigma_2, F(x)]+D_1(x)[\sigma_1, F(x)]- \\ -iD_2(x)\{\sigma_2, F(x)\}+2F_x(x)]\,\Psi(x,k)\end{aligned} \tag{3.2.3}$$

with

$$B=-\begin{pmatrix} 0 & \beta^{(-)}(k)\beta^{(+)\prime}(k) \\ \beta^{(-)\prime}(k)\beta^{(+)}(k) & 0\end{pmatrix}. \tag{3.2.4}$$

We then introduce a sequence of matrices $F^{(n)}(x)$ through the recursion formula

$$\begin{aligned}\{\sigma_3, F^{n+1}(x)\}=\tfrac{1}{2}\,[S_1(x)\{\sigma_1, F^{(n)}(x)\}-iS_2(x)[\sigma_2, F^{(n)}(x)]+ \\ +D_1(x)[\sigma_1, F^{(n)}(x)]-iD_2(x)\{\sigma_2, F^{(n)}(x)\}]+F^{(n)}_x(x)\,,\end{aligned} \tag{3.2.5}$$

so that, if $F^{(n)}(x)$ satisfies the conditions (3.2.1) and (3.2.2), we may rewrite eq. (3.2.3) as

$$\begin{aligned}[F_1^{(n)}(-\infty)+F_2^{(n)}(-\infty)\,\sigma_3]\,B+ik\int_{-\infty}^{+\infty}\mathrm{d}x\,\Psi'^T(x,k)\{\sigma_3, F^{(n)}(x)\}\,\Psi(x,k) = \\ =\int_{-\infty}^{+\infty}\mathrm{d}x\,\Psi'^T(x,k)\{\sigma_3, F^{(n+1)}(x)\}\,\Psi(x,k)\,.\end{aligned} \tag{3.2.6}$$

To analyse the recursion relations (3.2.5) we set

$$F^{(n)}(x) = F_0^{(n)}(x) + F_1^{(n)}(x)\,\sigma_1 + F_2^{(n)}(x)\,i\sigma_2 + F_3^{(n)}(x)\,\sigma_3\,. \tag{3.2.7}$$

Insertion of this expression in (3.2.5) shows that $F_0^{(n)}$, $F_3^{(n)}$ determine $F_1^{(n)}$, $F_2^{(n)}$ and $F_0^{(n+1)}$, $F_3^{(n+1)}$. Thus it is convenient to introduce the sequence of spinors $v^{(n)}$ through the definition

$$v^{(n)}(x) = [F_0^{(n)}(x) + F_3^{(n)}(x)]\,\chi_+ + [F_0^{(n)}(x) - F_3^{(n)}(x)]\,\chi_-\,, \tag{3.2.8}$$

or, equivalently,

$$\{\sigma_3, F^{(n)}(x)\} = 2\eta v^{(n)}(x)\,, \tag{3.2.9}$$

where we have introduced the formal operator η that transforms a spinor into a diagonal matrix

$$\eta\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}. \tag{3.2.10}$$

With these definitions we get

$$F_1^{(n)}(x) = -\int\limits_x^{+\infty}\mathrm{d}\xi\,\big(v_+(\xi), i\sigma_2\,v^{(n)}(\xi)\big)\,, \tag{3.2.11a}$$

$$F_2^{(n)}(x) = -\int\limits_x^{+\infty}\mathrm{d}\xi\,\big(v_-(\xi), \sigma_1\,v^{(n)}(\xi)\big)\,, \tag{3.2.11b}$$

where we have defined (note that the second equality is consistent with (2.2.3))

$$v_\pm(x) = \tfrac{1}{2}[r'(x) \pm r(x)]\,\chi_+ + \tfrac{1}{2}[q'(x) \pm q(x)]\,\chi_- = \tfrac{1}{2}[v'(x) \pm v(x)]\,, \tag{3.2.12}$$

and

$$v^{(n+1)}(x) = i\Lambda v^{(n)}(x)\,, \tag{3.2.13}$$

where we have introduced the integro-differential matrix operator

$$\Lambda = \frac{1}{2i}\left[\sigma_3\frac{\partial}{\partial x} + \begin{pmatrix} r'Iq' + rIq & -r'Ir - rIr' \\ q'Iq + qIq' & -q'Ir' - qIr\end{pmatrix}\right]. \tag{3.2.14}$$

In this last formula I is the integral operator of eq. (1.9), so that, for instance,

$$rIqf = r(x)\int\limits_x^{+\infty}\mathrm{d}\xi\,q(\xi)\,f(\xi)\,. \tag{3.2.15}$$

From eq. (3.2.13) we immediately get

(3.2.16)
$$v^{(n)}(x) = (i\Lambda)^n v^{(0)}(x) ,$$

while from eqs. (3.2.6) and (3.2.9) we get

(3.2.17)
$$V_{n+1} = ikV_n + \tfrac{1}{2}[F_1^{(n)}(-\infty) + F_2^{(n)}(-\infty)\sigma_3]B$$

with

(3.2.18)
$$V_n = \int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta v^{(n)}(x)]\,\Psi(x, k) .$$

The solution of the recursion formula (2.2.17) for the quantities V_n is a simple task, and the result thus obtained can be rewritten, by means of eqs. (3.2.11) and (3.2.16), as follows:

(3.2.19)
$$\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta(i\Lambda)^n v^{(0)}(x)]\Psi(x, k) = (ik)^n \int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta\, v^{(0)}(x)]\Psi(x, k) +$$
$$+ \frac{1}{2}\int_{-\infty}^{+\infty} dx \left\{\left(v_+(x), i\sigma_2\left[\frac{(ik)^n - (i\Lambda)^n}{ik - i\Lambda}\right] v^{(0)}(x)\right) + \left(v_-(x), \sigma_1\left[\frac{(ik)^n - (i\Lambda)^n}{ik - i\Lambda}\right] v^{(0)}(x)\right)\sigma_3\right\} B .$$

It is easily seen that the condition of validity of this formula for all n is that all derivatives of $v^{(0)}(x)$ vanish asymptotically:

(3.2.20)
$$\lim_{x\to\pm\infty}\left[\frac{d^p}{dx^p} v^{(0)}(x)\right] = 0 , \qquad p = 0, 1, 2, \dots .$$

The structure of eq. (3.2.19) implies immediately the more general formula

(3.2.21)
$$\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta f(\Lambda)\, v^{(0)}(x)]\Psi(x, k) = f(k)\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta v^{(0)}(x)]\Psi(x, k) +$$
$$+ \frac{1}{2}\int_{-\infty}^{+\infty} dx \left\{\left(v_+(x), \sigma_2\left[\frac{f(k) - f(\Lambda)}{k - \Lambda}\right] v^{(0)}(x)\right) - i\left(v_-(x), \sigma_1\left[\frac{f(k) - f(\Lambda)}{k - \Lambda}\right] v^{(0)}(x)\right)\sigma_3\right\} B ,$$

where $f(z)$ is an arbitrary entire function. It should be emphasized that in this formula (whose validity is a consequence of (2.2.1) and (2.2.4)) the spinor $v^{(0)}(x)$ is arbitrary, except for the restriction (3.2.20).

We now return to the generalized Wronskian relation (3.1.1), inserting again, in the limit $x_1 = -\infty$, $x_2 = +\infty$, two solutions Ψ and Ψ' of eq. (2.2.1) characterized by the boundary conditions (2.2.5), but with the two special choices $F(x) = i\sigma_2$ and $F(x) = \sigma_1$. We thus get

$$(3.2.22)\quad \begin{pmatrix} \alpha^{(+)'}(k) - \alpha^{(+)}(k) & \alpha^{(+)'}(k)\alpha^{(-)}(k) + \beta^{(+)'}(k)\beta^{(-)}(k) - 1 \\ 1 - \alpha^{(+)}(k)\alpha^{(-)'}(k) - \beta^{(+)}(k)\beta^{(-)'}(k) & \alpha^{(-)}(k) - \alpha^{(-)'}(k) \end{pmatrix} = 2\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta\sigma_3 v_-(x)]\Psi(x, k)\,,$$

$$(3.2.23)\quad \begin{pmatrix} \alpha^{(+)'}(k) + \alpha^{(+)}(k) & 1 + \alpha^{(+)'}(k)\alpha^{(-)}(k) - \beta^{(+)'}(k)\beta^{(-)}(k) \\ 1 + \alpha^{(+)}(k)\alpha^{(-)'}(k) - \beta^{(+)}(k)\beta^{(-)'}(k) & \alpha^{(-)'}(k) + \alpha^{(-)}(k) \end{pmatrix} = 2\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta v_+(x)]\Psi(x, k)\,,$$

where we have used eqs. (3.2.10) and (3.2.12).

The last step is to set in eq. (3.2.21) $v^{(0)}(x) = \sigma_3 v_-(x)$ respectively $v^{(0)}(x) = v_+(x)$, and use (3.2.22) respectively (3.2.23). We thus obtain the two final formulae

$$(3.2.24)\quad 2\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta f(\Lambda)\sigma_3 v_-(x)]\Psi(x, k) = f(k)\begin{pmatrix} \alpha^{(+)'}(k) - \alpha^{(+)}(k) & \alpha^{(+)'}(k)\alpha^{(-)}(k) + \beta^{(+)'}(k)\beta^{(-)}(k) - 1 \\ 1 - \alpha^{(+)}(k)\alpha^{(-)'}(k) - \beta^{(+)}(k)\beta^{(-)'}(k) & \alpha^{(-)}(k) - \alpha^{(-)'}(k) \end{pmatrix} - i[f_+(k) + f_-(k)\sigma_3]B\,,$$

$$(3.2.25)\quad 2\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k)[\eta g(\Lambda) v_+(x)]\Psi(x, k) = g(k)\begin{pmatrix} \alpha^{(+)'}(k) + \alpha^{(+)}(k) & 1 + \alpha^{(+)'}(k)\alpha^{(-)}(k) - \beta^{(+)'}(k)\beta^{(-)}(k) \\ 1 + \alpha^{(+)}(k)\alpha^{(-)'}(k) - \beta^{(+)}(k)\beta^{(-)'}(k) & \alpha^{(-)'}(k) + \alpha^{(-)}(k) \end{pmatrix} - i[g_+(k) + g_-(k)\sigma_3]B\,,$$

where

$$(3.2.26a)\quad f_\pm(k) = \int_{-\infty}^{+\infty} dx \left(v_\pm(x), \sigma_\pm \left[\frac{f(k) - f(\Lambda)}{k - \Lambda}\right]\sigma_3 v_-(x)\right), \qquad \sigma_+ \equiv i\sigma_2,\ \sigma_- \equiv \sigma_1\,,$$

$$(3.2.26b)\quad g_\pm(k) = \int_{-\infty}^{+\infty} dx \left(v_\pm(x), \sigma_\pm \left[\frac{g(k) - g(\Lambda)}{k - \Lambda}\right] v_+(x)\right), \qquad \sigma_+ \equiv i\sigma_2,\ \sigma_- \equiv \sigma_1\,.$$

The symbols appearing in these formulae, which constitute the main result of this section and provide the main tool for our treatment, are defined by eqs. (2.2.1), (2.2.2), (2.2.3), (2.2.5), (3.2.4), (3.2.10), (3.2.12) and (3.2.14). The two functions $f(z)$ and $g(z)$ are entire, but otherwise arbitrary; they might of course depend parametrically on other variables besides z (not on x or k).

3'3. *Application to the discrete spectrum.* – A procedure, analogous to that described above for the continuum-spectrum regime, can be applied in the discrete-spectrum situation. We report here only the final formulae, whose derivation follows closely the pattern set above. There are, however, now two different possibilities, that must be analysed separately.

We treat first the case when the problem with the « potential » r, q has a discrete eigenvalue $k^{(+)}$ (or $k^{(-)}$), while the problem with the potential r', q' does not have the same eigenvalue. We then consider the generalized Wronskian built with $\psi'_+(x, k^{(+)})$ (or $\psi'_-(x, k^{(-)})$) and $\psi^{(+)}(x)$ (or $\psi^{(-)}(x)$), where by definition $\psi'_+(x, k^{(+)})$ is the first column of the matrix $\Psi'(x, k)$ characterized by the boundary conditions (2.2.5) (with $k = k^{(+)}$; an analytic continuation off the real axis in the k-plane is implied here), and $\psi'_-(x, k)$ is instead the second column of the same matrix (with $k = k^{(-)}$). The (normalized) eigenfunctions $\psi^{(\pm)}(x)$ have been defined, eqs. (2.2.6)-(2.2.8). The formulae read

$$(3.3.1)\qquad 2\int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\psi'_\pm(x, k^{(\pm)}),\,[\eta f(\Lambda)\,\sigma_3\, v_-(x)]\,\psi^{(\pm)}(x)\big) = \mp\,\gamma^{(\pm)} f(k^{(\pm)})\,,$$

$$(3.3.2)\qquad 2\int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\psi'_\pm(x, k^{(\pm)}),\,[\eta g(\Lambda)\, v_+(x)]\,\psi^{(\pm)}(x)\big) = \gamma^{(\pm)} g(k^{(\pm)})\,,$$

with $f(z)$ and $g(z)$ arbitrary entire functions and the other symbols defined by eqs. (3.2.10), (3.2.12), (3.2.14) and (2.2.7a). It should be emphasized that the condition that $k^{(\pm)}$ not be an eigenvalue of (2.2.1) with r', q' is essential for the validity of this formula.

The second type of formulae obtains from the consideration of the generalized Wronskian built out of $\psi^{(+)\prime}(x)$ and $\psi^{(+)}(x)$, these being the normalized eigenfunctions corresponding to the eigenvalues $k^{(+)\prime}$ respectively $k^{(+)}$, of (2.2.1) with the potentials r', q' respectively r, q. They read

$$(3.3.3)\qquad \int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\psi^{(+)\prime}(x),\,[\eta f(\Lambda)\,\sigma_3\, v_-(x)]\,\psi^{(+)}(x)\big) =$$

$$= ik_D^{(+)}\Big\{\tfrac{1}{2} f(k_S^{(+)})\int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\psi^{(+)\prime}(x),\,\sigma_1\,\psi^{(+)}(x)\big) + i\int_{-\infty}^{+\infty}\mathrm{d}x\,\big(\psi^{(+)\prime}(x),\,F(k_S^{(+)}, x)\,\psi^{(+)}(x)\big)\Big\}\,,$$

$$\int_{-\infty}^{+\infty} dx\, (\psi^{(+)'}(x),\, [\eta g(\Lambda)\, v_+(x)]\, \psi^{(+)}(x)) = \tag{3.3.4}$$

$$= ik_D^{(+)}\left\{\tfrac{1}{2}\, g(k_S^{(+)}) \int_{-\infty}^{+\infty} dx\, (\psi^{(+)'}(x),\, i\sigma_2\, \psi^{(+)}(x)) + i\int_{-\infty}^{+\infty} dx\, (\psi^{(+)'}(x),\, G(k_S^{(+)}, x)\, \psi^{(+)}(x))\right\}$$

with

$$k_S^{(+)} = \tfrac{1}{2}\,(k^{(+)'} + k^{(+)})\,, \qquad k_D^{(+)} = \tfrac{1}{2}\,(k^{(+)'} - k^{(+)})\,, \tag{3.3.5}$$

and

$$F(k, x) = i\sigma_2 f_+(k, x) + \sigma_1 f_-(k, x)\,, \tag{3.3.6a}$$

$$G(k, x) = i\sigma_2 g_+(k, x) + \sigma_1 g_-(k, x)\,, \tag{3.3.6b}$$

$$f_\pm(k, x) = \int_x^{+\infty} d\xi \left(v_\pm(\xi),\, \sigma_\pm \left[\frac{f(k) - f(\Lambda)}{k - \Lambda}\right] \sigma_3\, v_-(\xi)\right), \qquad \sigma_+ \equiv i\sigma_2\,,\ \sigma_- \equiv \sigma_1\,, \tag{3.3.7a}$$

$$g_\pm(k, x) = \int_x^{+\infty} d\xi \left(v_\pm(\xi),\, \sigma_\pm \left[\frac{g(k) - g(\Lambda)}{k - \Lambda}\right] v_\mp(\xi)\right), \qquad \sigma_+ \equiv i\sigma_2\,,\ \sigma_- \equiv \sigma_1\,. \tag{3.3.7b}$$

The symbols in these formulae are defined by eqs. (3.2.10), (3.2.12) and (3.2.14) (with Λ in eq. (3.3.7) acting of course on the variable ξ); the entire functions $f(z)$ and $g(z)$ are arbitrary. A completely analogous formula also holds, with the superscript $^{(+)}$ replaced everywhere by $^{(-)}$.

Formulae that involve $\psi^{(+)'}(x)$ and $\psi^{(-)}(x)$ (or *vice versa*) might also be derived, but they do not seem to be useful.

Equations (3.3.3) and (3.3.4) remain valid even if only one of the two quantities $k^{(+)'}$, $k^{(+)}$ corresponds to an eigenvalue of the corresponding problem (2.2.1), provided its imaginary part is larger than the imaginary part of the other; for the case with superscript $^{(-)}$ in place of $^{(+)}$, the requirement is analogous, *i.e.* the value (of $k^{(-)'}$ or $k^{(-)}$) corresponding to the discrete eigenvalue must have an imaginary part larger in modulus than that of the other, if this does not also correspond to a discrete eigenvalue.

It should be noted that the conditions under which eqs. (3.3.1) and (3.3.2), respectively (3.3.3) and (3.3.4), have been derived are different; this explains why eqs. (3.3.3) or (3.3.4), in the special case $k^{(+)'} = k^{(+)}$, does not reproduce eq. (3.3.1) or (3.3.2); for the validity of the former it is indeed required, if $k^{(+)'} = k^{(+)}$, that this value be a (discrete) eigenvalue of both problems (2.2.1) with r', q' and r, q, while for the validity of eqs. (3.3.1) or (3.3.2) it is instead required that only one of the two ψ-functions that appear in the formulae corresponds to a (discrete) eigenvalue.

4. – Results.

4'1. NLPDE's *solvable by the* IST. – Assume now that, in (2.2.1), the fields r, q (and therefore also ψ, $\alpha^{(\pm)}$, etc.) depend on other variables besides x (and/or k). Let y be one of these variables, and consider eq. (3.2.24) with $v(x)=v(x,y)$ and $v'(x)=v(x,y+\Delta y)$, in the limit $\Delta y\to 0$. There follows first of all the « unitarity » equation

$$\alpha^{(+)}(k,y)\,\alpha^{(-)}(k,y)+\beta^{(+)}(k,y)\,\beta^{(-)}(k,y)=1\,, \tag{4.1.1}$$

and then, if we keep terms linear in Δy (and use (4.1.1)), the relation

$$\begin{aligned}2\int\limits_{-\infty}^{+\infty}\mathrm{d}x\,\Psi'^{T}(x,k,y)[\eta f(L)\sigma_3 v_y(x,y)]\Psi(x,k,y) = \\ -f(k)\begin{pmatrix}\alpha_y^{(+)}(k,y) & \alpha^{(-)}(k,y)\alpha_y^{(+)}(k,y)+\beta^{(-)}(k,y)\beta_y^{(+)}(k,y)\\ -\alpha^{(+)}(k,y)\alpha_y^{(-)}(k,y)-\beta^{(+)}(k,y)\beta_y^{(-)}(k,y) & -\alpha_y^{(-)}(k,y)\end{pmatrix}+\\ +2i\beta^{(+)}(k,y)\beta^{(-)}(k,y)\sigma_1\int\limits_{-\infty}^{+\infty}\mathrm{d}x\left(v(x,y),\,i\sigma_2\left[\frac{f(k)-f(L)}{k-L}\right]\sigma_3 v_y(x,y)\right)\end{aligned} \tag{4.1.2}$$

with

$$L=\frac{1}{2i}\left[\sigma_3\frac{\partial}{\partial x}+2\begin{pmatrix}rIq & -rIr\\ qIq & -qIr\end{pmatrix}\right]. \tag{4.1.3}$$

In this last formula I is of course the integral operator of eq. (1.9) (see eqs. (3.2.14) and (3.2.15)), and we have, for simplicity, not indicated the arguments of r, q. Note that (4.1.1) implies that the matrix in the r.h.s. of (4.1.2) is symmetrical.

Let us re-emphasize that the entire function $f(z)$ in (4.1.2) is arbitrary; if v depends on several variables, one can write as many equations similar to (4.1.2), with every variable playing the rôle of y, and with a different function f in each case. Indeed the reader should imagine that we have done just that, once with $y=t$, and M times with $y=y_j$, $j=1,2,\ldots,M$, under the assumption that v (and ψ, $\alpha^{(\pm)}$, etc.) depends on the scalar t and on the M-dimensional vector $\boldsymbol{y}$, besides x (and/or k).

Together with all these equations, one should also consider the single equation that obtains in the $\Delta y \to 0$ limit from (3.2.25), namely

$$(4.1.4)\quad 2\int_{-\infty}^{+\infty} dx\, \Psi'^T(x, k, y)[\eta g(L)v(x, y)]\Psi(x, k, y) = \\ = g(k)\begin{pmatrix} \alpha^{(+)}(k, y) & \alpha^{(+)}(k, y)\alpha^{(-)}(k, y) \\ \alpha^{(+)}(k, y)\alpha^{(-)}(k, y) & \alpha^{(-)}(k, y)\end{pmatrix} + \\ + 2i\beta^{(+)}(k, y)\beta^{(-)}(k, y)\sigma_1 \int_{-\infty}^{\infty} dx \left(v(x, y), i\sigma_2\left[\frac{g(k)-g(L)}{k-L}\right]v(x, y)\right).$$

We also recall that the arbitrary entire function $g(z)$ in this equation, as well as the analogous functions in the equations described above, might depend on other variables besides z (except x and k).

By taking a simple linear combination of all these equations there immediately then follows that validity of the NLPDE for the field v

$$(4.1.5)\quad f(L, \boldsymbol{y}, t)\sigma_3 v_t(x, \boldsymbol{y}, t) + \boldsymbol{h}(L, \boldsymbol{y}, t)\frac{\partial}{\partial \boldsymbol{y}}\sigma_3 v(x, \boldsymbol{y}, t) + g(L, \boldsymbol{y}, t)v(x, \boldsymbol{y}, t) = 0$$

implies validity of the linear equations for $\alpha^{(\pm)}$, $\beta^{(\pm)}$

$$(4.1.6)\quad f(k, \boldsymbol{y}, t)\alpha_t^{(\pm)}(k, \boldsymbol{y}, t) + \boldsymbol{h}(k, \boldsymbol{y}, t)\frac{\partial}{\partial \boldsymbol{y}}\alpha^{(\pm)}(k, \boldsymbol{y}, t) \pm g(k, \boldsymbol{y}, t)\alpha^{(\pm)}(k, \boldsymbol{y}, t) = 0,$$

$$(4.1.7)\quad f(k, \boldsymbol{y}, t)\beta_t^{(\pm)}(k, \boldsymbol{y}, t) + \boldsymbol{h}(k, \boldsymbol{y}, t)\frac{\partial}{\partial \boldsymbol{y}}\beta^{(\pm)}(k, \boldsymbol{y}, t) \pm \bar{\varphi}(k, \boldsymbol{y}, t)\beta^{(\pm)}(k, \boldsymbol{y}, t) = 0$$

with

$$(4.1.8)\quad \bar{\varphi}(k, \boldsymbol{y}, t) = 2i\int_{-\infty}^{+\infty} dx \left(v(x), i\sigma_2\left\{\left[\frac{f(k)-f(L)}{k-L}\right]\sigma_3 v_t(x) + \right.\right. \\ \left.\left. + \left[\frac{\boldsymbol{h}(k)-\boldsymbol{h}(L)}{k-L}\right]\frac{\partial}{\partial \boldsymbol{y}}\sigma_3 v(x) + \left[\frac{g(k)-g(L)}{k-L}\right]v(x)\right\}\right).$$

Equations (4.1.6) for $\alpha^{(\pm)}$ follow from the diagonal part of the matrix equation obtained from the linear combination described above, while eqs. (4.1.7) for $\beta^{(\pm)}$ follows from the nondiagonal part by means of (4.1.6) (and (4.1.1)). In r.h.s. of the last equation we have, for notational simplicity, not indicated the dependence upon the variables $\boldsymbol{y}$, t.

We may thus conclude that, if the fields r, q evolve according to the NLPDE (4.1.5), the quantities $\alpha^{(\pm)}$ and $\beta^{(\pm)}$ evolve according to the linear equations (4.1.6) and (4.1.7). This, together with the possibility to reconstruct r, q from $\alpha^{(\pm)}$, is the basis for the solvability of the NLPDE (4.1.5) by the IST (see below). Note that eqs. (4.1.1), (4.1.6) and (4.1.7) imply that, if r, q evolve according to (4.1.5),

$$\left[f(k, y, t)\frac{\partial}{\partial t} + h(k, y, t)\frac{\partial}{\partial y}\right]\alpha^{(+)}(k, y, t)\alpha^{(-)}(k, y, t) = 0\,, \tag{4.1.9a}$$

$$\left[f(k, y, t)\frac{\partial}{\partial t} + h(k, y, t)\frac{\partial}{\partial y}\right]\beta^{(+)}(k, y, t)\beta^{(-)}(k, y, t) = 0\,. \tag{4.1.9b}$$

To discuss more specifically the solvability by the IST we prefer to rewrite these equations in a manner that singles out the variable t, so that they take the form of evolution equations ([24]). This is simply achieved by setting

$$\gamma(z, y, t) = g(z, y, t)/f(z, y, t)\,, \tag{4.1.10a}$$

$$\nu(z, y, t) = h(z, y, t)/f(z, y, t)\,, \tag{4.1.10b}$$

so that the functions γ and ν are now ratios of entire functions of z ([25]). In place of eqs. (4.1.5)-(4.1.8) we then get

$$\sigma_3 v_t(x, y, t) + \nu(L, y, t)\frac{\partial}{\partial y}\sigma_3 v(x, y, t) + \gamma(L, y, t) v(x, y, t) = 0\,, \tag{4.1.11}$$

$$\alpha_t^{(\pm)}(k, y, t) + \nu(k, y, t)\frac{\partial}{\partial y}\alpha^{(\pm)}(k, y, t) \pm \gamma(k, y, t)\alpha^{(\pm)}(k, y, t) = 0\,, \tag{4.1.12}$$

$$\beta_t^{(\pm)}(k, y, t) + \nu(k, y, t)\frac{\partial}{\partial y}\beta^{(\pm)}(k, y, t) \pm \varphi(k, y, t)\beta^{(\pm)}(k, y, t) = 0\,, \tag{4.1.13}$$

$$\varphi(k, y, t) = 2i\int_{-\infty}^{+\infty}\mathrm{d}x\left(v(x, y, t), i\sigma_2\left[\frac{\nu(k, y, t) - \nu(L, y, t)}{k - L}\right]\frac{\partial}{\partial y}\sigma_3 v(x, y, t)\right). \tag{4.1.14}$$

The derivation of (4.1.14) (from (4.1.8); clearly $\varphi = \tilde{\varphi}/f$) requires the use of (4.1.11) and of some properties of the operator L that are discussed in the appendix, where we also show that, in spite of the presence of the integral operator in L, the expression $L^n v$, with n any positive integer, contains only powers of r, q and of their derivatives up to the order n.

([24]) For an outline of the difficulties that might originate from this formal step we refer to previous works, such as ref. ([3g,3h]), postponing a more detailed discussion to subsequent papers of this series.

([25]) A condition, that we have not, for simplicity, mentioned previously ([16]), but that is clearly implied by (4.1.10), is that γ and ν have the same singularity structure in the finite part of the complex z-plane. Let us however also mention at this point that the requirement that these be entire (or ratios of entire) functions is sufficient, but not necessary, for the validity of all these results, that might indeed also hold for nonentire functions provided a suitable definition is given of the operator that obtains after replacing the argument of such a function by an operator.

The NLPDE (4.1.11) coincides with that presented in the introduction, eq. (1.7). Its solvability by the IST is accomplished as follows: given r, q at time t_0,

$$r(x, y, t_0) = \bar{r}(x, y), \qquad q(x, y, t_0) = \bar{q}(x, y), \tag{4.1.15a}$$

or equivalently

$$v(x, y, t_0) = \bar{v}(x, y), \tag{4.1.15b}$$

one computes

$$\alpha^{(\pm)}(k, y, t_0) = \bar{\alpha}^{(\pm)}(k, y) \tag{4.1.16}$$

solving the direct problem, *i.e.* through eqs. (2.2.1), (2.2.2) and (2.2.5) (with y a fixed parameter); one obtains then $\alpha^{(\pm)}(k, y, t)$ from $\alpha^{(\pm)}(k, y, t_0)$ integrating the linear partial differential equation (4.1.11); and one reconstructs finally r, q at time t from $\alpha^{(\pm)}(k, y, t)$ solving the inverse problem, *i.e.* using eqs. (2.3.1)-(2.3.4) (with y, t fixed parameters).

To perform the last step it is required to know also the parameters of the discrete spectrum. The equations characterizing their time evolution follow, under the assumption mentioned above, directly from eq. (4.1.6), by inserting the ansatz

$$\alpha^{(\pm)}(k, y, t) \approx \varrho^{(\pm)}(y, t)/[k - k^{(\pm)}(y, t)], \tag{4.1.17}$$

and then taking the limit $k \to k^{(\pm)}(y, t)$. In this manner one gets

$$k_t^{(\pm)}(y, t) + \nu\big(k^{(\pm)}(y, t), y, t\big)\frac{\partial}{\partial y} k^{(\pm)}(y, t) = 0, \tag{4.1.18}$$

$$\varrho_t^{(\pm} + \nu \frac{\partial}{\partial y}\varrho^{(\pm)} + \left[\nu_k \frac{\partial}{\partial y} k^{(\pm)} \pm \gamma\right]\varrho^{(\pm)} = 0. \tag{4.1.19}$$

In the last equation $\varrho^{(\pm)}$ is a function of y and t, ν_k is the derivative of ν with respect to its first argument and, together with ν and γ, has arguments $k^{(\pm)}$, y and t, while everywhere $k^{(\pm)} \equiv k^{(\pm)}(y, t)$. A derivation of these evolution equations that does not rely on the relationship with the singularity structure of $\alpha^{(\pm)}$ in the complex k-plane is also possible; for instance, eq. (4.1.18) is obtained by writing the equations analogous to (4.1.2) and (4.1.4), that read

$$2\int_{-\infty}^{+\infty} dx\,\big(\psi^{(\pm)}(x, y), [\eta f(L)\sigma_3 v_y(x, y)]\psi^{(\pm)}(x, y)\big) = -\frac{i}{2} f\big(k^{(\pm)}(y)\big) k_y^{(\pm)}(y), \tag{4.1.20a}$$

$$2\int_{-\infty}^{+\infty} dx\,\big(\psi^{(\pm)}(x, y), [\eta g(L) v(x, y)]\psi^{(\pm)}(x, y)\big) = 0, \tag{4.1.20b}$$

and follow from (3.3.4) and (3.3.5), and then proceeding as above.

Equations (4.1.18) and (4.1.19) allow one in principle to compute the time evolution of the discrete-spectrum parameters. Note however that eq. (4.1.18) is nonlinear, while eq. (4.1.19) is linear (once (4.1.18) has been solved).

In the slightly less general case that obtains if $\boldsymbol{\nu}$ and γ are independent of $\boldsymbol{y}$, eqs. (4.1.12), (4.1.18) and (4.1.19) that describe the time evolution of the spectral parameters can be integrated in closed form:

$$\alpha^{(\pm)}(k, \boldsymbol{y}, t) = \exp\left[\mp\int_{t_0}^{t} dt'\,\gamma(k, t')\right] \bar{\alpha}^{(\pm)}\left(k, \boldsymbol{y} - \int_{t_0}^{t} dt'\,\boldsymbol{\nu}(k, t')\right), \tag{4.1.21a}$$

$$k^{(\pm)}(\boldsymbol{y}, t) = \bar{k}^{(\pm)}\left(\boldsymbol{y} - \int_{t_0}^{t} dt'\,\boldsymbol{\nu}\left(k^{(\pm)}(\boldsymbol{y}, t), t'\right)\right), \tag{4.1.21b}$$

$$\varrho^{(\pm)}(\boldsymbol{y}, t) = \exp\left[\mp\int_{t_0}^{t} dt'\,\gamma(k^{(\pm)}, t')\right] \cdot \left[1 + \bar{k}_{\boldsymbol{y}}^{(\pm)}\left(\boldsymbol{y} - \int_{t_0}^{t} dt'\,\boldsymbol{\nu}(k^{(\pm)}, t')\right)\int_{t_0}^{t} dt'\,\boldsymbol{\nu}_k(k^{(\pm)}, t')\right]^{-1} \bar{\varrho}^{(\pm)}\left(\boldsymbol{y} - \int_{t_0}^{t} dt'\,\boldsymbol{\nu}(k^{(\pm)}, t')\right), \tag{4.1.21c}$$

where of course

$$\bar{k}^{(\pm)}(\boldsymbol{y}) = k^{(\pm)}(\boldsymbol{y}, t_0), \tag{4.1.22a}$$

$$\bar{\varrho}^{(\pm)}(\boldsymbol{y}) = \varrho^{(\pm)}(\boldsymbol{y}, t_0) \tag{4.1.22b}$$

are fixed by the initial condition at time t_0 (as well as $\bar{\alpha}^{(\pm)}(k, \boldsymbol{y})$; see eq. (4.1.16)). In eq. (4.1.21c) we have used the shorthand notation $\bar{k}_{\boldsymbol{y}}^{(\pm)}(\boldsymbol{y}')$ for the gradient of $\bar{k}^{(\pm)}$ with respect to $\boldsymbol{y}$ evaluated at $\boldsymbol{y}'$; the symbol $\boldsymbol{\nu}_k$ has been defined above. Note that in this equation $k^{(\pm)}$ stands for $k^{(\pm)}(\boldsymbol{y}, t)$ (the last argument is t, not t', even when $k^{(\pm)}$ enters under the integral sign).

It is remarkable that the NLPDE (4.1.18) is, in this case, exactly integrable; although of course this is implied by consistency from the integrability of eq. (4.1.12) (indeed (4.1.21b) and (4.1.21c) may be obtained from (4.1.21a) by using (4.1.17); it can also be explicitly verified that they satisfy (4.1.18) and (4.1.19)).

Equation (4.1.21b) does not however provide the explicit expression of $k^{(\pm)}(\boldsymbol{y}, t)$, being instead a transcendental equation for this quantity, whose structure depends on the initial conditions (in special cases it reduces to an algebraic equation).

In the special case when $\boldsymbol{\nu} = 0$, all reference to the variable $\boldsymbol{y}$ disappears from the NLPDE (4.1.11) (a possible presence of $\boldsymbol{y}$ as an argument in γ is of course trivial in this case). In this case the time evolution of the spectral parameters is particularly simple:

$$\alpha^{(\pm)}(k,t) = \exp\left[\mp\int_{t_0}^{t} dt'\,\gamma(k,t')\right]\bar{\alpha}^{(\pm)}(k)\,, \tag{4.1.23a}$$

$$\beta^{(\pm)}(k,t) = \bar{\beta}^{(\pm)}(k)\,, \tag{4.1.23b}$$

$$k^{(\pm)}(t) = \bar{k}^{(\pm)}\,, \tag{4.1.23c}$$

$$\varrho^{(\pm)}(t) = \exp\left[\mp\int_{t_0}^{t} dt'\,\gamma(k^{(\pm)},t')\right]\bar{\varrho}^{(\pm)}\,, \tag{4.1.23d}$$

where of course $\bar{\alpha}^{(\pm)}(k)$, $\bar{\beta}^{(\pm)}(k)$, $\bar{k}^{(\pm)}$ respectively $\bar{\varrho}^{(\pm)}$ are the values taken by $\alpha^{(\pm)}(k,t)$, $\beta^{(\pm)}(k,t)$, $k^{(\pm)}(t)$ respectively $\varrho^{(\pm)}(t)$ for $t=t_0$. Equation (4.1.23b) follows immediately from (4.1.13) and (4.1.14). These equations had been already given (with the more stringent assumption that γ be time independent) by AKNS, since this case coincides with that treated by them.

The time independence of the spectral parameters $\beta^{(\pm)}(k)$ that obtains in this case implies the existence of an infinite number of conservation laws; for the derivation of these we refer to the literature [3,26]. For another approach to this problem see subsect. **4'3** below. The fact that, if the y-dependence is instead present, both $k^{(\pm)}$ and $\beta^{(\pm)}$ vary with time underscores the nontrivial nature of this generalization. A more detailed analysis is postponed to subsequent papers of this series.

4'2. *Bäcklund transformations.* – Let $f(z)$ and $g(z)$ be two entire, but otherwise arbitrary, functions of z. It is then clear that the basic equations (3.2.24) and (3.2.25) imply that if two fields v, v' are related by the formula

$$f(\Lambda)\,\sigma_3\,v_- + g(\Lambda)\,v_+ = 0\,, \tag{4.2.1}$$

the corresponding spectral parameters are related by the formula

$$\pm f(k)[\alpha^{(\pm)\prime}(k) - \alpha^{(\pm)}(k)] + g(k)[\alpha^{(\pm)\prime}(k) + \alpha^{(\pm)}(k)] = 0\,, \tag{4.2.2}$$

and by another formula for the betas, that obtains from the nondiagonal terms in the r.h.s. of (3.2.24) and (3.2.25) and an appropriate use of the unitarity relation (4.1.1) (both for primed and unprimed variables). The operator Λ

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[26] The basic idea that is used to extract the conserved quantities is a fairly old one in potential scattering theory, that may be traced back to papers by N. LEVINSON, R. G. NEWTON and L. D. FADDEEV; see, for instance, F. CALOGERO and A. DEGASPERIS: *Journ. Math. Phys.*, **9**, 90 (1968).

in eq. (4.2.1) (and in the following equations) is of course that defined in the previous section, eq. (3.2.14).

It is more elegant to write these equations in the form

$$H_+(\Lambda)\,v'(x) + H_-(\Lambda)\,v(x) = 0 \tag{4.2.3}$$

with (note that the order in the last term is important)

$$H_\pm(z) = g(z) \pm f(z)\,\sigma_3\,, \tag{4.2.4}$$

and

$$\alpha^{(\pm)\prime}(k) = \{[f(k) \mp g(k)]/[f(k) \pm g(k)]\}\,\alpha^{(\pm)}(k)\,, \tag{4.2.5}$$

$$\beta^{(\pm)\prime}(k) = \{[f(k) \mp g(k)]/[f(k) \pm g(k)]\}\,[\Theta^{(\pm)}(k)/\Theta^{(\mp)}(k)]\,\beta^{(\pm)}(k) \tag{4.2.6}$$

with

$$\Theta^{(\pm)}(k) = f_+(k) + g_+(k) \mp [f_-(k) + g_-(k)]\,, \tag{4.2.7}$$

where $f_\pm(k)$, $g_\pm(k)$ are defined by eqs. (3.2.26).

Note that these equations imply

$$\alpha^{(+)\prime}(k)\,\alpha^{(-)\prime}(k) = \alpha^{(+)}(k)\,\alpha^{(-)}(k)\,, \tag{4.2.8}$$

$$\beta^{(+)\prime}(k)\,\beta^{(-)\prime}(k) = \beta^{(+)}(k)\,\beta^{(-)}(k)\,. \tag{4.2.9}$$

To obtain eq. (4.2.6) we have also used the important relation

$$i[\Theta^{(\mp)}(k)]^2 = -2g(k)[f_+(k) + g_+(k)] \pm 2f(k)[f_-(k) + g_-(k)] \tag{4.2.10}$$

that is a consequence of the unitarity equation (4.1.1).

The equations written above remain of course valid even if v and v' depend on other variables, as in the preceding subsection. The two functions f and g might also depend on these variables, and in the following subsection we shall take advantage of this possibility. Here we assume that they do not, namely that they are functions of their argument z only. Then eq. (4.2.5) (where the reader should now imagine that both $\alpha^{(\pm)}$ and $\alpha^{(\pm)\prime}$ depend on y and t besides k) implies that, if $\alpha^{(\pm)}$ satisfies the linear partial differential equation (4.1.12), so does $\alpha^{(\pm)\prime}$, since it coincides with $\alpha^{(\pm)}$ up to a factor of proportionality that is independent of y and t. But we know from the development of the preceding subsection that the linear equation (4.1.12) corresponds to the NLPDE (4.1.11) for v. We may therefore conclude that two pair of fields r, q and r', q' related by (4.2.3) have the property that, if r, q satisfy the NLPDE (4.1.11), r', q' satisfy the same equation.

Thus eq. (4.2.3) is a Bäcklund transformation, *i.e.* a relation that connects two fields v and v' that satisfy the same NLPDE (4.1.11). It should be emphasized that the functions $f(z)$ and $g(z)$ in (4.2.3) and (4.2.4) are arbitrary, as well as the functions $\boldsymbol{\nu}$ and γ in (4.1.11); the only connection between (4.2.3) and (4.1.11) is the structure of the operators Λ and L, with the latter being the limit of the former for $v' = v$.

The significance of the Bäcklund transformations (4.2.3) is directly evident from eq. (4.2.5), that displays their effect on the spectral parameters $\alpha^{(\pm)}$. The implications for the discrete spectrum, as long as it corresponds to the singularities of $\alpha^{(\pm)}$, can also be evinced from this formula. It is also possible to study more directly the effects of these Bäcklund transformations on the parameters of the discrete spectrum using the results given above; for instance eqs. (3.3.12) imply that, if $k^{(\pm)}$ is a discrete eigenvalue for v' and not for v, then

$$f(k^{(\pm)}) \pm g(k^{(\pm)}) = 0\,, \tag{4.2.11}$$

consistently with eq. (4.2.5).

The Bäcklund transformations (4.2.3) have the same, quite general, structure for all the class of NLPDE's (4.1.11); note moreover that they contain no explicit dependence on the variables $\boldsymbol{y}$ and t. If they are used to generate a new solution v' of (4.1.11) out of a given solution v, they yield of course a dependence on $\boldsymbol{y}$ and t that obtains from the dependence of v from these variables (if any) and moreover from the « constants of integration » that arise on solving (4.2.3) for v'; these in fact depend generally on $\boldsymbol{y}$ and t (their constancy refers only to the x-dependence), in a manner that is characteristic of the particular NLPDE considered, and that may be ascertained by substituting the solution into it [27].

As is clear from eq. (4.2.5), it is a general property of the Bäcklund transformations (4.2.3) to commute; this highly nontrivial property has important implications [4], that shall be discussed in a subsequent paper of this series (except for a terse treatment in some special cases in the next two following subsections). We also defer a discussion of the general structure of (4.2.3), limiting our treatment here to a display of the very simplest cases that obtain with the simpler choices of the functions f and g.

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[27] The relation (4.2.3) is a generalized version of the formulae often referred to in the literature as « one half » of a Bäcklund transformation; see the papers of ref. [3] and, more specifically, those of ref. [4,19,20]. *Note added in proofs.* – The fact that the same Bäcklund transformation applies to all the equations of the AKNS class had been previously noted by H. H. CHEN: *Phys. Rev. Lett.*, **33**, 925 (1974) (but he only consideredt he simple Bäcklund transformations that are included in the class of eq. (4.2.13a) below, since the more general Bäcklund transformations introduced here, eq. (4.2.1), were not known, nor their spectral significance, eq. (4.2.2), understood).

For f and g constant, eqs. (4.2.3)-(4.2.5) yield

(4.2.12) $$r' = \lambda r\,, \qquad q' = \lambda^{-1} q\,, \qquad \alpha^{(\pm)\prime} = \lambda^{\pm 1}\alpha^{(\pm)}\,.$$

For f and g linear (and chosen so as to eliminate a simultaneous « scale » transformation such as (4.2.12)),

(4.2.13a) $$f(z) = p_+ + 2iz\,, \qquad g(z) = p_-$$

with (here and below)

(4.2.13b) $$p_\pm = \tfrac{1}{2}(p^{(+)} \pm p^{(-)})\,,$$

we get

(4.2.14a) $$r'_x + p^{(+)} r' + r' J = r_x + p^{(-)} r - rJ\,,$$

(4.2.14b) $$q'_x - p^{(-)} q' + q' J = q_x - p^{(+)} q - qJ\,,$$

where we have not explicitly indicated the x-dependence and

(4.2.15) $$J = J(x) = \int\limits_x^{+\infty} \mathrm{d}\xi\,[r'(\xi)\,q'(\xi) - r(\xi)\,q(\xi)]\,.$$

The two constants $p^{(+)}$ and $p^{(-)}$ are required to satisfy the conditions

(4.2.16) $$\pm\,\mathrm{Re}\,p^{(\pm)} > 0\,,$$

if one assumes that r and q vanish faster than exponentially as $x \to \pm\infty$, since eqs. (4.2.14) (together with the integral relation (4.2.20) given below) then imply

(4.2.17) $$r'(x) \sim \exp[-p^{(\pm)} x]\,, \qquad q'(x) \sim \exp[p^{(\mp)} x] \qquad \text{as } x \to \pm\infty\,.$$

This is consistent with the corresponding formula for the alphas, that reads

(4.2.18) $$\alpha^{(\pm)\prime}(k) = [(k - k^{(\mp)})/(k - k^{(\pm)})]\,\alpha^{(\pm)}(k)$$

with

(4.2.19) $$k^{(\pm)} = \frac{i}{2}\,p^{(\pm)}\,.$$

If one assumes that r', q' vanish faster than exponentially, then the signs in eq. (4.2.16) are reversed.

It should be noted that in this case $g_\pm = f_- = 0$, so that $\Theta^{(\pm)} = f_+$, and moreover $f_+ = -iJ(-\infty)$. Thus eq. (4.2.10) yields the integral identity

(4.2.20) $$J(-\infty) = p^{(-)} - p^{(+)}\,.$$

From eqs. (4.2.14) one easily obtains the formulae for the « soliton » solution, that were already given and discussed (in the case without γ-dependence) by AKNS. The explicit expression for r', q' is in fact easily obtained by assuming that r, q vanish. It is convenient to write these formulae in the form

$$(4.2.21a)\qquad r'(x) = -i\varrho^{(+)} \exp[-p^{(+)}x_0] \exp[-p_+(x-x_0)]/\cosh[p_-(x-x_0)]\,,$$

$$(4.2.21b)\qquad q'(x) = i\varrho^{(-)} \exp[p^{(-)}x_0] \exp[p_+(x-x_0)]/\cosh[p_-(x-x_0)]$$

with

$$(4.2.22)\qquad p_-^2 \exp[2p_-x_0] = -\varrho^{(+)}\varrho^{(-)}\,,$$

since the dependence upon the variables y and t can then be obtained directly from the formulae of the preceding subsection (see eqs. (4.1.18) and (4.1.19) and, if appropriate (4.1.21b) and (4.1.21c), and recall (4.2.19)) [28].

4'3. *Functional equation.* – Let us restrict our attention in this subsection to the case when $\nu = 0$, so that the NLPDE (4.1.11) reduces to the form (already considered by AKNS, but in the slightly less general case of time-independent γ)

$$(4.3.1)\qquad \sigma_3 v_t(x, t) + \gamma(L, t)\, v(x, t) = 0\,,$$

and the corresponding evolution of the spectral parameters is given by the simple formulae (4.1.23). Consider then the transformation (4.2.3), but now with functions f and g that depend also on time, and in such a manner that

$$(4.3.2)\qquad f(k, t) \mp g(k, t) = \exp\left[\mp \tfrac{1}{2}\int_{t_0}^{t} dt'\, \gamma(k, t')\right].$$

Comparison of eq. (4.2.5) (with this choice for f and g) to eq. (4.1.23a) implies that the field v' related to v by (4.2.3) and (4.2.4) (with this choice of f and g) is just the field into which v, given at time t_0, has evolved at time t, following the NLPDE (4.3.1). In other words (and after a little algebra) we have found that the remarkable functional equation

$$(4.3.3)\qquad \begin{pmatrix} r(x, t) \\ q(x, t') \end{pmatrix} = \exp\left[\int_{t}^{t'} d\tau\, \gamma(\Lambda, \tau)\right] \begin{pmatrix} r(x, t') \\ q(x, t) \end{pmatrix}$$

---

(28) Note that, since in this case both $\alpha^{(\pm)}$ and $\alpha^{(\pm)'}$ vanish, we are in fact extrapolating our results to a case in which the discrete spectrum cannot be obtained by analytic continuation from the alphas. A discussion of this point is deferred to a subsequent paper, as well as a more detailed analysis of this « soliton » solution when a nontrivial γ-dependence is present (in which case in general it does not behave like a soliton at all).

relates the solution $r(x, t)$, $q(x, t)$ of the NLPDE (4.3.1) at time t to the same solution at time t'. Of course the operator Λ in (4.3.3) is given by eq. (3.2.14), with $r' = r(x, t')$, $q' = q(x, t')$, $r = r(x, t)$, $q = q(x, t)$.

The functional equation (4.3.3) is an intriguing mathematical construct, and we propose to investigate it in some detail in a subsequent paper of this series. It yields nontrivial results even in the simplest cases, as shown by the following two examples that we report, for completeness, from ref. ([16a]).

i) If $\gamma(z, t) = 1$, eq. (4.1.11) becomes

$$r_t + r = 0\,, \qquad q_t - q = 0\,, \tag{4.3.4}$$

and eq. (4.3.3) yields directly the solution of this equation

$$r(t') = r(t)\exp[t - t']\,, \qquad q(t') = q(t)\exp[t' - t]\,. \tag{4.3.5}$$

ii) If $\gamma(z, t) = 2iz$, eq. (4.1.11) becomes

$$r_t + r_x = 0\,, \qquad q_t + q_x = 0\,, \tag{4.3.6}$$

and has therefore the solution

$$r(x, t) = f(x - t)\,, \qquad q(x, t) = g(x - t) \tag{4.3.7}$$

with $f(z)$ and $g(z)$ arbitrary functions (vanishing for $z \to \pm\infty$). Inserting this solution in (4.3.3) we get the remarkable nonlinear operator identity

$$\begin{pmatrix} f(z) \\ g(z+a) \end{pmatrix} = \exp[-aC]\begin{pmatrix} f(z+a) \\ g(z) \end{pmatrix} \tag{4.3.8}$$

with

$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial z} + \begin{pmatrix} f'Ig' + fIg & -f'If - fIf' \\ g'Ig + gIg' & -g'If' - gIf \end{pmatrix}, \tag{4.3.9}$$

where we have written for short $f' = f(z + a)$, $g' = g(z + a)$, $f = f(z)$, $g = g(z)$ and, as above, $I = \int_z^{+\infty} dz'$. The arbitrariness of $f(z)$ and $g(z)$, that are only required to vanish asymptotically and to be infinitely differentiable, should be emphasized. The special choice $g(z) = \lambda f(z)$ yields

$$f(z + a) = E_+^{-1}(1 - \lambda E_-)f(z)\,, \tag{4.3.10}$$

where the operators $E_{\pm}$ are defined by

$$E_{\pm} = (\chi_{+}, \exp[-aC]\,\chi_{\pm}) \tag{4.3.11}$$

with $\chi_{\pm}$ defined by eq. (2.1.2) and of course C given by (4.3.9) with $g' = \lambda f'$, $g = \lambda f$. For $\lambda = 0$ eq. (4.3.11) goes over into the well-known linear operator formula

$$f(z + a) = \exp\left[a\frac{\partial}{\partial z}\right] f(z) . \tag{4.3.12}$$

A comparison of eqs. (4.3.2), (4.2.5) and (4.1.23*a*) has allowed us to conclude that the functional equation (4.3.3) relates the same solution of eq. (4.3.1) at different times. We are therefore now also allowed to conclude, from eqs. (4.1.23*b*) and (4.2.6), that if f and g are given by eq. (4.3.2), we have

$$(f \pm g)\,\Theta^{(\mp)} = (f \mp g)\,\Theta^{(\pm)} . \tag{4.3.13}$$

This equation, together with (4.2.10), implies

$$\Theta^{(\pm)} = 0 , \tag{4.3.14}$$

or equivalently (see (4.2.7))

$$f_{\pm} + g_{\pm} = 0 . \tag{4.3.15}$$

The two equations (4.3.15) may be rewritten in terms of the definitions (3.2.26) and the formula defining the Bäcklund transformation (most conveniently in the form (4.2.1)). In this manner one gets

$$\int_{-\infty}^{+\infty} dx \left(v_{\pm}(x, t, t'), \sigma_{\pm}(z - \Lambda)^{-1}\left\{\cosh\left[\tfrac{1}{2}\int_t^{t'} d\tau\, \gamma(z, \tau)\right]\sigma_3 v_{-}(x, t, t') + \right.\right. \\ \left.\left. + \sinh\left[\tfrac{1}{2}\int_t^{t'} d\tau\,\gamma(z, \tau)\right] v_{+}(x, t, t')\right\}\right) = 0 , \tag{4.3.16}$$

where of course

$$v_{\pm}(x, t, t') = \tfrac{1}{2}[v(x, t') \pm v(x, t)] , \qquad \sigma_{+} \equiv i\sigma_2 , \qquad \sigma_{-} \equiv \sigma_1 , \tag{4.3.17}$$

and the operator Λ is defined by eq. (3.2.14) with $r' = r(x, t')$, $q' = q(x, t')$, $r = r(x, t)$, $q = q(x, t)$.

Also these equations are functional relations connecting the same solution of the NLPDE (4.3.1) at different times; note that they contain the parameter z, that may take any value. They constitute in some sense a generalization, for finite time intervals, of the infinitely many conservation laws that are known to characterize the NLPDE's of the class (4.3.1). Indeed these conservation laws can be easily derived by taking the limit of (4.3.16) for $t'-t\to 0$ (and using the results of the appendix); but we prefer to defer a discussion of this question to a subsequent paper of this series.

4'4. *Special cases.* – Using the results of subsect. **2'4** one can easily analyse possible subclasses of solutions of the NLPDE (4.1.11), as well as subclasses of NLPDE's involving only one field. This we do in this subsection, that ends with a terse treatment of some specific results in the special case of the sine-Gordon equation, singled out as an example in view of its special importance.

In the following formulae ε is always such that $\varepsilon^2=1$, *i.e.* $\varepsilon=+1$ or $\varepsilon=-1$. We do not list below the properties of the spectral parameters (alphas and betas), since they can in each case be easily evinced from the formulae of subsect. **2'4**. These properties are of course instrumental for the derivation of the results reported below.

Subclasses of solutions. We list below 3 cases.

i) If in (4.1.11)

$$\boldsymbol{\nu}(z, y, t)=\boldsymbol{\nu}(-z^*, y, t)\,, \qquad \gamma(z, y, t)=\gamma^*(-z^*, y, t)\,, \tag{4.4.1}$$

the relation

$$r(x, y, t)=\varepsilon r^*(x, y, t)\,, \qquad q(x, y, t)=\varepsilon q^*(x, y, t) \tag{4.4.2}$$

is consistent with (4.1.11), namely, if true at $t=t_0$, it remains true for $t>t_0$. Moreover, if this relation is true for r, q, it may also be true for r', q' obtained from r, q via a Bäcklund transformation (4.2.3), provided the functions f and g that characterize it satisfy the relation

$$f(z)\,g^*(-z^*)=f^*(-z^*)\,g(z)\,. \tag{4.4.3}$$

Note that this condition is necessary but not sufficient, since additional restrictions (easy to ascertain and to implement) must of course be imposed on the constants of integration that obtain by « solving » the Bäcklund transformation for r', q'. In the case of the linear Bäcklund transformation (4.2.13) this condition implies $p^{(\pm)}=p^{(\pm)*}$, namely the poles are constrained to occur on the imaginary axis. The corresponding restrictions for the residues, in the case of the soliton solution (4.2.21), is $\varrho^{(\pm)}=-\varepsilon\varrho^{(\pm)*}$.

ii) If in (4.1.11)

$$\boldsymbol{\nu}(z, y, t)=\boldsymbol{\nu}(-z, y, t)\,, \qquad \gamma(z, y, t)=\gamma(-z, y, t)\,, \tag{4.4.4}$$

the relation

$$(4.4.5)\qquad r(x, y, t) = \varepsilon r(-x + a, y, t)\,, \qquad q(x, y, t) = \varepsilon q(-x + a, y, t)$$

is consistent with (4.1.11). The corresponding (in the sense detailed above) condition for the Bäcklund transformations is

$$(4.4.6)\qquad f(z)\, g(-z) = f(-z)\, g(z)\,.$$

No linear Bäcklund transformation exists consistent with this condition (except the trivial one with constants f, g corresponding to simple scaling of r, q).

iii) If in (4.1.11)

$$(4.4.7)\qquad \nu(z, y, t) = \nu(z^*, y, t)\,, \qquad \gamma(z, y, t) = \gamma^*(z^*, y, t)\,,$$

the relation

$$(4.4.8)\qquad r(x, y, t) = \varepsilon r^*(-x + a, y, t)\,, \qquad q(x, y, t) = \varepsilon q^*(-x + a, y, t)$$

is consistent with (4.1.11). The corresponding condition for the Bäcklund transformation is

$$(4.4.9)\qquad f(z)\, g^*(z^*) = f^*(z^*)\, g(z)\,.$$

Thus a linear Bäcklund transformation may be compatible with (4.4.7) only if it yields poles on the real axis.

This completes our list of possible special solutions of (4.1.11). A more interesting class of special cases obtains from transformations that include (2.4.3), and therefore relate r to q, since one obtains in this manner NLPDE's for one field only. We list below the 4 cases that obtain in this manner.

Subclasses of NLPDE's *for one field.*

i) If in eq. (4.1.11)

$$(4.4.10)\qquad \nu(z, y, t) = \nu(-z, y, t)\,, \qquad \gamma(z, y, t) = -\gamma(-z, y, t)\,,$$

it is consistent to set

$$(4.4.11)\qquad r(x, y, t) = \varepsilon q(x, y, t)\,,$$

obtaining thereby a single NLPDE for the field q (or rather 2 different NLPDE's, depending on whether $\varepsilon = +1$ or $\varepsilon = -1$). A necessary condition for the

Bäcklund transformations (4.2.3) to be consistent with (4.4.11) is

$$f(z)\,g(-z) = -f(-z)\,g(z)\,. \tag{4.4.12}$$

In the special case of linear Bäcklund transformations this implies $p^{(+)} = -p^{(-)}$, or equivalently $p_{+} = 0$. The corresponding restriction for the residues in the case of the soliton solution is $\varrho^{(+)} = -\varepsilon\varrho^{(-)}$. This case is a particularly interesting one, since it includes the sine-Gordon and modified KdV equations; it is further discussed below.

ii) If in eq. (4.1.11)

$$\boldsymbol{\nu}(z, \boldsymbol{y}, t) = \boldsymbol{\nu}(z^*, \boldsymbol{y}, t)\,, \qquad \gamma(z, \boldsymbol{y}, t) = -\gamma^*(z^*, \boldsymbol{y}, t)\,, \tag{4.4.13}$$

it is consistent to set

$$r(x, \boldsymbol{y}, t) = \varepsilon q^*(x, \boldsymbol{y}, t)\,. \tag{4.4.14}$$

The necessary condition on the Bäcklund transformation (4.2.3) to be consistent with (4.4.14) is

$$f(z)\,g^*(z^*) = -f^*(z^*)\,g(z)\,. \tag{4.4.15}$$

In the linear case this implies $p^{(+)} = -p^{(-)*}$, and, for the soliton solution, one has the restriction $\varrho^{(+)} = \varepsilon\varrho^{(-)*}$. This case is also important, since it includes the nonlinear Schrödinger equation (it coincides in fact with the case originally studied by ZAKHAROV and SHABAT ([3f])).

iii) If in eq. (4.1.11)

$$\gamma(z, \boldsymbol{y}, t) = 0\,, \tag{4.4.16}$$

it is consistent to set

$$r(x, \boldsymbol{y}, t) = \varepsilon q(-x + a, \boldsymbol{y}, t)\,. \tag{4.4.17}$$

The corresponding limitation on the Bäcklund transformations is $g = 0$ (namely, there is no Bäcklund transformation consistent with (4.4.17)). Clearly this is not an interesting case.

iv) If in eq. (4.1.11)

$$\boldsymbol{\nu}(z, \boldsymbol{y}, t) = \boldsymbol{\nu}(-z^*, \boldsymbol{y}, t)\,, \qquad \gamma(z, \boldsymbol{y}, t) = -\gamma^*(-z^*, \boldsymbol{y}, t)\,, \tag{4.4.18}$$

it is consistent to set

$$r(x, \boldsymbol{y}, t) = \varepsilon q^*(-x + a, \boldsymbol{y}, t)\,, \tag{4.4.19}$$

obtaining thereby a nonlinear functional evolution equation for the field q, that contains its values at x and the values of its complex conjugate at $-x+a$. The subclass of Bäcklund transformations (4.2.3) consistent with (4.4.19) is constrained by the necessary condition

$$f(z)\,g^*(-z^*) = -f^*(-z^*)\,g(z)\,. \tag{4.4.20}$$

The poles are however constrained to occur on the real axis, since one has the condition $p^{(+)*} = p^{(-)}$. Clearly this is a rather peculiar example; we are not aware of its having been considered by other authors.

This concludes our list of the cases in which one obtains from (4.1.11) a solvable NLPDE for a single field. We have not written these equations explicitly, since the simpler procedure is to obtain them in each case from (4.1.11).

We end this paper discussing the implications of the permutability of Bäcklund transformations in the special case of the equation for the single field q of case i) above (see (4.4.10)). We note first of all that, after a little algebra [29], the linear Bäcklund transformation consistent with (4.4.11) yields

$$q'(x) = q(x) + p\sin[Q'(x) + Q(x)]\,. \tag{4.4.21}$$

In this formula

$$p = p^{(+)} = -p^{(-)}\,, \tag{4.4.22}$$

$$Q(x) = \int_x^{+\infty} d\xi\, q(\xi)\,, \qquad Q'(x) = \int_x^{+\infty} d\xi\, q'(\xi)\,, \tag{4.4.23}$$

and we have assumed $\varepsilon = -1$ (if $\varepsilon = +1$, the sine is replaced by the hyperbolic sine; we do not write explicitly the results for this case).

We exploit now the permutability of two Bäcklund transformations of this kind, characterized by parameters p_1, p_2, obtaining, with an obvious meaning of the symbols, the nonlinear superposition formula [4]

$$\sin Q_3 = \{\sin Q_1[(p_1^2 + p_2^2)\cos(Q_2 - \bar{Q}_2) - 2p_1p_2] + (p_1^2 - p_2^2)\cos Q_1 \sin(Q_2 - \bar{Q}_2)\}[p_1^2 + p_2^2 - 2p_1p_2\cos(Q_2 - \bar{Q}_2)]^{-1}\,. \tag{4.4.24}$$

It is immediately seen that this formula implies $Q_3 = Q_1 \pmod{2\pi}$ if $p_2 = -p_1$.

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(29) Write explicitly the linear Bäcklund transformation for the fields (using eqs. (4.2.3) and (4.2.13a)), introduce the function $w(x) = \int_x^\infty dx'\, q^2(x')$ (and $w'(x)$, similarly related to $q'(x)$), solve for $w' - w$ in terms of $q' + q$ (choosing appropriately the sign in the solution of the second-degree equation), differentiate, simplify, and finally integrate using the asymptotic boundary conditions $Q(+\infty) = Q'(+\infty) = q(+\infty) = q'(+\infty) = 0$.

If instead $p_2 = p_1 = p$, one obtains the interesting formula

$$\sin Q'' = \sin Q + 2pQ'_p(\cos Q - pQ'_p \sin Q)(1 + p^2 Q'^2_p)^{-1}, \tag{4.4.25}$$

where Q' is the field obtained from Q by the Bäcklund transformation with parameter p, and Q'_p indicates the partial derivative of Q' with respect to p. Q'' indicates of course the field obtained from Q by a double application of the Bäcklund transformation (so that the corresponding spectral parameters $\alpha^{(\pm)''}$ contain generally a double pole).

If $Q = 0$, eq. (4.4.25) provides an explicit solution of the NLPDE, since in this case Q' is known (see eqs. (4.2.21) and (4.4.22)):

$$Q' = 2 \operatorname{arctg} \{\exp [p(x - x_0)]\}. \tag{4.4.26}$$

Note however that, to get Q'_p, account must also be taken of the p-dependence of x_0, that is characteristic of the particular NLPDE being considered (while the soliton expression (4.4.26) is instead common to all the equations of the class (4.1.11) with (4.4.10) and (4.4.11)). For instance, for the special case $\nu = 0$, $\gamma(z, y, t) = -(2iz)^{-1}$, that is an interesting one since it yields for

$$\varphi = 2Q \tag{4.4.27}$$

the sine-Gordon equation

$$\varphi_{xt} = \sin \varphi, \tag{4.4.28}$$

one finds $x_0 = t/p^2$, so that eq. (4.4.25) yields

$$\varphi = 2 \operatorname{arctg} \{2u_+ \cosh u_- / [\cosh^2 u_- - u_+^2]\} \tag{4.4.29}$$

with

$$u_\pm = px \pm t/p. \tag{4.4.30}$$

The physical significance of this solution is best discussed going over to the variables $X = x - t$, $T = x + t$. Note that, in terms of these variables,

$$u_- = (1 - v^2)^{-\frac{1}{2}}(X - vT), \tag{4.4.31a}$$

$$u_+ = -(1 - v^2)^{-\frac{1}{2}}(vX - T) \tag{4.4.31b}$$

with

$$v = (1 - p^2)/(1 + p^2); \tag{4.4.32}$$

while the sine-Gordon equation takes its proper (relativistically invariant) form

$$\varphi_{TT} - \varphi_{XX} = \sin\varphi \,. \tag{4.4.33}$$

Assuming that this originates from a classical field theory with Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(-\varphi_T\varphi_T + \varphi_X\varphi_X) - (1-\cos\varphi) \tag{4.4.34}$$

one obtains for the energy density

$$\mathscr{H} = \tfrac{1}{2}(\varphi_T\varphi_T + \varphi_X\varphi_X) + (1-\cos\varphi) \tag{4.4.35}$$

corresponding to the special solution (4.4.29) the expression

$$\mathscr{H} = 4(\cosh^2 u_- + u_+^2)^{-2}[p^2(\cosh u_- - u_+ \sinh u_-)^2 + \\ + p^{-2}(\cosh u_- + u_+ \sinh u_-)^2 + 2u_+^2 \cosh^2 u_-] \,, \tag{4.4.36}$$

which clearly vanishes as $T\to\infty$, even if X diverges (in contrast to the soliton solution (4.4.26), which yields

$$\mathscr{H} = 4(1-v^2)^{-1}\cosh^{-2} u_- , \tag{4.4.37}$$

so that in this well-known case $\mathscr{H}$ remains constant if T and X both diverge keeping u_- constant; this solution represents of course a disturbance, the soliton, moving with velocity V).

Note added in proofs.

The special solution of the sine-Gordon equation discussed here had been previously obtained by G. L. LAMB jr.: *Rev. Mod. Phys.*, **43**, 99 (1971); it can also be recovered from the « breather » solution by an appropriate limiting procedure (private communication by L. D. FADDEEV, P. P. KULISH and L. A. TAKHTAJAN).

APPENDIX

In this appendix we show some important properties of the operator L, which follow directly from its definition

$$L = \frac{1}{2i}\left[\sigma_3 \frac{\partial}{\partial x} + 2v(x)\int_x^\infty d\xi\,[i\sigma_2 v(\xi)]^T\right]. \tag{A.1}$$

Here $v(x)$ is the vector (2.2.3) and the matrix operator L has been expressed in terms of the well-known dyadic notation.

Let us introduce the operator-valued function of the complex variable z

$$A(z) = (1-2izL)^{-1} \tag{A.2}$$

that satisfies the equation

$$A(z) = 1 + 2izLA(z), \tag{A.3}$$

and the vector-valued function

$$\lambda(z; x) = A(z)v(x). \tag{A.4}$$

Note that all coefficients of the power expansion in z of $\lambda(z; x)$ vanish as $x \to \pm\infty$. Equation (A.3) then implies that

$$\lambda(z; x) = [1 + 2zJ(z; x)]v(x) + z\sigma_3\lambda_x(z; x), \tag{A.5}$$

where we have defined

$$J(z; x) = \int_x^{+\infty} d\xi \left(i\sigma_2 v(\xi), \lambda(z; \xi)\right). \tag{A.6}$$

Note that, for $z = 0$, $\lambda = v$ and $J = 0$ (due to the antisymmetry of σ_2).

The following differential relation can be easily derived from (A.5):

$$(z_2 - z_1)\left(\lambda(z_1), i\sigma_2\lambda(z_2)\right) = -\frac{d}{dx}\left[z_1 z_2\left(\lambda(z_1), \sigma_1\lambda(z_2)\right) + z_1 J(z_1) + z_2 J(z_2) + 2z_1 z_2 J(z_1) J(z_2)\right]. \tag{A.7}$$

Here we have, for notational simplicity, not indicated the argument x.

If we set now $z_1 = z_2 = z$ in this expression, it follows that the complex function

$$C(z) \equiv z\left(\lambda(z), \sigma_1\lambda(z)\right) + 2J(z) + 2zJ^2(z) \tag{A.8}$$

is independent of x, and therefore the coefficients of its power expansion in z are also x-independent (for those values of x such that this power expansion is meaningful). On the other hand, these coefficients can be expressed in terms of $\left(L^n v(x), \sigma_1 L^m v(x)\right)$ and $\int_x^{+\infty} d\xi \left(i\sigma_2 v(\xi), L^n v(\xi)\right)$ only (n, m nonnegative integers), that vanish as $x \to +\infty$ since we assume $v(x)$ to vanish in this limit with all its derivatives. We therefore conclude that the function $C(z)$ vanishes or, equivalently, that

$$J(z) = -z\left(\lambda(z), \sigma_1\lambda(z)\right)\left[1 + \left(1 - 2z^2\left(\lambda(z), \sigma_1\lambda(z)\right)\right)^{\frac{1}{2}}\right]^{-1}. \tag{A.9}$$

From this formula and the remark (A.4) it then follows that the coefficients of the power expansion in z of $J(z, x)$ vanish in the $x \to -\infty$ limit, so that

$$\text{(A.10)} \qquad \int_{-\infty}^{+\infty} dx\,\big(v(x), i\sigma_2 L^n v(x)\big) = 0\,, \qquad n = 0, 1, 2, \ldots .$$

An immediate consequence of these integral relations is the complete equality of the class of NLPDE's given by AKNS to the class given in this paper (or rather to a subclass of these, as explained above). To show this it suffices to prove that

$$\text{(A.11)} \qquad L^n v(x) = L_-^n v(x)\,,$$

where L_- is the integro-differential operator defined by (1.3) and introduced by AKNS ([39]). We begin by noting that

$$\text{(A.12)} \qquad L_- = L + T\,, \qquad T = iv(x)\int_{-\infty}^{+\infty} d\xi\,[i\sigma_2 v(\xi)]^T\,,$$

and that the operator T annihilates all vectors obtained from $v(x)$ by repeated application of L

$$\text{(A.13)} \qquad TL^n v(x) = 0\,, \qquad n = 0, 1, 2, \ldots .$$

The equality (A.11) then follows immediately from this formula and the definition (A.12).

Equation (A.9) obtains upon solving (A.8) for J^{-1}. If one solves instead for J and then substitutes in (A.5), one gets

$$\text{(A.14)} \qquad \lambda(z; x) = \big[1 - 2z^2\big(\lambda(z; x), \sigma_1 \lambda(z; x)\big)\big]^{\frac{1}{2}} v(x) + z\sigma_3 \lambda_x(z; x)\,.$$

Equating the coefficients of the expansion in powers of z of this formula one concludes (by recursion) that the vectors $L^n v(x)$ do not contain any integral expression of the functions $r(x)$ and $q(x)$, being instead expressed only in terms of products of these two functions and of their derivatives. This result had been reported already in sect. **4**.

Another interesting formula is the equality

$$\text{(A.15)} \qquad \int_{-\infty}^{+\infty} dx\,\big(L^n v(x), i\sigma_2 L^m v(x)\big) = 0\,, \qquad n, m = 0, 1, 2, \ldots,$$

that shall play an important rôle in the discussion of the infinitely many constants of motion associated with the class of solvable NLPDE's with only one space co-ordinate (see the last part of subsect. **4**·3). In order to prove this equation we perform a double expansion in powers of z_1 and z_2 of (A.7), and equate the coefficients; in this way one concludes that the quantities $\big(L^n v(x), i\sigma_2 L^m v(x)\big)$ are exact differentials, and this, together with (A.10), implies (A.15).

Finally, using (A.10), we obtain from the definition (4.1.8) of the function $\bar{\varphi}(k, y, t)$ the expression (4.1.14) given in subsect. 4.1. In fact, by using the NLPDE (4.1.5), the equation (4.1.8) may be reduced to

$$\text{(A.16)}\qquad \bar{\varphi}(k, y, t) = 2if(k)\int_{-\infty}^{+\infty} dx \left(v(x), i\sigma_2(k-L)^{-1}\left[\sigma_3 v_t(x) + \nu(k)\frac{\partial}{\partial y}\sigma_3 v(x)\right]\right),$$

since the contribution of the integral $\int_{-\infty}^{+\infty} dx\,\big(v(x), i\sigma_2(k-L)^{-1}v(x)\big)$ vanishes as a consequence of (A.10). If the evolution equation is then used to eliminate $\sigma_3 v_t(x)$ in (A.16), we are left with the final expression

$$\text{(A.17)}\qquad \varphi(k, y, t) = \bar{\varphi}(k, y, t)/f(k) = 2i\int_{-\infty}^{+\infty} dx \left(v(x), i\sigma_2\left[\frac{\nu(k)-\nu(L)}{k-L}\right]\frac{\partial}{\partial y}\sigma_3 v(x)\right),$$

where once again the integral $\int_{-\infty}^{+\infty} dx\big(v(x), i\sigma_2(k-L)^{-1}\gamma(L)v(x)\big)$ has been eliminated using (A.10).

● RIASSUNTO

Questo lavoro è il primo di una serie dedicata ad un metodo generale per trovare e studiare equazioni non lineari alle derivate parziali risolubili per mezzo della tecnica della trasformata spettrale inversa. In questo articolo si presentano i risultati che si ottengono applicando questo metodo al problema lineare generalizzato di Zakharov-Shabat. Si dà una classe di equazioni di evoluzione nonlineari, solubili con la trasformata spettrale inversa, che è più generale di quella presentata da Ablowitz, Kaup, Newell e Segur, poiché si includono anche equazioni contenenti coefficienti non costanti e più di una variabile spaziale. Riportiamo inoltre una classe molto generale di trasformazioni di Bäcklund che contiene tutte le trasformazioni già note e ne chiarisce il significato. Infine otteniamo, per una classe più ristretta di equazioni nonlineari di evoluzione (contenenti solo una variabile spaziale), un'interessante equazione funzionale che lega la soluzione al tempo t alla stessa soluzione al tempo t'. Questo articolo è dedicato ad una presentazione generale del metodo ed alla dimostrazione dei risultati principali (alcuni dei quali sono già stati pubblicati senza dimostrazione). Sebbene l'analisi di equazioni particolari e di soluzioni speciali è rimandata ai lavori successivi di questa serie, alcuni risultati di questo tipo sono già presenti in questo lavoro, tra i quali l'espressione esplicita della soluzione esatta, non di tipo solitone, dell'equazione sine-Gordon, che corrisponde ad un polo doppio dei corrispondenti parametri spettrali.

Нелинейные уравнения эволюции, решаемые с помощью обратного спектрального преобразования - I.

Резюме (*). — Эта статья является первой статьей из серии, основанной на общем методе для исследования нелинейных дифференциальных уравнений в частных производных, решаемых с помощью техники обратного спектрального преобразования. Результаты, полученные в этой статье, аналогичны результатам, которые получаются при применении этого метода к обобщенной линейной проблеме Захарова-Шабата. Мы приводим класс нелинейных уравнений эволюции, решаемых с помощью обратного спектрального преобразования. Этот класс является более общим, чем класс, введенный Абловитцем, Каупом, Невеллом и Сегуром, т.к. он содержит уравнения, включающие более чем одну пространственную переменную и содержащие коэффициенты, которые не являются постоянными. Мы также рассматриваем очень общий класс преобразований Беклунда, который содержит все такие преобразования, которые были рассмотрены ранее. Проводится анализ физического смысла этих преобразований. Для случая менее общего класса нелинейных уравнений эволюции (включающего только одну пространственную переменную) мы получаем функциональное уравнение, которое связывает решение в момент времени t с тем же решением в момент времени t'. Основное внимание в статье уделяется общему подходу и доказательству основных результатов (некоторые из которых были приведены ранее без доказательств). Хотя анализ специальных уравнений и специальных решений отложен на последующие статьи этой серии, в этой работе приводится несколько результатов такого рода, которые включают точное несолитонное решение уравнения Гордона, соответствующего двойному полюсу ассоциированного спектрального параметра.

(*) *Переведено редакцией.*