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On the Phase Factors in Inversions (*).

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Summary. — The phase factors which can appear in the definition of the inversions, $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{T}$ and their products are discussed. It is shown that because of the existence of «physically equivalent» Hamiltonians, the phases in $\boldsymbol{C}$, $\boldsymbol{CP}$, $\boldsymbol{T}$ and $\boldsymbol{TP}$ for complex fields are unmeasurable. For the remaining inversions, it is possible to construct interactions which require more general phases for complex fields than the usual ± 1, $\pm i$, when and only when the theory contains certain discrete multiplicative symmetries. Examples of such interactions are given.

1. – Introduction.

The transformations of the field operators of a quantized field theory, which are generated when the state vectors undergo inversion operations such as space reflection ($\boldsymbol{P}$), charge conjugation ($\boldsymbol{C}$) and time reversal ($\boldsymbol{T}$) are not entirely determined *a priori*. In particular, when these operators act on the complex fields which represent particles different from their antiparticles, there is the possibility of introducing arbitrary complex numbers of modulus 1 (phase factors) into the action of the inversion on such fields, while still maintaining their unitary character, the invariance of the free field Lagrangian and the

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required transformation properties of free field observables such as currents, momenta, etc. The existence of this formal possibility has long been realized, but there has been some dispute as to its physical content, particularly in the case of space reflection. The purpose of this paper is to discuss the restrictions on the phase factors which may exist in field theories and the physical content of such restrictions. A consequence of our discussion is a specification of which phases are purely conventional and which are fixed by a particular theory, and hence may, at least in principle, be determined by experiment. The main results are contained in the following two statements:

1) In a theory invariant under charge conjugation the phase factors which occur in the transformation of complex fields under $\boldsymbol{C}$ can always be chosen to have any value for any such field without altering the physical content of a theory, and so are not observable quantities.

A similar result holds for the phase factors occurring in time reversal. However, the product of the phases in $\boldsymbol{C}$ and $\boldsymbol{T}$ cannot be chosen arbitrarily, and is related to the phase factor in $\boldsymbol{P}$ for each field, in a way to be discussed, provided that the $\boldsymbol{TCP}$ theorem is satisfied.

2) In a theory invariant under space reflection it is not in general possible to restrict the phase factor in $\boldsymbol{P}$ to the «usual» values (± 1 for boson fields, and ± 1, $\pm i$ for fermion fields). That is, it is possible to construct theories which are invariant under space reflection operations involving more general phase factor than these, but which are not invariant with any of the usual phase factors.

Such theories are characterized by the existence of new multiplicative symmetry operations which are not part of continuous gauge groups. No examples are known for any of the commonly accepted interactions of the known elementary particles. While the formal theory of inversions can be carried through most easily by working with the fields and the corresponding phase factors, the physically interesting quantities are the phase factors occurring in the transformation of states, the so-called intrinsic parities. It will be shown that intrinsic parities may be compared only for states which have the same transformation under all multiplicative symmetry operations.

In the second section of this paper, we will define the operations $\boldsymbol{P}$, $\boldsymbol{C}$, $\boldsymbol{T}$ as well as certain other operators and discuss several of their properties. In the third section, we discuss the ambiguity in inversions due to the existence of many physically equivalent Hamiltonians, and justify the first statement above. In the fourth section we discuss the consequence for inversions of the existence of multiplicative symmetries. In the final section we will prove certain relations among products of the inversions.

2. – Inversion operators and multiplicative operators.

In this paper, we consider only theories in which the degeneracy of the 1 particle states is completely specified by the spins and the particle-antiparticle character. This means that the transformations representing the inversions can only take the field operator into itself or into its Hermitian conjugate, rather than permuting fields which refer to different particles. Theories in which there is additional degeneracy and in which the space reflection and time reversal operators are more general have been considered by WIGNER and by MICHEL and WIGHTMAN (1).

The form of the inversion operators is chosen to make the observables transform in accordance with the classical interpretation of these operations. For example, we make the *a priori* requirement that space reflection should not change particles to antiparticles but should change the sign of momentum, while charge conjugation should change particle to antiparticle without changing momentum. The use of the term parity for an operation which does not satisfy the first criterion appears unwarranted on the basis of the classical concept of space reflection.

Let $\varphi_m(X)$ be the operators for spinless boson fields, $\varphi_{\mu m}(X)$ the operators for spin 1 fields and $\psi_m(X)$ the operators for spin $\frac{1}{2}$ fields. The three inversions are defined by the following equations:

$$(1)\qquad \begin{cases} \boldsymbol{C}\varphi_m(X)\boldsymbol{C}^{-1} = n_m^c\varphi_m^\dagger(X)\,; & \boldsymbol{C}\varphi_m^\dagger(X)\boldsymbol{C}^{-1} = n_m^{*c}\varphi_m(X)\,,\\ \boldsymbol{C}\varphi_{\mu m}(X)\boldsymbol{C}^{-1} = n_m^c\varphi_{\mu m}^\dagger(X)\,; & \boldsymbol{C}\varphi_{\mu m}^\dagger(X)\boldsymbol{C}^{-1} = n_m^{*c}\varphi_{\mu m}(X)\,,\\ \boldsymbol{C}\psi_m(X)\boldsymbol{C}^{-1} = n_m^c\, C\,\overline{\psi}_m^T(X)\,; & \boldsymbol{C}\overline{\psi}_m(X)\boldsymbol{C}^{-1} = -\,n_m^{*c}\psi_m^T(X)C^+. \end{cases}$$

Here C is the usual charge conjugation matrix satisfying

$$C\gamma_\mu^T C^{-1} = -\gamma_\mu\,.$$

We use hermitian γ_μ satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\,, \qquad (\mu = 1,\ 2,\ 3,\ 4,\ \text{always})\,.$$

The superscript T on a matrix or a field indicates transposition in the spin space. The dagger operation means hermitian conjugation in Hilbert space,

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(1) See L. MICHEL and A. WIGHTMAN: *Lecture notes at Princeton University* (unpublished).

together with transposition of spinor indices. The operator $\boldsymbol{C}$ is linear and unitary. For spin 1 fields we shall let μ take the values 0, 1, 2, 3, where $\varphi_0 \equiv -i\varphi_4$.

$$(2)\quad \begin{cases} \boldsymbol{P}\varphi_m(X)\boldsymbol{P}^{-1} = n_m^P\varphi_m(\varrho X); & \boldsymbol{P}\varphi_m^\dagger(X)\boldsymbol{P}^{-1} = n_m^{P*}\varphi_m^\dagger(\varrho X)\,, \\ \boldsymbol{P}\varphi_{\mu m}(X)\boldsymbol{P}^{-1} = n_m^P\varrho\varphi_{\mu m}(\varrho X); & \boldsymbol{P}\varphi_{\mu m}^\dagger(X)\boldsymbol{P}^{-1} = n_m^{P*}\varrho\varphi_{\mu m}^\dagger(\varrho X)\,, \\ \boldsymbol{P}\psi_m(X)\boldsymbol{P}^{-1} = in_m^P\gamma_4\psi_m(\varrho X); & \boldsymbol{P}\overline{\psi}_{\mu m}(X)\boldsymbol{P}^{-1} = -in_m^{P*}\overline{\psi}_m(\varrho X)\gamma_4\,. \end{cases}$$

Here ϱX represents the transformed coordinates, *i.e.* $\varrho(\boldsymbol{X}, t) = (-\boldsymbol{X}, t)$. Similarly $\varrho\varphi_{im} = -\varphi_{im}$ for the space components of a vector field, and $\varrho\varphi_{4m} = \varphi_{4m}$ for the time component of a vector field. $\boldsymbol{P}$ is also a linear unitary operator.

$$(3)\quad \begin{cases} \boldsymbol{T}\varphi_m(X)\boldsymbol{T}^{-1} = n_m^T\varphi_m(\tau X); & \boldsymbol{T}\varphi_m^\dagger(X)\boldsymbol{T}^{-1} = n_m^{T*}\varphi_m^\dagger(\tau X)\,, \\ \boldsymbol{T}\varphi_{\mu m}(X)\boldsymbol{T}^{-1} = n_m^T\tau\varphi_m(\tau X); & \boldsymbol{T}\varphi_{\mu m}^\dagger(X)\boldsymbol{T}^{-1} = n_m^T\tau\varphi_{\mu m}^\dagger(\tau X)\,, \\ \boldsymbol{T}\psi_m(X)\boldsymbol{T}^{-1} = n_m^T C^{-1}\gamma_5\psi_m(\tau X); & \boldsymbol{T}\overline{\psi}_m(X)\boldsymbol{T}^{-1} = n_m^{T*}\overline{\psi}_m(\tau X)\gamma_5 C\,. \end{cases}$$

$\boldsymbol{T}$ is an antiunitary operator, *i.e.* $\boldsymbol{T}\lambda\boldsymbol{T}^\dagger = \lambda^*$ for any c-number. τX again represents the transformed coordinate, *i.e.* $\tau(X, t) = (X, -t)$.

The conditions that $\boldsymbol{C}, \boldsymbol{P}, \boldsymbol{T}$ should be unitary (antiunitary) and that the free field Lagrangian is invariant restrict n_m^C, n_m^P, n_m^T to be phase factors, *i.e.* $|n_m^{C,P,T,}| = 1$ for any field.

For the «real» fields which represent particles identical to their antiparticles, one has the additional conditions

$$(4)\quad \begin{cases} \varphi_r = \varphi_r^\dagger \equiv \varphi_r^C\,, \\ \varphi_\mu = \varphi_{\mu r}^\dagger \equiv \varphi_{\mu r}^C\,, \\ \psi_r = C\overline{\psi}_r^T \equiv \psi_r^C\,. \end{cases}$$

These conditions just insure the identity of the particle and antiparticle states. By substituting these conditions into equations (1), (2), (3), it is easily seen that n_m^C, n_m^P, n_m^T must all be real for such fields, and thus are restricted to the values ± 1.

These are the only conditions on the phase factors which are imposed by the general requirements on the inversions, which involve the free field observables. A theory will be said to be invariant under one of the inversions whenever the phase factors n_m for the fields appearing in the theory can be chosen so that the inversion as defined with these phases commutes with the total Hamiltonian. Furthermore, if more than one choice of phase lead to inversion operators which commute with H, then any of the distinct operators which are defined by the different choices of phase can be chosen to represent the inversion.

The physical content of this definition can be illustrated for the case of space reflection. A theory is invariant under space reflection if no experiment enables one to distinguish between left and right. This will be the case if and only if in all transitions from an initial state with a definite orbital parity to final states containing a specified set of particles, the orbital parities of the final states are always the same. It is easy to see that this condition is satisfied whenever there is invariance in the sense we have used above. Furthermore, the assignment of intrinsic parities to particles by the use of the phase factors n_m^P is from this point of wiev a way of keeping track of the fact that when transitions take place between certain particles, the orbital parities change in a certain way. Any choice of phase consistent with the unitarity requirement and the invariance of the Hamiltonian is to be admitted, whenever such a choice is required to summarize a physical consequence of the theory. In the fourth section we will see an example of how this possibility of arbitrary phases is actually necessary for space reflection. We have stated these rather trivial points here at such lenght because they are in conflict with views which have sometimes been advanced. We believe that any requirement to be made on the inversion operators beyond the ones we have stated must involve additional physical assumptions which should always be emphasized clearly.

In the subsequant discussion a class of unitary operators to be referred to as multiplicative operators will be met with frequently. A multiplicative operator U is defined by

$$(5) \qquad U\varphi_m(X)U^{-1} = n_m(U)\varphi_m(X); \qquad U\varphi_m^\dagger U^{-1} = n_m^*(U)\varphi_m^\dagger ,$$

where the φ_m are boson or fermion fields, and the $n_m(U)$ are phase factors. Each U is prescribed by giving the quantities n_m for all the fields, which in general are different for the different fields. Since U is assumed unitary the n_m again satisfy $|n_m| = 1$. Also for real fields $n_r = \pm 1$.

Several properties of multiplicative operators follow from the definition.

a) The product of any number of multiplicative operators is a multiplicative operator.

b) The inverse of a multiplicative operator is a multiplicative operator, whose phases satisfy $n_m(U^{-1}) = n_m^*(U)$.

c) Define $\sqrt{U}$, a square root of a multiplicative operator U, by $\sqrt{U}\varphi_m\sqrt{U^\dagger} = \sqrt{n_m(U)}\varphi_m$.
Then $\sqrt{U}$ is also a multiplicative operator satisfying $\sqrt{U}\sqrt{U} = U$. There

are a number of such square roots, corresponding to the choice of sign of $\sqrt{n_m(U)}$ for each m.

d) Any two multiplicative operators commute.

e) Let U be any multiplicative operator. Then

$$\left\{\begin{aligned} &\boldsymbol{U}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{U}\,, \\ &\boldsymbol{U}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{U}^\dagger\,, \\ &\boldsymbol{U}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{U}^\dagger\,. \end{aligned}\right. \tag{6}$$

We prove the results of eq. (6) for a spinless boson, the proof being trivially generalized to other cases

$$\boldsymbol{U}\boldsymbol{P}\varphi_m(X)\boldsymbol{P}^{-1}U^{-1} = Un_m^P\varphi_m(\varrho X)U^{-1} = n_m^P n_m(U)\varphi_m(\varrho X)\,,$$

$$\boldsymbol{P}U\varphi_m(X)U^{-1}\boldsymbol{P}^{-1} = \boldsymbol{P}n_m\varphi_m\;(X)\,\boldsymbol{P}^{-1} = n_m^P n_m(U)\varphi_m(\varrho X)\,,$$

so $$\boldsymbol{U}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{U}\,,$$

$$\boldsymbol{U}\boldsymbol{C}\varphi_m\,(X)\;\boldsymbol{C}^{-1}U^{-1} = Un_m^C\varphi_m^\dagger(X)U^{-1} = n_m^C n_m^*\varphi_m^\dagger(X)\,,$$

$$\boldsymbol{C}\boldsymbol{U}^\dagger\varphi_m(X)U^{\dagger-1}\boldsymbol{C}^{-1} = \boldsymbol{C}n_m^*\varphi_m\,(X)\boldsymbol{C}^{-1} = n_m^C n_m^*\varphi_m^\dagger(X)\,,$$

so $$\boldsymbol{U}\boldsymbol{C} = \boldsymbol{C}\boldsymbol{U}^\dagger\,,$$

$$\boldsymbol{U}\boldsymbol{T}\varphi_m\,(X)\boldsymbol{T}^{-1}U^{-1} = Un_m^T\varphi_m(\tau X)U^{-1} = n_m^T n_m\varphi_m(\tau X)\,,$$

$$\boldsymbol{T}\boldsymbol{U}^\dagger\varphi_m(X)\boldsymbol{U}\boldsymbol{T}^{-1} = \boldsymbol{T}n_m^*\varphi_m\,(X)\,\boldsymbol{T}^{-1} = n_m^T n_m\varphi_m(\tau X)\,,$$

where the last eq. follows because $\boldsymbol{T}$ is antilinear;

so $$\boldsymbol{U}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{U}^\dagger.$$

If a multiplicative operator commutes with the total Hamiltonian, it will be called a multiplicative symmetry operator.

A well known class of multiplicative operators is given by the gauge transformations, U_λ defined by

$$U_\lambda\varphi_m U_\lambda^{-1} = \exp[iq_m\lambda]\varphi_m\,,$$

where λ is an arbitrary number which is the same for all fields φ_m, while q_m is a number, usually an integer, which varies from field to field and represents the « charge » of the particle which is conserved if the gauge transformations commute with the Hamiltonian. Certain multiplicative operators occur naturally when discussing inversions. In particular, it may be seen from the physical requirement of how inversions act on states that the square of each inversion operator should be a multiplicative operator, as the double application of an inversion must take each state into a physically equivalent state. Furthermore, the « commutator » of the inversions I_1 and I_2 defined by $I_1 I_2 I_1^{-1} I_2^{-1}$ should also be a multiplicative operator.

It is easy to verify this with the definitions given for the inversions in (1), (2), (3). In particular

(7a) $$\boldsymbol{C}^2 = 1 \, ,$$

(7b) $$\boldsymbol{P}^2 = GF \, ,$$

(7c) $$\boldsymbol{T}^2 = F \, ,$$

Here 1 is the identity operator, and F and G are multiplicative operators defined by

(8) $$G\varphi_m G^{-1} = (n_m^P)^2 \varphi_m \, ,$$

for any field, fermion or boson

(9) $$\begin{cases} F\varphi_m F^{-1} = \varphi_m \, , & \text{for boson fields} \, , \\ F\psi_m F^{-1} = -\psi_m \, , & \text{for fermion fields} \, . \end{cases}$$

F will be recognized as the operator $(-1)^{N_F}$ where N_F is the total number of fermions. It is also the operator of rotation through 360° about any axis, and so will commute with the Hamiltonian in any theory invariant under proper Lorentz transformations. Note that $F^2 = 1$.

It should be stressed that eqs. (7a) and (7c) are true for all choices of the phase factor n_m^C, n_m^T whereas by (8), the form of $\boldsymbol{P}^2$ depends on n_m^P.

If the inversions $P(T)$ commute with H for some particular theory, then so will their squares $FG(F)$, and the theory will at least contain these as multiplicative symmetry operators.

If a theory is invariant under all three inversions it will in general process an additional multiplicative symmetry E. To see this, consider the operator $\boldsymbol{CPT}$.

(10) $$\begin{cases} \boldsymbol{CPT}\varphi_m(X)\,(\boldsymbol{CPT})^{-1} = n_m^P n_m^C n_m^T \varphi_m(\varrho\tau X) \, , \\ \boldsymbol{CPT}\varphi_{\mu m}(X)\,(\boldsymbol{CPT})^{-1} = n_m^P n_m^C n_m^T \varrho\tau\varphi_{\mu m}(\varrho\tau X) \, , \\ \boldsymbol{CPT}\psi_m(X)\,(\boldsymbol{CPT})^{-1} = -i n_m^P n_m^C n_m^T \gamma_5^T \psi_m^{\dagger T}(\varrho\tau X) \, . \end{cases}$$

If $\boldsymbol{C}$, $\boldsymbol{P}$ and $\boldsymbol{T}$ each commute with H then so will this product, where the phases n^P, n^C, n^T are the same ones that make the separate inversions commute.

In the proof of the ***TCP*** theorem (2), it is shown that the product θ of the operations of strong reflections and hermitian conjugation, defined by:

$$(11)\qquad \begin{cases} \theta\varphi_m(X)\;\theta^{-1} = \varphi_m(\varrho\tau X) \\ \theta\varphi_{\mu m}(X)\theta^{-1} = \varrho\tau\varphi_{\mu m}(\varrho\tau X) \\ \theta\psi_m(X)\;\theta^{-1} = -i\gamma_5^T\psi_m^{\dagger T}(\varrho\tau X)\,. \end{cases}$$

will always commute with H for a local, Lorentz invariant theory with the usual connection between spin and statistics. Comparing this with the definition of ***CPT***, we conclude that for theories where the TCP theorem is true, if $\boldsymbol{T}$, $\boldsymbol{C}$ and $\boldsymbol{P}$ separately commute with $\boldsymbol{H}$, then the multiplicative operator $\boldsymbol{E}$ defined by

$$(12)\qquad E\varphi_m E^{-1} = n_m^P n_m^C n_m^T \varphi_m\,,$$

will commute with H where n_m^P, n_m^C, n_m^T are any phases for which the separate inversions commute with H .

3. – Ambiguities due to physically equivalent Hamiltonians.

It is been recognized by PAULI (3), and others, that the relation between observable quantities such as transition probabilities, and the interaction Hamiltonian, is in general not one to one, but rather one to many. In particular, there may be many different interaction Hamiltonians involving the same particles, which lead to identical transition probabilities between any two states.

A theorem expressing this possibility can be stated as follows (3): Let U be a unitary transformation which leaves invariant (up to a multiplicative phase factor) the initial and final states for some process. Then the two interaction Hamiltonians H and UHU^{-1} lead to the same transition probabilities for the process.

In particular, if a transformation U multiplies all free particle states (« in » states) by arbitrary phase factors (which may vary from state to state) then the physical consequences (as expressed by transition probabilities) of

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(2) G. LÜDERS: *Ann. Phys.*, **2**, 1 (1957).

(3) W. PAULI: *Nuovo Cimento*, **6**, 204 (1957). See also D. PURSEY: *Nuovo Cimento*, **6**, 266 (1957).

the theories in which H or UHU^{-1} are the interactions, are identical. Note that we are not simply expressing the unitary equivalence of the two theories, which is trivial. The transition probability is to be computed between the same states for each Hamiltonian, rather than between unitarily transformed states.

The multiplicative transformations U defined in the previous section are examples of such transformations. This is because the « in » states are defined as eigenstates of the free particle Hamiltonian, with quantum numbers given by the free particle observables. But the multiplicative operators commute with the free particle Hamiltonian and all the free particle observables. It therefore follows that for any « in » state $|\psi\rangle$, and any multiplicative operator U,

$$U|\psi\rangle = n_\psi(U)|\psi\rangle \,, \tag{13}$$

where $|n_\psi| = 1$ and in general depends on the state $|\psi\rangle$, as well as on U.

We will now demonstrate the above stated theorem for the operators U. Consider the two interaction Hamiltonians H and $H' = UHU^{-1}$. The S-matrices calculated from H and H' are clearly related by

$$S' = USU^{-1} \,.$$

Then if $|a\rangle$ and $|b\rangle$ are any two « in » states, it follows that

$$\langle a|S'|b\rangle = \langle a|USU^{-1}|b\rangle \,. \tag{14}$$

But by the above,

$$U^{-1}|a\rangle = n_a^*|a\rangle \,,$$

$$U^{-1}|b\rangle = n_b^*|b\rangle \,,$$

$$\langle a|S'|b\rangle = n_a n_b^* \langle a|S|b\rangle \,. \tag{15}$$

It follows from this that the two transition probabilities

$$\begin{cases} P_{ab} = |\langle a|S|b\rangle|^2 \,, \\ P'_{ab} = |\langle a|S'|b\rangle|^2 \,, \end{cases} \tag{16}$$

are equal for any states $|a\rangle$, $|b\rangle$, which is Pauli's theorem in this case. The Hamiltonians H and H' therefore describe the same physical system, and so can be used interchangeably without any change in the states. Two such Hamiltonians will be called equivalent.

This holds for states which contain fixed numbers of particles of each type.

States which are superpositions of such states, such as:

$$\sqrt{\tfrac{2}{3}}\,|n\pi^+\rangle + \exp[i\lambda]\sqrt{\tfrac{1}{3}}\,|p\pi^0\rangle$$

or

$$|K^0\rangle + \exp[i\lambda]\,|\overline{K}^0\rangle\,,$$

where λ is any real number, will not be transformed into themselves by arbitrary multiplicative transformations (4) but rather into states with different values of λ. This does not contradict our contention that the Hamiltonians H and UHU^{-1} are physically indistinguishable. This is because there are no experiments which directly determine the mixing phases λ for states like the above. To see this, we note that the matrix elements of all the free particle observables are independent of λ. Indeed, one can regard the introduction of the superposition states as merely a mathematical convenience. Only after a particular interaction Hamiltonian is chosen is it possible to distinguish between states with different mixing phases. But for a given choice of λ, the properties of the state will depend on which of the equivalent Hamiltonians is chosen, and so in the absence of an independent way of distinguishing between the different values of λ, this cannot be used to determine which of the equivalent Hamiltonian is correct.

For example, the eigenstates of total isotopic spin are superpositions of the above type. However, the isotopic spin operator can be specified only after choosing a particular set of coupling constants in the interaction Hamiltonian. When one transforms to an equivalent Hamiltonian, the isotopic spin operator, I, will also change to UIU^{-1}, unlike the free field observables. The U-transformed isotopic spin states will be eigenstates of the transformed isotopic spin operator. This implies that transition probabilities between states of definite isotopic spin are also unchanged by the multiplicative transformations. It may therefore be seen that the use of isotopic spin as an observable does not allow one to distinguish between equivalent Hamiltonians.

For comparison, instead of a multiplicative transformation consider an operator like P. The interactions H and PHP^{-1} are not equivalent unless P commutes with H, because some free particle observables, such as momentum, are not invariant under P, and therefore H and PHP^{-1}, give different transition probabilities for states of fixed momentum, which is an observable distinction. For example, if a Hamiltonian contains a spinor field always in the

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(4) This was pointed out to the authors by Dr. G. C. WICK. We thank Dr. WICK for very helpful discussions of this and many other points in this paper.

form $(1+\gamma_5)\psi$, so that only left-handed particles interact, then PHP^{-1} will contain $(1-\gamma_5)\psi$ and here only right-handed particles interact.

In general U will not commute with H, and so H and H' will be different functions of the field operators. In particular, the phase factors of certain coupling constants may be altered after transforming with U. A corollary of this theorem is therefore that the absolute phase of the coupling constants for interactions which involve a complex field linearly is unobservable, since it can always be changed by transforming the Hamiltonian with a multiplicative operator without changing the physical content of the theory. Of course, the relative phase of coupling constants for several interactions involving the same fields may be measureable.

Suppose now that some Hamiltonian H is invariant under any inversion I, satisfying $IU=U^\dagger I$ (*e.g.*, $\boldsymbol{C}$, $\boldsymbol{T}$, $\boldsymbol{CP}$, $\boldsymbol{PT}$), with a particular set of phase factors n_m^I for the fields φ_m involved in H. This will mean that the coupling constants appearing in H will satisfy certain reality conditions, involving also the phase n_m^I. We can construct a Hamiltonian, equivalent in the sense defined previously, which is invariant under a new inversion in which the phase factors for all complex fields are $+1$ or any other number we choose, whereas the phase factors for real fields are unchanged.

To do this define a multiplicative operator U_I by

$$\left\{\begin{array}{ll} U_I\varphi_m U_I^\dagger = n_m(U_I)\varphi_m\,, & \\ n_m(U_I) = \sqrt{n_m^I} & \text{for } \varphi_m \text{ any « complex » field}\,, \\ n_m(U_I) = \sqrt{1} & \text{for } \varphi_m \text{ any « real » field}\,. \end{array}\right. \tag{17}$$

Either square root may be chosen for each m. The definition of $n_m(U_I)$ for φ_m real is forced upon us by the condition that U_I be unitary, as discussed in Section **2**.

By hypothesis $[I, H]=0$. Therefore

$$[U_I I U_I^\dagger,\; U_I H U_I^\dagger]=0\,, \tag{18}$$

so that $I'=U_I I U_I^\dagger$ would be a suitable operator to represent the inversion with $U_I H U_I^\dagger$ as the Hamiltonian. But $U_I H U_I^\dagger$ is equivalent to H, and so we could adopt it as the Hamiltonian without changing the result of any experiment. It follows from eqs. (6) that the transformed inversion operator is

$$I'=U_I I U_I^{-1}=U_I^2 I\,. \tag{19}$$

This operator will have all phase factors $+1$ for complex fields. To see this in the case of $\boldsymbol{C}$ for example, one has for a spinless field

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(20)\qquad \left\{\begin{aligned} \boldsymbol{C}'\varphi\boldsymbol{C}'^{-1} &= U_c^2\boldsymbol{C}\varphi\boldsymbol{C}^{-1}(U_c^{-1})^2 \\ &= n^c U_c^2\varphi^\dagger(U_c^{-1})^2 \\ &= n^c n^{c*}\varphi^\dagger, \\ &= \varphi^{\dagger}. \end{aligned}\right.
$$

so the phase is $+1$. Clearly, any other phase factor can be obtained for any complex field by choice of U_I. On the other hand, for real fields, $U_1^2=1$ so that $I'=I$ which means that the inversion is unchanged.

Since the results of experiments are invariant under transformations which change the phase factors in these inversions in an arbitrary way, the phase factors for complex fields must be unobservable, either absolutely or relative to each other. This is not so for real fields as we have seen. In particular the phases for the photon field $n^C=-1$, $n^P=1$, $n^T=-1$, which make the electromagnetic interaction invariant, cannot be changed by such transformations, and thus can be determined by experiment.

While the phases appearing in $\boldsymbol{C}$, $\boldsymbol{T}$, $\boldsymbol{CP}$, $\boldsymbol{TP}$ are unobservable they cannot be simultaneously changed in an arbitrary way. This is because such inversions as $\boldsymbol{P}$, $\boldsymbol{CT}$ and $\boldsymbol{PCT}$ commute with multiplicative operators, and therefore the phases for these inversions will be unaltered by the transformation to an equivalent Hamiltonian. That is, for the equivalent Hamiltonian UHU^{-1}, the parity operator is

$$U\boldsymbol{P}U^{-1}=\boldsymbol{P},$$

and

$$U\boldsymbol{CT}U^{-1}=\boldsymbol{CT}.$$

These phase factors are then in principle measurable. The restrictions on measurements of such phases will be discussed in the next section.

The above results show that it is meaningless to ask for the relative n^C even for particles like Σ^0 and Λ^0 which can decay into each other by interactions which conserve C. Similarly, the relative n^T of the neutron and Λ^0 is not measurable even if the decay $\Lambda^0\to n+\pi^0$ conserves T. Experiments to measure these quantities therefore can not be devised.

4. – Ambiguities due to conservation laws.

In this Section we consider those inversion phases which are the same for all equivalent Hamiltonians. These include the phases in P, CT and PCT for any field, since these inversions commute with multiplicative transformations, and all inversion phases for real fields.

If I is any inversion that commutes with the Hamiltonian, and U is any multiplicative symmetry of the theory, then IU also is an inversion that transforms the « in » and « out » states in the way required by physical considerations, and which commutes with H. Furthermore, the inversions I and IU will differ only in their phase factor, according to

$$n_m^{IU} = n_m^I n_m(U) . \tag{21}$$

It is not possible to distinguish by experiment between the choice of I or IU to represent the inversion. The physical reason for this is that the matrix elements of these operators between two states differ only when the states transform differently under the multiplicative symmetry U, and transitions, either real or virtual, between such states are forbidden by the conservation of U. This fact was first pointed out by WICK, WIGHTMAN and WIGNER (5).

The converse of this result also holds for these inversions. That is, if a particular Hamiltonian commutes with two inversions I and I', both of which transform the free particle observables in the same way, then the Hamiltonian also commutes with the quotient operation $I^{-1}I'$, which is a multiplicative operation, and so the theory contains at least one multiplicative symmetry. This leads directly to the main problem of this Section, which is the question of what phases can arise in a physical theory, and what properties in the theory allow for the use of « unconventional » phases.

We will illustrate the discussion by referring to the parity operation, which is the most familiar and most often discussed (6). According to eqs. (7), (8), (9), if a theory is invariant under space reflection, it will be invariant under the multiplicative operator $\boldsymbol{P}^2 = GF$. It has sometimes been argued (6) that since $\boldsymbol{P}^2$ is the operator representing double reflection, it must be the identity operator for bosons, and either the identity operator or F for fermions. This is based on a principle that observable quantities should be unchanged by double reflection. It was pointed out in the fundamental paper by WICK, WIGHTMAN and WIGNER that such a principle cannot be used without some way of specifying what quantities are observable. These authors have given examples of some hermitian operators which occur in field theories and yet cannot be measured if the theory contains certain symmetries. A detailed analysis of which quantities appearing in field theories are observable would be difficult, although quite interesting. However, it appears reasonable that only such

(5) G. C. WICK, A. WIGHTMAN and E. P. WIGNER: *Phys. Rev.*, **88**, 101 (1952). This will be referred to as WWW.

(6) See ref. (5), and also C. N. YANG and J. TIOMNO: *Phys. Rev.*, **79**, 495 (1950); P. T. MATTHEWS: *Nuovo Cimento*, **6**, 642 (1957).

quantities that are invariant under all of the multiplicative symmetries of a theory can be observed. Since the field operators themselves are not in general invariant under multiplicative transformations, they will not be observables in theories containing such symmetries. Quantities such as the momentum, spin and charge, which are constructed from the free particle Lagrangian, are invariant under all multiplicative transformations, since they involve products like $\varphi_m^\dagger \varphi_m$. Thus, for these quanities the principle that observable quantities should be invariant under double reflection therefore does not restrict the operator $\boldsymbol{P}^2$ at all, and such restrictions can only be obtained by examining the interactions. But these will only require that the phases be chosen to give invariance of the Hamiltonian, and we will show below examples of interactions which require arbitrary phases to give invariance. We conclude that no *a priori* restrictions on the phases for space reflection, etc., can be admitted.

Suppose that the Hamiltonian commutes with a parity operator $\boldsymbol{P}$ for some choice of phases n_m^P. In general, since $\boldsymbol{P}^2 \neq 1$, the operator $\boldsymbol{P}$ will have complex eigenvalues. Thus with this choice of phases, the « intrinsic parities » of the particles created by the fields φ_m, ψ_m will be complex numbers of modulus one. We examine the circumstances under which these complex intrinsic parities can be eliminated by a redefinition of the parity operator.

Since $\boldsymbol{P}$ commutes with H, and $\boldsymbol{P}^2 \neq 1$, the theory necessarily contains at least one multiplicative symmetry $\boldsymbol{P}^2 = GF$. We consider theories invariant under rotation, which also have the multiplicative symmetry F. The general condition under which intrinsic parities may be chosen real is that $\sqrt{G^+F}$ should commute with the Hamiltonian (7). For if this happens, it is possible to define a new parity operator $\boldsymbol{P}' = \sqrt{G^+F}\,P$, which satisfies

$$\boldsymbol{P}'^2 = G^+F\boldsymbol{P}^2 = G^{\dagger}F^{\dagger}GF = 1 \tag{22}$$

and thus has real eigenvalues. Furthermore, $\boldsymbol{P}'$ commutes with H, since it is the product of operators which commute with H. Conversely, the condition is a necessary one, because if there exists a conserved parity operator $\boldsymbol{P}'$ satisfying $\boldsymbol{P}'^2 = 1$, then $\boldsymbol{P}' = U\boldsymbol{P}$, where U is a multiplicative symmetry operator, and $U^2 = G^\dagger F$. Therefore, having once found a parity operator which commutes with H, involving complex phases, it is possible to test whether the use of such phases is essential by seeing whether for the G defined by these phases, $\sqrt{G^\dagger F}$ is a symmetry of the theory. There are three general cases to be considered.

1) All of the multiplicative symmetries of the theory, including F, are parts of continuous gauge groups. This is believed to be the case in the present

(7) Here $\sqrt{G^+F}$ refers to any of the square roots defined above.

theory of elementary particles, assuming that strangeness is an additive quantum number for strong interactions, rather than a multiplicative one [8]. For such theories, since $G^{\dagger}F^{\prime}$ ($= P^{-2}$) is a number of a gauge group which commutes with the Hamiltonian, $\sqrt{G^{\dagger} F}$ is also a member of the gauge group and so also commutes with the Hamiltonian. It is then always possible to make the intrinsic parity of fermions and bosons real in theories satisfying assumption 1. As indicated, this is probably the case in the present theory of elementary particles.

2) The theory contains apart from gauge transformations the additional invariance F, such that $\sqrt{F}$ is not a multiplicative symmetry. In such theories there is no additive conservation of fermions, or else $\sqrt{F}$ would be part of the fermion gauge group. If there is parity conservation with $P^2 = GF$ then since G commutes with H, there are two possibilities for G. Either

a) $G =$ a gauge transformation, γ, so that $P^2 = \gamma F^{\prime}$,

or

b) $G = \gamma F$, so that $P^2 = \gamma$.

In case *a*), the conserved operator $P' = \sqrt{\gamma^{\dagger}}P$ satisfies

$$P'^2 = F^{\prime}\,.$$

Thus up to a gauge transformation, the intrinsic parities of all bosons are real, and of all fermions are imaginary in this case. An example of such a theory is given by the interaction

$$H_{\text{int}} = \bar{\psi}^C \psi\varphi + \bar{\psi}\psi\varphi + \text{h. c.} \tag{23}$$

Here φ is a real boson field and ψ a complex fermion field. It is easy to see that for invariance under P, $n_{\psi} = \pm 1$, $n_{\varphi} = 1$, so that $G = 1$, $P^2 = F$ or the intrinsic parity of the fermion is imaginary, while that of the boson is real. Since this theory has no gauge invariances, there is no freedom in choosing these parities, except that coming from F, which accounts for the $\pm$ sign in n_{ψ} and makes the relative parity of the boson and fermion unobservable.

Theories containing real fermion fields and satisfying assumption 2 must fall under case *a*) if they conserve parity, since according to Section **2**, $P^2 = F^{\prime}$ for such fields.

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[8] The possibility that strangeness conservation might be multiplicative was suggested by W. HEISENBERG and W. PAULI (preprint). See also K. M. CASE, R. KARPLUS and C. N. YANG: *Phys. Rev.*, **101**, 874 (1956).

In case *b*), the parity operator can again be redefined as $P' = \gamma^{\mathcal{F}} P$, and $P'^2 = 1$, so that up to a gauge transformation, all parities, fermion or boson, are real.

A Hamiltonian giving such a theory is

$$H_{\text{int}} = \overline{\psi}^{C}\psi\varphi + \overline{\psi}\gamma_5\psi\varphi + \text{h. c.} \tag{24}$$

with the same symbols as before. Now $n_\psi = \pm i$, $n_\varphi = -1$, so that $G = F$, $P^2 = 1$.

It should be emphasized that if assumption 2*a* is satisfied, then all fermions have imaginary parity, whereas if 2*b* is satisfied, all have real parity, modulo gauge transformations. This type of theory does not have enough symmetry to allow some fermion to have irremovably real parity while others have irremovably imaginary parity. This is because the only non-gauge multiplicative invariance we have allowed is F, which does not distinguish between fermions.

3) The theory contains multiplicative symmetries U, other gauge transformations and F, such that $\sqrt{U^\dagger}$ and $\sqrt{U^\dagger F}$ do not commute with H. In this case, if there is parity conservation with $G = U$, then it is impossible to find a parity operator which commutes with H and satisfies $P^2 = 1$, or $P^2 = F$. Then we expect that the intrinsic parities of bosons and fermions might be arbitrary complex numbers, providing that the theory has sufficiently complicated multiplicative symmetries. It is clear that the existence of « discrete » multiplicative symmetries is only a necessary condition that use of complex n^P should be unavoidable in a theory, rather than sufficient. This is because there is never a conservation law for intrinsic parities alone, without specification of the orbital states involved. This is illustrated by the interactions (23) and (24) which have the same multiplicative symmetry F.

We continue the discussion by reference to a particular example. Consider the interaction of a fermion field ψ with a complex boson field φ, given by

$$H = \overline{\psi} O \psi\varphi^2 + \overline{\psi}\,\overline{O}\psi\varphi^{\dagger 2} \qquad (\overline{O} = \gamma_4 O^\dagger \gamma_4). \tag{25}$$

We wish to consider two cases

$$O = \gamma_5 \,. \tag{a}$$

Here the theory is invariant under space reflection transformations with the following phases

$$n_\psi^P = \text{any phase factor}\,,$$

$$n_\varphi^P = \pm i\,.$$

Thus

$$\begin{cases} G\psi G^{-1} = -(n_\psi^P)^2\psi\,, \\ G\varphi G^{-1} = -\varphi\,. \end{cases} \tag{26}$$

Furthermore, the theory possesses the following multiplicative invariances:

$$B_\lambda \psi B_\lambda^{-1} = \exp[i\lambda]\psi\,, \qquad \text{for all real } \lambda \tag{27}$$

a gauge transformation on the spinor field, and

$$U\varphi U^{-1} = -\varphi$$

a discrete transformation on the boson field.

It is clear that the phase factor in the transformation of the spinor field is only conventional, and can be removed by a redefinition of the parity operator. However, the phase factor $\pm i$ for φ cannot be removed. This is because the operator $\sqrt{U^\dagger}$, defined by

$$\sqrt{U^\dagger}\varphi\sqrt{U} = \pm i\varphi\,,$$

does *not* commute with the Hamiltonian, so that $\sqrt{U^\dagger}P$ is not a conserved operator. The existence of the phase factor $\pm i$ is essential in the physical interpretation of the theory. The interaction (25) involves, among other processes, the annihilation of two S wave φ quanta together with a transition of the fermion from an S state to a P state. Such process cannot be consistent with invariance under space reflection unless the intrinsic parity of the φ quanta is $\pm i$. This follows immediately from the conservation law

$$(\eta_\varphi^P)^2(-1)^{\Sigma l_{\text{initial}}} = (-1)^{\Sigma l_{\text{final}}}\,. \tag{28}$$

b) Consider next $\boldsymbol{O}=1$. The Hamiltonian has the same multiplicative invariances as before, but now it is invariant under space reflections with

$$n_\varphi^P = \text{any complex number}\,,$$
$$n_\varphi^P = \pm 1$$

and so here by a suitable redefinition of the parity operator the intrinsic parities can be made real. This indicates as stated that the existence of discrete multiplicative symmetries only allows the possibility of irremovably complex intrinsic parities, without requiring them.

Next we construct a Hamiltonian requiring $\sqrt{i}$ for the space reflection of a fermion field. To do this, consider a complex spinor field ψ interacting with a real boson field φ.

$$H_{\text{int}} = g\bar{\psi}\gamma_5\psi\varphi + h\bar{\psi}^c\psi\bar{\psi}^c\psi\varphi + \text{h. c.} \tag{29}$$

From the definition of $\bar{\psi}^c$, it is easy to see that

$$P\bar{\psi}^c P^{-1} = -\, i n_\psi^P \gamma_4 \bar{\psi}^c \cdot$$

The interaction (29) is invariant under space reflection with

$$n_\varphi^P = -1\,, \qquad n_\psi^P = \pm\sqrt{i}\,. \tag{30}$$

It is also invariant under the multiplicative transformations F and

$$W_\pm \psi W_\pm^{-1} = \pm\, i\psi\,.$$

However, it is not invariant under $\sqrt{W^+}$, and therefore the factors $\pm\sqrt{i}$ in the space reflection of ψ are not removable.

From these examples, it may be seen that one can construct Hamiltonians which require any n-th root of 1 as a phase factor in the transformation of complex fields under space reflection. These Hamiltonians will be characterized by the existence of discrete multiplicative invariances, whose square roots are not invariances of the theory. It is also possible to write « interactions » which require other complex phase factors, but these will involve irrational operations, on the field operators, whose meaning is questionable.

In the light of our discussion, we can conclude the following about the four classes of spinors introduced by YANG and TIOMNO [6], and used by many other authors. In any theory invariant under space rotations, $F = (-1)^{N_F}$ is a multiplicative symmetry. Any such theory could possibly be invariant under a parity operation in which $P^2 = F$. As we have stressed, this would mean that all fermions have parity $\pm i$. However, the invariance under F does not by itself allow the relative parity of two fermions to be imaginary. Such a possibility is connected with the existence of other discrete multiplicative symmetries, which do not act the same way on all fermion fields.

It may be further noted that the use of discrete multiplicative symmetries or of space reflection invariance to forbid unwanted processes as is sometimes done involves the difficulty that these can only give conservation laws « modulo n » and not the absolute conservation laws associated with gauge groups [9]. If one accepts the usual conservation laws as absolute (conservation of charge, baryons, leptons, and strangeness in strong interactions) the use of phases $\pm i$ for some fields is unnecessary, and can be removed by a redefinition of the parity operator. On the other hand, if strangeness conservation only held modulo 4, for instance, it might be necessary to use complex phase factors for strange particles. This would happen, *e.g.*, if four Λ^0 in S states could go into three S state neutrons and one P state neutron.

We conclude this discussion with some comments about the conditions under which the relative parity of two states is measurable. Our conclusions here are in essential agreement with those of WWW. The general result may be stated as follows

The relative parity of two states is measurable only if the states transform the same way under all the multiplicative symmetry operations of the theory.

These is because, if P is a conserved parity operator, then so is UP, where U is any multiplicative symmetry operator. But if $|\psi_1\rangle$ and $|\psi_2\rangle$ are two eigenstates of P, with

$$(31)\qquad \begin{cases} P|\psi_1\rangle = \varepsilon_1|\psi_1\rangle\,, \\ P|\psi_2\rangle = \varepsilon_2|\psi_2\rangle\,. \end{cases}$$

Then

$$(32)\qquad \begin{cases} UP|\psi_1\rangle = \varepsilon_1 n_1(U)|\psi_1\rangle\,, \\ UP|\psi_2\rangle = \varepsilon_2 n_2(U)|\psi_2\rangle\,, \end{cases}$$

where

$$U|\psi_1\rangle = n_1(U)|\psi_1\rangle\,,$$

$$U|\psi_2\rangle = n_2(U)|\psi_2\rangle\,.$$

Then unless $n_1(U) = n_2(U)$ for all U, the two parity operators, which according to our previous remarks are physically indstinguishable, will have different relative eigenvalues for the two states.

Equivalently, the phase factor in the inversion of a field, or a product of fields, is measurable only if the field or product of fields is invariant under all multiplicative symmetries of the theory.

As an example of this, we note that the quantity which is measurable is the relative parity of a Ξ^-p) system compared to a $2\,\Lambda^0$ system, rather than the relative parity of Ξ and nucleon. If strangeness is an additive quantum

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(9) This is true unless one assumes in addition specific forms for the interaction, such as Yukawa couplings. If the latter is done, the discrete multiplicative invariance of H may imply a continuous gauge invariance.

number, then the intrinsic parity of the Λ may be chosen real by convention, and then there is no difference in the two statements. However, if strangeness were multiplicative, the statements are not equivalent.

The above discussion of parity can also be applied to $\boldsymbol{CT}$, $\boldsymbol{CPT}$ and all inversions of real fields, which commute with all multiplicative transformations.

5. – Products of inversions.

In this Section we discuss some of the relations among products of the inversions, and the multiplicative operators E, F, G. We consider a theory invariant under $\boldsymbol{C}$, $\boldsymbol{P}$ and $\boldsymbol{T}$ simultaneously. Then according to Section **2**, it will be invariant under E, F and G. The following results for the products of inversions can easily be demonstrated.

$$(33)\qquad \left\{\begin{array}{ll} (a) & \boldsymbol{P}^2 = FG\,, \\ (b) & \boldsymbol{C}^2 = 1\,, \\ (c) & \boldsymbol{T}^2 = F\,, \\ (d) & (\boldsymbol{CP})^2 = (\boldsymbol{PC})^2 = F\,, \\ (e) & (\boldsymbol{TP})^2 = (\boldsymbol{PT})^2 = F\,, \\ (f) & (\boldsymbol{CT})^2 = (\boldsymbol{TC})^{-2} = FGE^{-2} \\ (g) & \boldsymbol{CP} = \boldsymbol{PC}G\,, \\ (h) & \boldsymbol{TP} = \boldsymbol{PT}FG\,, \\ (i) & \boldsymbol{CT} = \boldsymbol{TC}GE^{-2}\,, \\ (j) & (\boldsymbol{TCP})^2 = FE^2\,, \end{array}\right.$$

As an example, we derive the relation $\boldsymbol{CT} = \boldsymbol{TC}GE^{-2}$ for a spinor field. From (1), (3)

$$\begin{aligned} \boldsymbol{CT}\psi\boldsymbol{T}^{-1}\boldsymbol{C}^{-1} &= n_C n_T \gamma_5 \overline{\psi}^T\,, \\ \boldsymbol{CT}\psi\boldsymbol{C}^{-1}\boldsymbol{T}^{-1} &= n_C^* n_T^* \gamma_5 \overline{\psi}^T\,, \\ \boldsymbol{CT}\psi\boldsymbol{T}^{-1}\boldsymbol{C}^{-1} &= n_C^2 n_T^2 \boldsymbol{TC}\psi\boldsymbol{C}^{-1}\boldsymbol{T}^{-1} \\ &= \boldsymbol{TC} n_C^{*2} n_T^{2*}\psi\boldsymbol{C}^{-1}\boldsymbol{T}^{-1} \\ &= \boldsymbol{TC}GE^{-2}\psi(GE^{-2})^{-1}\boldsymbol{C}^{-1}\boldsymbol{T}^{-1}\,, \\ \boldsymbol{CT} &= \boldsymbol{TC}GE^{-2}\,. \end{aligned}$$

Because of the relations (6) between $\boldsymbol{C}$, $\boldsymbol{T}$ and multiplicative operators, the relations (*b*), (*c*), (*d*), (*e*) cannot be changed by redefinition of $\boldsymbol{C}$, $\boldsymbol{P}$ or $\boldsymbol{T}$. Since E commutes with H, one can redefine P by

$$\boldsymbol{P}' = E^{-1}\boldsymbol{P}$$

and obtain a conserved parity operator for which

$$\boldsymbol{P}'^2 = FGE^{-2} = FG' \,, \tag{33a}$$

$$(\boldsymbol{CT})^2 = FG' \,, \tag{33f}$$

$$\boldsymbol{CP}' = \boldsymbol{P\,C}G' \,, \tag{33g}$$

$$\boldsymbol{TP}' = \boldsymbol{P}'\boldsymbol{T}FG' \,, \tag{33h}$$

$$\boldsymbol{CT} = \boldsymbol{TC}G' \,, \tag{33i}$$

$$(\boldsymbol{TCP}')^2 = F \,. \tag{33j}$$

This is the general result in a theory containing some discrete multiplicative invariances. If, however, $\sqrt{G^{\dagger\prime} F}$ is a symmetry of the theory, then it is possible to again redefine P and C so that other relations becomes simplified,

$$P'' = \sqrt{G^{\dagger\prime} F}\, P' \,,$$

$$C' = \sqrt{G^{\dagger\prime} F}\, C$$

and we drop primes. Then $\boldsymbol{P}$, $\boldsymbol{C}$ still commute with H and

$$\boldsymbol{P}^2 = 1 \,, \tag{33a''}$$

$$(\boldsymbol{CT})^2 = 1 \,, \tag{33f''}$$

$$\boldsymbol{TP} = \boldsymbol{PT} \,, \tag{33h''}$$

$$\boldsymbol{CP} = \boldsymbol{PC}F \,, \tag{33g''}$$

$$\boldsymbol{CT} = \boldsymbol{TC}F \,, \tag{33i}$$

$$(\boldsymbol{TCP})^2 = F \,. \tag{33j''}$$

It is also possible, by omitting the $\sqrt{F}$ in the definition of P'', to remove the factor F from the relations *g*, *i* at the price of restoring it to the others. The form used here is that usually adopted, while the latter is used in the Majorana neutrino theory, where $\boldsymbol{C}$ is a multiplicative operator.

* * *

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Note added in proof.

We have been informed by Dr. G. LÜDERS that problems similar to those treated in our Section **4** were discussed by him at the Summer School in Varenna in July 1959.

RIASSUNTO (*)

Si discutono i fattori di fase che possono comparire nella definizione delle inversioni $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{T}$, e i loro prodotti. Si dimostra che, a causa dell'esistenza di Hamiltoniane « fisicamente equivalenti », le fasi in ω, $\boldsymbol{CP}$, $\boldsymbol{T}$ e $\boldsymbol{TP}$ non sono misurabili per campi complessi. Per le restanti inversioni si possono costruire delle interazioni che richiedano fasi più generali delle usuali ± 1, $\pm i$ per i campi complessi; ciò è possibile se, e solo se, la teoria contiene certe discrete simmetrie moltiplicative. Si danno esempi di tali interazioni.

(*) *Traduzione a cura della Redazione.*