Soliton Surfaces (*).

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As is well known some soliton equations admit the surface-geometric interpretation. Two oldest examples (dating back to the middle of the 19th century) are the sine-Gordon and the Liouville equations. The Gauss-Mainardi-Codazzi (GMC) system of differential geometry of surfaces in E^3 when applied to any pseudospherical surface endowed with the so-called asymptotic co-ordinates is reducible to the sine-Gordon equation ([1-3]). Likewise, the same GMC system applied to any minimal surface in the so-called curvature co-ordinates is reducible to the Liouville equation ([4]). A well-known Pohlmeyer-Lund-Regge-Getmanov system ([5]) can be written as the GMC system as well ([6]).

An idea of the path leading from surfaces to solitons originated in the Lund-Regge work ([5]) has been subsequently developed by LUND ([6]). See also ([4,7]). The Lund-Regge approach is in a sense a surface-geometric way to generate soliton systems ([8]).

The GMC system of differential geometry of surfaces in E^3 ([2,9]) can be generalized for the case of n-dimensional manifold embedded into N-dimensional flat space ([10]). In the sequel the term «surface» means any 2-dimensional manifold embedded into N-dimensional flat space.

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(10) L. P. EISENHART: *Riemannian Geometry* (Princeton, N. J., 1949).

In the context of the Lund-Regge approach two obvious questions arise. Suppose we are given any 2-dimensional (nondiscrete) soliton system with the corresponding linear problem.

A) First question: is it always possible to put the soliton system in a form of the GMC system for some surfaces? In other words, can any soliton system be reached in the Lund-Regge framework?

In ([11]) we proved that the answer to this question is positive, provided that a Lie algebra g of the associated linear problem is semi-simple. Moreover, the resulting surfaces are mebedded into g equipped with the Killing-Cartan form (or scalar product) converting g ($\dim g = N$) into Euclidean (or pseudo-Euclidean) N-dimensional flat space. The resulting surfaces will be called soliton surfaces.

B) Second question: what is a general description of all possible soliton surfaces?

This question remains still open. One may conjecture that the answer to this question will be of some importance in a possible unifying approach to solitons.

In this paper we discuss general properties of soliton surfaces. As an example we consider a class of soliton equations with SU_2—linear problem in the ZS-AKNS gauge ([12]). Finally, we present a generalization of the so-called Bianchi-Lie transformation for any 2-dimensional (nondiscrete) soliton system. Originally, the Bianchi-Lie transformation ([13,14]) has been introduced as a surface-geometric analog of the Bäcklund transformation for the sine-Gordon equation.

The paper is based upon the following results: 1) the 19th century differential geometry ([2,9,14]) with the special stress on the Italian School (G. Mainardi, D. Codazzi, E. Beltrami, U. Dini, L. Bianchi), 2) the Killing-Cartan form ([15]), 3) the Pohlmeyer transformation ([5,16]) known in the chiral context.

We begin with a construction of soliton surfaces. Concerning details see ([11]). Here we use the following conventions: $x = (x^1, x^2)$ (two real variables) and $\partial u/\partial x^\mu = u_{,\mu}$ etc. Let us consider any 2-dimensional (nondiscrete) soliton system. It is a system of nonlinear partial differential equations for real fields $\varphi(x)$, $\psi(x)$, ... that admits the following representation:

$$g_{1,2} - g_{2,1} + [g_1, g_2] = 0 \,, \tag{1}$$

where g_μ ($\mu = 1, 2$) are functions of x (through $\varphi, \psi, \ldots; \varphi_{,\mu}, \ldots$) and some real (called spectral) parameter ζ taking values in a fixed ($d \times d$)-matrix Lie algebra g ($\dim g = N$)

$$g_\mu = g_\mu(\varphi, \psi, \ldots; \varphi_{,\mu}, \ldots; \zeta) \,. \tag{2}$$

Equation (1) is a necessary and sufficient condition (under some general conditions imposed on g_μ) for a global existence and uniqueness of the single-valued solution Φ

([11]) A. Sym: *Soliton theory is surface theory*, preprint IFT/11/81.
([12]) M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur: *Stud. Appl. Math.*, **53**, 249 (1974).
([13]) R. L. Anderson and N. H. Ibragimov: *Lie-Bäcklund Transformation in Applications* (Siam, Philadelphia, Pa., 1979).
([14]) L. Bianchi: *Lezioni di geometria differenziale* (Pisa, 1922).
([15]) A. O. Barut and R. Rączka: *Theory of Group Representations and Applications* (Warsaw, 1977); B. G. Wybourne: *Classical Groups for Physicists* (New York, N.Y., London, Sydney, and Toronto, 1974).
([16]) S. J. Orfanidis: *Phys. Rev. D*, **21**, 1513 (1980).

to the following system called a linear problem of the inverse method ([12,17]):

$$\Phi_{,\mu} = g_\mu \Phi \,, \tag{3}$$

where $\Phi = \Phi(x, \zeta)$ is assumed to be a $(d \times d)$-matrix–valued function (columns of which form a basis of Jost functions in the scattering problem terminology ([12,17])). Assuming $\Phi(x_0, \zeta) \in G$ (Lie group of g) as an initial condition gives $\Phi \in G$ everywhere ([18]). In the sequel we assume $\Phi \in G$.

The G-valued function $R(x, \zeta) = \Phi(x, \zeta_0)^{-1}\Phi(x, \zeta)$ (ζ_0-fixed) is known in the chiral context as the Pohlmeyer transformation ([5,16]). Equation (3) yields

$$R_{,\mu} = \Phi(x, \zeta_0)^{-1}[g_{\mu,\zeta}(x, \zeta_0)(\zeta - \zeta_0) + \ldots]\,\Phi(x, \zeta_0)R \,. \tag{4}$$

One of the consequences of the integrability conditions of eq. (4) ($R_{,\mu\nu} = R_{,\nu\mu}$) is

$$(\Phi^{-1}g_{1,\zeta}\Phi)_{,2} = (\Phi^{-1}g_{2,\zeta}\Phi)_{,1} \,. \tag{5}$$

Equation (5) implies there exists a g-valued function $r = r(x, \zeta)$ such that

$$r_{,\mu} = \Phi^{-1}g_{\mu,\zeta}\Phi \,. \tag{6}$$

Equation (6) can be easily integrated ([19])

$$r = \Phi^{-1}\Phi_{,\zeta} + \text{const} \,, \tag{7}$$

where const $\in g$. We put const $= 0$ ([20]) .

The equation

$$g \ni r = r(x, \zeta) = \Phi^{-1}(x, \zeta)\,\Phi_{,\zeta}(x, \zeta) \tag{8}$$

is interpreted as a co-ordinate representation of the ζ-family of surfaces embedded into the N-dimensional affine space g: the independent variables in the original soliton system (1) $x = (x^1, x^2)$ turn out to be co-ordinates upon resulting surfaces, the real spectral parameter ζ enumerates copies of ζ-family and the function $r = \Phi^{-1}\Phi_{,\zeta}$ is a position vector of the resulting surfaces. Moreover, the affine space g is equipped with a nondegenerate scalar product (the Killing-Cartan form of the semi-simple Lie algebra g) that converts g into a flat space.

Thus, for any solution φ, ψ, ... of the soliton system (1), there exists a ζ-family of surfaces (8) with tangent vectors given by (6).

These surfaces are called soliton surfaces (corresponding to the solution φ, ψ, ...).

The main properties of soliton surfaces are listed below.

a) The GMC system when applied to any soliton surface is reducible to the original soliton system (1) ([11]). This result shows that soliton systems can always be interpreted in a surface-geometric fashion.

([17]) V. E. Zakharov, S. V. Manakov, S. P. Novikov and L. P. Pitaevski: *Theory of Solitons* (Moscow, 1980) (in Russian); S. Manakov: *Sov. Sci. Rev. Phys. Rev.*, **1**, 133 (1979).
([18]) A. Sym and J. Corones: *Phys. Lett. A*, **68**, 305 (1978).
([19]) This important observation is due to J. Tafel.
([20]) The equation $\Phi_{,\zeta} = \Phi r$ is, modulo the order of Φ and r matrices, the second equation of the deformation theory, see for example H. Flaschka and A. C. Newell: *Commun. Math. Phys.*, **76**, 65 (1980). This remark is due to D. Levi.

b) Soliton surfaces are invariant with respect to ζ-independent gauge-transformations [3,17,18,21] performed on the soliton system (1).

c) The metric tensor $g_{\mu\nu}$ of soliton surfaces (induced by the surrounding flat space g) is given by

$$g_{\mu\nu} = \mathrm{Tr}\,\mathrm{ad}\,g_{\mu,\zeta}\,\mathrm{ad}\,g_{\nu,\zeta}\,, \tag{9}$$

where Tr = trace and « ad » is the adjoint representation of g [15]. Hence, all intrinsic (metric) properties of soliton surfaces can be calculated explicitly.

d) For g-compact the corresponding soliton surfaces are embedded into E^N, whereas for g-noncompact the corresponding soliton surfaces are embedded into N-dimensional pseudo-Euclidean space [15].

Consider as an example a class of soliton systems with SU_2-linear problem in the ZS-AKNS gauge [12]. Since SU_2 is a 3-dimensional compact Lie algebra the corresponding soliton surfaces are embedded into E^3. Thus, in this case we are in a position to make use of the rich harvest of the 19-th century differential geometry [2,9,14]. Two well-known soliton equations belonging to the discussed class are

$$iq_{,2} + q_{,11} + 1/2|q|^2 q = 0 \qquad \text{(nonlinear Schrödinger equation)}\,, \tag{10}$$

$$\varphi_{,12} = \sin\varphi \qquad \text{(sine-Gordon equation)}\,. \tag{11}$$

According to the classical Bonnet theorem [9] any surface in E^3 is implicitly defined (modulo position) by its two (I and II) fundamental quadratic forms. These are defined as follows:

$$\mathrm{I} = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{12}$$

$$\mathrm{II} = d_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{13}$$

where $g_{\mu\nu}$ and $d_{\mu\nu}$ are the metric and the second fundamental tensor of a surface, respectively [2,9]. Without entering into technical details, we present the forms I and II in the case of our interest (SU_2-linear problem in the ZS-AKNS gauge).

$$\mathrm{I} = (\mathrm{d}x^1)^2 - 2\,\mathrm{Tr}\,\sigma g_{2,\zeta}\,\mathrm{d}x^1\,\mathrm{d}x^2 + \det g_{2,\zeta}(\mathrm{d}x^2)^2\,, \tag{14}$$

$$\begin{aligned}\mathrm{II} = -\det{}^{-\frac{1}{2}}[\sigma, g_{2,\zeta}]\{&\mathrm{Tr}\,[\sigma, g_1][\sigma, g_{2,\zeta}](\mathrm{d}x^1)^2 + 2\,\mathrm{Tr}\,[\sigma, g_2][\sigma, g_{2,\zeta}]\,\mathrm{d}x'\,\mathrm{d}x^2 + \\ &+ \tfrac{1}{2}\,\mathrm{Tr}\,([g_{2,\zeta}, g_2] + g_{2,2\zeta})[\sigma, g_{2,\zeta}](\mathrm{d}x^2)^2\}\,,\end{aligned} \tag{15}$$

where $\sigma = -i/2\sigma_3$ (σ_3 is the Pauli matrix). These formulae may be also used in verifying directly that soliton systems are equivalent to the GMC system. To this purpose, we need an especially convenient form of the GMC system. For surfaces in E^3 it consists of the Gauss equation and two Mainardi-Codazzi (MC) equations [2,9,14]. Geometers of the 19-th century have discovered many forms of the Gauss equation. To our knowledge the most useful form for soliton purposes is the Liouville-Beltrami formula [22]

$$\begin{aligned}2\sqrt{\det g_{\mu\nu}}\,K = {}&[\det{}^{-\frac{1}{2}} g_{\mu\nu}(-g_{22,1} + g_{12}g_{11}^{-1}g_{11,2})]_{,1} + \\ &+ [\det{}^{-\frac{1}{2}} g_{\mu\nu}(2g_{12,1} - g_{11,2} - g_{12}g_{11}^{-1}g_{11,1})]_{,2}\,,\end{aligned} \tag{16}$$

(21) M. CRAMPIN: *Phys. Lett. A*, **66**, 170 (1978).
(22) W. A. BLASCHKE: *Einführung in die Differentialgeometrie* (Berlin, 1930).

where $K = \det d_{\mu\nu}/\det g_{\mu\nu}$ is the Gaussian curvature. Putting $K = 0$ (the so-called developable surfaces [2,9], for instance plane), one sees that the Gauss equation (16) is in the form of the conservation law so characteristic for one-field soliton equations. Indeed, for instance the modified Korteweg-de Vries equation can be associated with a plane [11]. The most convenient form of the MC equations has been found by BIANCHI:

$$(17a)\qquad (\det{}^{-\frac{1}{2}} g_{\mu\nu} d_{22})_{,1} - (\det{}^{-\frac{1}{2}} g_{\mu\nu} d_{12})_{,2} + \det{}^{-\frac{1}{2}} g_{\mu\nu} (\Gamma^1_{22} d_{11} - 2\Gamma^1_{12} d_{12} + \Gamma^1_{11} d_{22}) = 0\,,$$

$$(17b)\qquad (\det{}^{-\frac{1}{2}} g_{\mu\nu} d_{11})_{,2} - (\det{}^{-\frac{1}{2}} g_{\mu\nu} d_{12})_{,1} + \det{}^{-\frac{1}{2}} g_{\mu\nu} (\Gamma^2_{22} d_{11} - 2\Gamma^2_{12} d_{12} + \Gamma^2_{11} d_{22}) = 0\,,$$

where $\Gamma^{\gamma}_{\alpha\beta}$ are the Christoffel symbols of the metric $g_{\mu\nu}$.

The I and II forms (14), (15) for the nonlinear Schrödinger equation (10) become

$$(18)\qquad \mathrm{I} = (\mathrm{d}x^1)^2 + 8\zeta\, \mathrm{d}x^1 \mathrm{d}x^2 + (16\zeta^2 + \varrho^2)(\mathrm{d}x^2)^2\,,$$

$$(19)\qquad \mathrm{II} = \varrho(\mathrm{d}x^1)^2 + (4\zeta\varrho - 2\varrho\varphi_{,1})\,\mathrm{d}x^1\mathrm{d}x^2 + (4\zeta^2\varrho - 4\zeta\varrho\varphi_{,1} + 1/2\varrho^3 - \varrho\varphi_{12})(\mathrm{d}x^2)^2\,,$$

where $q = \varrho \exp[i\varphi]$. Inserting $g_{\mu\nu}$ and $d_{\mu\nu}$ of (18) and (19) into the Gauss equation (16), one sees that the Gauss equation becomes the real part of the nonlinear Schrödinger equation. Likewise, inserting the same $g_{\mu\nu}$ and $d_{\mu\nu}$ into the MC equation (17) gives that both MC equations become the imaginary part of the nonlinear Schrödinger equation.

The I and II forms (14), (15) for the sine-Gordon equation (11) become

$$(20)\qquad \mathrm{I} = (\mathrm{d}x^1)^2 + 1/2\zeta^{-2}\cos\varphi\, \mathrm{d}x^1\mathrm{d}x^2 + \zeta^{-4}16^{-1}(\mathrm{d}x^2)^2\,,$$

$$(21)\qquad \mathrm{II} = -\zeta^{-1}\sin\varphi\, \mathrm{d}x^1\mathrm{d}x^2\,.$$

In this case the Gaussian curvature K is constant and negative: $K = -4\zeta^2$ and the soliton surfaces for the sine-Gordon equation are pseudospherical surfaces. One can see the real spectral parameter ζ has a direct geometric meaning. Hence, the above-introduced concept of soliton surfaces is a far-reaching generalization of pseudospherical surfaces.

The second example concerns soliton systems with $SU_{1,1}$-linear problem. In this case corresponding soliton surfaces are embedded into $SU_{1,1} = M^3(++-)$ — 3-dimensional Minkowski space. The most important example of a soliton system of this class is the Ernst equation of general relativity [23-25]. Hence, all well-known exact solutions of the Ernst equation (like the Schwarzschild solution, the Kerr solution, the Weyl solution, the Tomimatsu-Sato family of solutions etc.) [25] may be interpreted as some surfaces in $M^3(++-)$. These are, of course, the soliton surfaces corresponding to the above-listed solutions.

The explicit co-ordinate representation (8) for multisoliton solutions can in principle be found by means of the so-called Zakharov-Shabat (or dressing) method [17]. 1-soliton surfaces, however, can be calculated in a purely geometric way. These are the so-called helicoids. A helicoid is a surface generated by a plane curve which is uniformly rotated about a fixed axis, and simultaneously uniformly translated in the axis direction.

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[23] F. J. CHINEA: *Phys. Rev. D*, **24**, 1053 (1981).
[24] V. A. BELINSKY and V. E. ZAKHAROV: *Ž. Èksp. Teor. Fiz.*, **75**, 1953 (1978); D. MAISON: *Phys. Rev. Lett.*, **42**, 521 (1978); G. NEUGEBAUER: *Phys. Lett. A*, **75**, 259 (1980).
[25] W. KINNERSLEY: in *General Relativity and Gravitation*, edited by G. SHAVIV and J. ROSEN (New York, N. Y., 1975).

For instance, the 1-soliton surface of the nonlinear Schrödinger equation is shown in fig. 1. The 1-kink surface of the sine-Gordon equation (11) is the so-called Dini pseudo-spherical surface with the tractrix [2,9] as a generator.

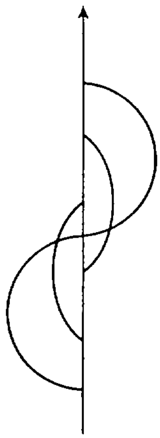

Fig. 1. – 1-soliton surface of the nonlinear Schrödinger equation (10) is generated by a semi-circle.

The above-presented surface-geometric setting allows us to interpret Bäcklund transformations [1,3,21,26,27] as a passage from an «old» soliton surface (defined by an «old» solution) to a « new » soliton surface (defined by a « new » solution) and the connection between the « old » and the « new » solution is a conventional (expressed by a first-order differential equation) Bäcklund transformation. This kind of passage (transformation) is a generalization of the mentioned at the beginning Bianchi-Lie transformation [13,14] for any 2-dimensional (nondiscrete) soliton system. More specifically, consider two solutions φ, ψ, ... and φ', ψ', ... to the soliton system (1). Φ and Φ' are corresponding solutions to the linear problem (2). G-valued function $\Phi'\Phi^{-1} = D$ is called sometimes a Darboux matrix. Any Darboux matrix satisfies the following first-order differential equation:

$$D_{,\mu} = g_\mu(\varphi', \psi', ...; \zeta)D - Dg_\mu(\varphi, \psi, ...; \zeta) . \tag{22}$$

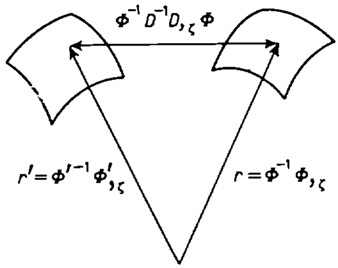

Fig. 2. – Bäcklund transformation as a generalized Bianchi-Lie transformation.

(26) K. Konno and M. Wadati: *Prog. Theor. Phys.*, **52**, 1652 (1975).
(27) D. Levi and O. Ragnisco: *Bäcklund transformations for chiral field equations*, preprint No. 268 1981, Istituto di Fisica, Università di Roma, Italy, *Phys. Lett. A* (in press).

Equation (22) is a conventional Bäcklund transformation [1,26] rewritten in a matrix form [3,21,27]. Thus, any Bäcklund transformation is defined by its Darboux matrix. The same Darboux matrix can be used in the explicit expression of the above-described generalized Bianchi-Lie transformation. See fig. 2. Observe that $D^{-1}D_{,\zeta} \in g$ and $\Phi^{-1}D^{-1}D_{,\zeta}\Phi \in g$ as well (Φ acts as an inner automorphism of g [15]).

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