Model of Quantum Statistics in Terms of a Fluid with Irregular Stochastic Fluctuations Propagating at the Velocity of Light: a Derivation of Nelson's Equations.

J. P. VIGIER

Equipe de Recherche Associée, au C.N.R.S. no. 533
Institut Henri Poicaré, 11 rue Pierre et Marie Curie - 75231 Paris Cedex 05

(ricevuto il 9 Novembre 1978)

Within the frame of the general discussion on the principles and physical content of quantum mechanics (QM) one the most interesting branches since 1952 deals with the possible stochastic nature of its associated statistics. An increasing set of results (1-3) have now established striking formal similarities with classical models of stochastic theory such as Markov processes (4,5).

Two basic obstacles remain however, which have prevented until now the completion of the main statistical interpretation of QM in terms of real physical stochastic motions.

The first obstacle is the existence of a wrong sign (from the classical point of view) in the stochastic version of Newton's second law: a sign which is clearly necessary to derive Schrödinger-type wave equations. For example in the notations of de la Peña and Cetto (3) Newton's law takes the form

$$m(D_c v + D_s u) = F^+ \tag{1}$$

for Brownian motion: in contrast with the form given by NELSON (2) *i.e.*

$$m(D_c v - D_s u) = F^+ , \tag{2}$$

from which he has deduced (combined with the continuity equation) a remarkable derivation of Schrödinger's equation.

The second obstacle is the relativistic generalization of these stochastic models. Indeed HAKIM (6) has shown that it is not enough to write a relativistic generalization

(1) D. BOHM and J. P. VIGIER: *Phys. Rev.*, **96**, 208 (1954).
(2) E. NELSON: *Phys. Rev.*, **150**, 1079 (1966).
(3) L. DE LA PEÑA and A. M. CETTO: *Found. Phys.*, **5**, 355 (1975).
(4) I. FENYES: *Zeits. Phys.*, **132**, 81 (1952); W. WEIZEL: *Zeits. Phys.*, **134**, 264 (1953); N. WIENER and A. SIEGEL: *Phys. Rev.*, **91**, 1551 (1953).
(5) F. GUERRA and P. RUGGIERO: *Phys. Rev. Lett.*, **31**, 1022 (1973).
(6) R. HAKIM: *Journ. Math. Phys.*, **9**, 1805 (1968).

of (2) since if $\Delta t \to 0$ the only value for the diffusion constant ν_0 (in $(\mathrm{d}x)^2 \simeq 2\nu_0 \mathrm{d}t$) compatible with relativistic invariance is $\nu_0 = 0$. As a consequence LEHR and PARK (7) have been led to add to eq. (2) two supplementary axioms *i.e.* *a*) the discretization of time in the stochastic model; *b*) the attribution of the speed of light c to the stochastic particle between interactions with the thermostat. Under these conditions they do indeed recover the Klein-Gordon equation provided antiparticles are considered as particles moving backward in time.

The aim of the present letter is to derive Nelson's equation and quantum statistics from a relativistic generalization of the hydrodynamical model of QM developed by MADELUNG (8), TAKABAYASI (9) and extended to spinning particles by various authors (10).

This classical relativistic model generalizes the nonrelativistic stochastic hydrodynamical model of QM of Bohm and Vigier on terms of a fluid with irregular fluctuations (1). It contains three new physical features.

I) the fluid elements (and the particles) which follow the lines of flow of the fluid with irregular fluctuations are built from extended elements in the sense discussed by Bohm (11) and Souriau (12).

II) The stochastic fluctuations occur at the velocity of light.

III) The fluid is a mixture of extended particles (and antiparticles): the latter being mathematically equivalent to particles moving backward in time (13,14).

The existence of such fluctuations (which induce in the particle a Markov type of Brownian motion) has been shown (1) to lead any initial distribution of the particles in the fluid into a limiting equilibrium distribution $\text{const}\cdot\varrho(x_\mu(\tau))$ proportional to the fluid's average conserved drift density $\varrho(x_\mu(\tau))$. This means that the fluctuations of our Madelung fluid induce on our particles stochastic jumps at the velocity of light (from one line of flow to another) and that such jumps can be decomposed into the regular drift motion $\boldsymbol{v}_\text{d}$ plus an apparent spacelike random part $\boldsymbol{u}_\text{s}$ with $\boldsymbol{v}_\text{d} = \mathrm{d}x_\mu(\tau)/\mathrm{d}\tau$, τ representing the proper time along the drift lines: so that $\boldsymbol{v}_\text{d}\cdot\boldsymbol{v}_\text{d} = -c^2$.

Indeed any velocity $\boldsymbol{w}$ represented by a point P (with $w_\mu w_\mu = 0$) of the light cone can be decomposed into the sum of two four-velocities $\boldsymbol{v}_\text{d}$ and $\boldsymbol{u}_\text{s}$ *i.e.* $\boldsymbol{w} = \boldsymbol{v}_\text{d} + \boldsymbol{u}_\text{s}$ with $\boldsymbol{u}_\text{s}\cdot\boldsymbol{u}_\text{s} \geqslant 0$. Since the three independrent components of $\boldsymbol{w}$ determine the four components of $\boldsymbol{u}_\text{s}$. As a consequence if one considers a particle of the preceding type it undergoes two independent types of motions: *a*) regular motions along the fluid's drift lines of flow with the fluids own velocity $\boldsymbol{v}_\text{d}$ *b*) stochastic jumps in any direction with the velocity of light with a four velocity $\boldsymbol{w}$ satisfying $\boldsymbol{w}\cdot\boldsymbol{w} = 0$.

To establish (*a*)) let us first recall that a particle or a regular fluid element (which can be compared with the stochastic particle and the thermostat's elements in the usual Brownian motion) are now represented in four dimensional space-time by time like hypertubes instead of timelike lines. These hypertubes can be naturally assumed to have a minimum spacelike radius $\bar{r}/2$ which yields the minimum distance $\bar{r}$ which separates two continuous particles in any spacelike section passing through their centre of mass. Independently of the stochastic jumps our drifting fluid is thus comparable

(7) W. LEHR and J. PARK: *Journ. Math. Phys.*, **18**, 1235 (1977).
(8) E. MADELUNG: *Zeits. Phys.*, **40**, 332 (1926).
(9) T. TAKABAYASI: *Prog. Theor. Phys.* (*Japan*), **8**, 143 (1952); **9**, 187 (1953).
(10) Summarized in F. HALBWACHS: *Theorie des fluides à spin* (Paris, 1960).
(11) D. BOHM and J. P. VIGIER: *Phys. Rev.*, **109**, 882 (1958).
(12) F. HALBWACHS, J. M. SOURIAU and J. P. VIGIER: *Journ. Phys. Radium*, **22**, 26 (1961).
(13) M. FLATO, G. RIDEAU and J. P. VIGIER: *Nucl. Phys.*, **61**, 250 (1965).
(14) YA. P. TERLETSKI and J. P. VIGIER: *Žurn. Èksp. Teor. Fiz.*, **13**, 356 (1961).

with a timelike set of extended fibers and the minimum time needed to pass from one of these hypertubes to the next is thus $r/c = \Delta t$ since the jumps occur at the velocity of light. This implies that the proper-time variable which corresponds to adjacent events in our stochastic model have nonzero minimum temporal separation $\Delta\bar{\tau}$.

The second step is just to generalize to our relativistic model the average velocities utilized by de la Peña and Cetto ([3]) to discuss the nonrelativistic theory of classical and quantum-mechanical systems. Let us start (fig. 1) from a four dimensional volume limited on the side by the fluid's regular lines of flow and, at both extremities, by two spacelike constant phase surfaces ([15]) S_1 and S_3. If the domain is small enough such surfaces are separated by an interval $2\Delta\tau$: an interval $\pm\Delta\tau$ separating S_1 and S_3 from a median section S_2. Of course $|\Delta\tau| \geqslant \overline{\Delta\tau}$.

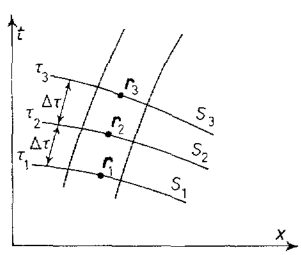

Fig. 1.

As a consequence of the assumed stochastic equilibrium we can treat on the same footing the fluid behaviour and an ensemble of similarly prepared particles characterized by the density $\varrho(\boldsymbol{x}, \tau)$ in configuration space where $\boldsymbol{x}$ represents a point in four dimensional space-time.

We shall now establish that the preceding model leads to the correct quantum-mechanical statistics (governed in our simplified case by the Klein-Gordon equation) in the simple case of a charged scalar particle. The simplification is justified since the introduction of spin complicates, but does not modify significantly, the various steps of our demonstration.

We can describe the average local motions of the elements of the ensemble by the selection of all particles that at proper time $\tau = \tau_2$ are contained in a small four-dimensional volume element around the point $\boldsymbol{r} = \boldsymbol{r}_2$ with co-ordinates $(r_2)_\mu$. This is necessary in our model, since if one starts from a particle in its local drift rest frame (*i.e.* the frame in which the neighbouring fluid element is practically at rest) its stochastic jumps along the light cone can bring it into any neighbouring line of flow: both in the forward and backward proper time direction. As a consequence our general stochastic model implies the use of a four-dimensional stochastic space-time volume element to recover all possible stochastic jumps of each drifting particle. We have thus made the new theoretical step of introducing along with the average space positions the new concept of an average time in a four dimensional volume element.

In order to describe the global motion of this element we select the particles that at proper time τ_2 are contained on a small section (space-volume element) of S_2 limited by the hypertubes boundary. According to our model it is possible to distinguish two different kinds of motion of this volume element during a short interval $\Delta\tau$. Besides its

([15]) J. P. Vigier: *Compt. Rend.*, **266**, 598 (1968).

motion as a whole in the hypertube (which preserves the fluid's scalar density ϱ) the element will suffer variations of ϱ due to the stochastic jumps which move matter from one line of flow to another and will bring fluid across the hypertubes' boundary. Generalizing de la Peña and Cetto ([3])'s ideas we can obtain a simplified description in terms of two quasilocal statistical velocities. If we take any one of the particles of our volume element and call $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ its average mean position at $\tau_1 = \tau_2 - \Delta\tau$ and $\tau_1 = \tau_2 + \Delta\tau$ we can calculate the average of $\boldsymbol{r}_3 - \boldsymbol{r}_2$ over the subensemble defined by the particles which belong to our small volume element. We call these average values the mean and denote them with $\langle\ \rangle$. We thus write

$$\boldsymbol{r}_3 - \boldsymbol{r}_2 = \langle \boldsymbol{r}_3 - \boldsymbol{r}_2 \rangle + \delta_+ \boldsymbol{r} \quad \text{and} \quad \boldsymbol{r}_2 - \boldsymbol{r}_1 = \langle \boldsymbol{r}_2 - \boldsymbol{r}_1 \rangle + \delta_- \boldsymbol{r}\,. \tag{3}$$

Since one must assume (in our model) the homogeneity, isotropy and time independence of our stochastic mechanism the change variable $\delta_\pm r_i$ must satisfy $\langle(\delta_+ r_i)\rangle = \langle(\delta_- r_i)\rangle$ so what we can omit the indexes from such expressions and write in general $\langle \delta \boldsymbol{r}_i \rangle = 0$.

We can now derive from (3) two different velocities *i.e.*

$$\boldsymbol{b}_+(2) = \big((\boldsymbol{r}_3 - \boldsymbol{r}_2)/\Delta\tau\big) \quad \text{and} \quad \boldsymbol{b}_-(2) = \big((\boldsymbol{r}_2 - \boldsymbol{r}_1)/\Delta\tau\big)\,,$$

whose mean values

$$\boldsymbol{v}_+(2) \equiv \langle \boldsymbol{b}_+(2) \rangle = \langle\big((\boldsymbol{r}_3 - \boldsymbol{r}_2)/\Delta\tau\big)\rangle \quad \text{and} \quad \boldsymbol{v}_-(2) \equiv \langle \boldsymbol{b}_-(2) \rangle = \langle\big((\boldsymbol{r}_2 - \boldsymbol{r}_1)/\Delta\tau\big)\rangle$$

are the relativistic generalization of the mean forward and backward velocities. From these one can derive the regular fluid's velocity $\boldsymbol{v}_d$ and a stochastic belocity $\boldsymbol{u}_s$ through the relations

$$\boldsymbol{v}_d(2) \equiv \langle\big((\boldsymbol{r}_3 - \boldsymbol{r}_1)/2\,\Delta\tau\big)\rangle = \tfrac{1}{2}(\boldsymbol{v}_+ + \boldsymbol{v}_-) \tag{4}$$

$$\boldsymbol{u}_s(2) \equiv \langle[(\boldsymbol{r}_3 - \boldsymbol{r}_2) - (\boldsymbol{r}_2 - \boldsymbol{r}_1)]/2\,\Delta\tau\rangle = \tfrac{1}{2}(\boldsymbol{v}_+ - \boldsymbol{v}_-) \tag{5}$$

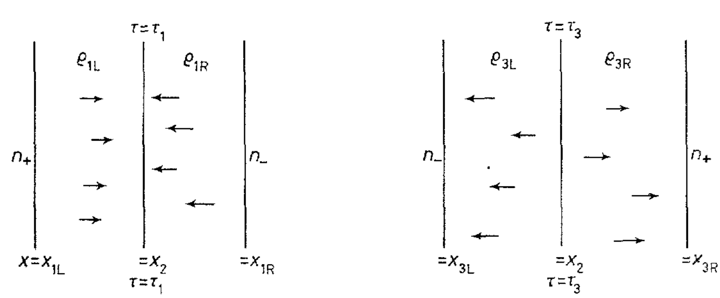

Fig. 2. – $\boldsymbol{x}_{1\mathrm{L}}(\boldsymbol{x}_{1\mathrm{R}})$ is the average position of the $n_+(n_-)$ particles at $\tau_1 = \tau_2 - \Delta\tau$ and $\boldsymbol{x}_{3\mathrm{R}}(\boldsymbol{x}_{3\mathrm{L}})$ is the average position of the same particle at $\tau_3 = \tau_2 + \Delta\tau$: $\varrho_\mathrm{L}(\varrho_\mathrm{R})$ being the densities of particles to the left (right) of $\boldsymbol{x} = x_2$.

Now the stochastic velocity $\boldsymbol{u}_s$ can be determined in any spacelike direction by calculating the flow between τ_1 and τ_3 of all elements which cross a drift timelike plane passing through $\boldsymbol{r}_2$ and orthogonal to a spacelike direction x. Indeed let us consider (see fig. 2) an ensemble of fluid elements (particles) which are at τ_2 in the neighbourhood of x. If $\varrho_{1L}(\varrho_{1R})$ then represents the scalar densities in the neighbourood of $x_{1L}(x_{1R})$ at $\tau=\tau_1$ we see that these densities are related to n_+ and n_- through

$$n_+=(x_{3R}-x_2)\,\varrho_{3R}=(x_2-x_{1L})\,\varrho_{1L} \quad \text{and} \quad n_-=(x_{1R}-x_2)\,\varrho_{1R}=(x_2-x_{3L})\,\varrho_{3L}\,.$$

This yields

$$x_1+x_3-2x_2=$$
$$=(1/(n_++n_-))[-\varrho_{1L}(x_2-x_{1L})^2+\varrho_{1R}(x_{1R}-x_2)^2+\varrho_{3R}(x_{3R}-x_2)^2-\varrho_{3L}(x_2-x_{3L})^2]$$

which can be averaged over the ensemble. Since each of the parentheses then become $\langle(\delta x)^2\rangle$ we can write to the first approximation (with $n_++n_-=2\varrho(\tau_2)\Delta x$):

$$\boldsymbol{u}_s=\frac{\langle x_1+x_3-2x_2\rangle}{2\,\Delta\tau}=\frac{\langle(\delta x)^2\rangle}{2\,\Delta\tau}\frac{1}{\varrho}\boldsymbol{\nabla}\varrho=D\frac{\boldsymbol{\nabla}\varrho}{\varrho}\,, \tag{6}$$

if we define as usual the diffusion coefficient as $D=\langle(\delta r_i)^2\rangle/2\,\Delta\tau$ and neglect higher-order terms in $\Delta\tau$. D is always >0 since our quantum jumps are spacelike.

This is exactly the relativistic generalization of Einstein's definition [16] of the stochastic velocity in Brownian motion. We have further $\boldsymbol{v}_\pm=\boldsymbol{v}_d\pm\boldsymbol{u}_s$ which connect out forward (particle) and backward (antiparticle) velocities with the fluids regular drift velocity $\boldsymbol{v}_d$ and its stochastic velocity $\boldsymbol{u}_s$.

The second step is to associate the two velocities needed to describe our motion to four accelerations required to describe the forward and backward changes of these velocities. To do this we require the existence of our minimum proper time interval $\Delta\tau$ which allows us to define the four accelerations

$$\begin{cases} \boldsymbol{b}_+(3)-\boldsymbol{b}_+(2)=\boldsymbol{a}_+^+ +\delta_+\boldsymbol{b}_+\,,\\ \boldsymbol{b}_-(3)-\boldsymbol{b}_-(2)=\boldsymbol{a}_-^+ +\delta_+\boldsymbol{b}_-\,,\\ \boldsymbol{b}_+(2)-\boldsymbol{b}_+(1)=\boldsymbol{a}_+^- +\delta_-\boldsymbol{b}_+\,,\\ \boldsymbol{b}_-(2)-\boldsymbol{b}_-(1)=\boldsymbol{a}_-^- +\delta_-\boldsymbol{b}_-\,,\end{cases} \tag{7}$$

which evidently lead to systematic drift and stochastic derivative operators. Indeed if we define as D_d and D_s the following operations on a general function $f(\boldsymbol{r})$ of the stochastic variable $\boldsymbol{r}$, *i.e.*

$$D_d f(\boldsymbol{r}_2)=\langle[f(\boldsymbol{r}_3)-f(\boldsymbol{r}_1)]/2\,\Delta\tau\rangle \quad \text{and} \quad D_s f(\boldsymbol{r}_2)=\langle[f(\boldsymbol{r}_1)+f(\boldsymbol{r})-2f(\boldsymbol{r}_2)]/2\,\Delta\tau\rangle\,,$$

which are evidently related with the forward (D^+) and backward (D^-) derivative operators through the relation: $D^\pm=D_d\pm D_s$ we see they thus correspond to scalar (proper time type) derivatives in timelike and spacelike directions ... and yield the drift and stochastic velocities through $\boldsymbol{v}_d=D_d\boldsymbol{r}$ and $\boldsymbol{u}_s=D_s\boldsymbol{r}$: where the dummy index 2 has been omitted. This generalizes $v_\mu=\mathrm{d}x_\mu/\mathrm{d}\tau$ and lead to the preceding mean accelerations through the expressions $\boldsymbol{a}_\pm^+=D^+\boldsymbol{v}_\pm$ and $\boldsymbol{a}_\pm^-=D^-\boldsymbol{v}_\pm$.

([16]) A. EINSTEIN: *Investigations on the Theory of Brownian Movement* (New York, N. Y., 1956).

Moreover a development in Taylor series yields

$$(8)\qquad \begin{cases} D_d f = \dfrac{\partial f}{\partial \tau} + (\boldsymbol{v}_d \cdot \boldsymbol{\nabla}) f + \dots , \\ D_s f = (\boldsymbol{u}_s \cdot \boldsymbol{\nabla}) f + D(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}) f + \dots , \end{cases}$$

where the diffusion coefficient D is given as before by the relation

$$\langle \delta r_i \cdot \delta r_j / 2\,\Delta\tau \rangle = D\delta_{ij} ,$$

in the drift rest frame: diffusion in time representing, as before, particle-antiparticle transition: δr_i and δr_j denoting any pair of Cartesian components of $\delta_\pm \boldsymbol{r}$ ... which are assumed to be statistically independent if $i \neq j$.

The third (essential) step is to derive the covariant generalization of Nelson's equation, in our model. To do that we recall that any detailed description must start from the general equation

$$m\ddot{\boldsymbol{r}} = \boldsymbol{f}_d + \boldsymbol{f}_s ,$$

where $\boldsymbol{f}_d$ represents the drift spacelike forces and $\boldsymbol{f}_s$ the purely random effects the $^\bullet$ denoting proper-time derivatives. The corresponding statistical theory must, according to our model, start from the ensemble of particles which at any proper time τ_2 lie in the neighbourhood of $\boldsymbol{r}_2$. The mean of the preceding relation thus becomes

$$(9)\qquad m\langle \ddot{\boldsymbol{r}} \rangle = \boldsymbol{F}_d + \boldsymbol{F}_s = \boldsymbol{F}, \quad \text{where } \boldsymbol{F}_d = \langle \boldsymbol{f}_d \rangle \text{ with } \boldsymbol{F}_s = \langle \boldsymbol{f}_s \rangle = 0.$$

Since the mean value of $\ddot{\boldsymbol{r}}$ is taken over the same ensemble utilized to define our average velocities and accelerations in the preceding steps, it must be expressed as a linear combination of $\boldsymbol{a}_\pm^\pm$. To determine these combinations, we remark that $\langle \ddot{\boldsymbol{r}} \rangle$ and $\langle f_d \rangle$ can be split into two parts *i.e.* a part $\langle \ddot{\boldsymbol{r}} \rangle^+$ (or $\langle f_d \rangle^+$) which is invariant under proper time reversal *i.e.* $\tau_3 - \tau_2 \rightarrow \tau_1 - \tau_2$ and a part $\langle \ddot{\boldsymbol{r}} \rangle^-$ (or $\langle f_d \rangle^-$) that changes sign under this discrete symmetry which changes $\boldsymbol{v}_d$ but conserves $\boldsymbol{u}_s$. Combining equation (9) with its counterpart obtained through a proper-time reversal operation we obtain the new set of equations

$$(10)\qquad m\langle \ddot{\boldsymbol{r}} \rangle^\pm = \boldsymbol{F}_d^\pm .$$

We now make the final step in our demonstration of Nelson's equation (2) by examining the implications of eq. (10). The first implication is the importance of the proper-time relation $m\langle \ddot{\boldsymbol{r}} \rangle^+ = \boldsymbol{F}_d^+$ which evidently represents the stochastic generalization of Newton's law for our model. Indeed the usual four-dimensional acceleration $\ddot{\boldsymbol{x}}$ of a classical point $\boldsymbol{x}$ satisfies $\dot{\boldsymbol{x}}\ddot{\boldsymbol{x}} = 0$ (since $\dot{\boldsymbol{x}} \cdot \dot{\boldsymbol{x}} = -c^2$) and is invariant under proper-time reversal. The same holds for our stochastic case since: *a*) the drift acceleration $\dot{\boldsymbol{v}}_d$ is orthogonal to $\boldsymbol{v}_d$; *b*) the stochastic spacelike velocity $\boldsymbol{u}_s$ is locally orthogonal to $\boldsymbol{v}_d$ so that the corresponding stochastic accelerations (which vanish on the average since $\langle \boldsymbol{F}_s \rangle = 0$) are thus always orthogonal to $\boldsymbol{v}_d$.

The second implication is that $\langle \ddot{\boldsymbol{r}} \rangle^+$ must be expressed by just the linear combination of relations (7) which are proper-time–inversion invariant *i.e.* $(\boldsymbol{a}_-^+ + \boldsymbol{a}_+^-)$ or $(\boldsymbol{a}_+^+ + \boldsymbol{a}_-^-)$ or a linear combination therefrom.

The third implication is that a mean acceleration (which corresponds mathematically to second-order proper-time derivatives) should be defined physically only by the motions of fluid elements surrounding $\boldsymbol{r}_2$ *i.e.* enclosed within the four-dimensional volume element limited by S_1 and S_3 utilized to define mean quantities. We deduce therefrom and from the explicit form of the $\boldsymbol{a}$'s given in eq. (13), that the only quantity of this type invariant under $\tau \to -\tau$ is $(\boldsymbol{a}^{\pm}_{-} + \boldsymbol{a}^{\mp}_{+})$. Indeed the definition of $(\boldsymbol{a}^{\pm}_{+} + \boldsymbol{a}^{-}_{-})$ implies knowledge of the behaviour of fluid elements which lie outside our volume since it contains four-velocities of elements which are crossing S_3 and S_1 in the backward and forward directions *i.e.* are leaving this volume. Moreover one sees that the combination $\langle \ddot{\boldsymbol{r}} \rangle = (\boldsymbol{a}^{\pm}_{-} + \boldsymbol{a}^{\mp}_{+})$ evidently represents the relativistic definition of the sum of the mean accelerations of antiparticles $(\boldsymbol{a}^{\pm}_{-})$ and particles $(\boldsymbol{a}^{\mp}_{+})$ passing through $\boldsymbol{r}_2$ at $\tau = \tau_2$:

As a consequence we must write relation (10) in the form

$$\tfrac{1}{2} m(\boldsymbol{a}^{\pm}_{-} + \boldsymbol{a}^{\mp}_{+}) = \boldsymbol{F}^{+}\,, \tag{11}$$

which is exactly the relativistic generalization of the form given by de la Peña and Cetto [3] to Nelson's equation. Clearly eq. (11) contains particle-antiparticle symmetry.

The same argument applies to the — part of (10). Indeed the only combinations of $a^{\pm}_{\pm}$ that change sign under proper-time reversal are $(\boldsymbol{a}^{\pm}_{+} - \boldsymbol{a}^{-}_{-})$ and $(\boldsymbol{a}^{\pm}_{-} - \boldsymbol{a}^{\mp}_{+})$ and the second only is exclusively defined by the motion of fluid elements between S_1 and S_3. We thus have $\frac{1}{2} m(\boldsymbol{a}^{\pm}_{-} - \boldsymbol{a}^{\mp}_{+}) = F^{-}$ which satisfies the continuity equation and is compatible with the introduction of the Lorentz force for charged fluid elements. Moreover these relations can be rewritten with the help of the definitions of D_{d} and D_{s} into the form

$$m(D_{\text{d}} \boldsymbol{v}_{\text{d}} - D_{\text{s}} \boldsymbol{u}_{\text{s}}) = \boldsymbol{F}^{+} \tag{12a}$$

and

$$m(D_{\text{d}} \boldsymbol{u}_{\text{s}} + D_{\text{s}} \boldsymbol{v}_{\text{d}}) = \boldsymbol{F}^{-}\,. \tag{12b}$$

In eq. (12*b*) both sides tend (as they should) to zero in the nonstochastic limit.

The last step of our demonstration is, of course, the derivation of the integrated stochastic equations which result from (11) and (12). This can evidently be done in two ways. The first is to start from the drift rest frame at $\boldsymbol{r}_2$ and define as usual Smoluchowski's densities ϱ and P_R. The interested reader can then check immediately that since we have demonstrated *a*) and Nelson's equation (11) one can just follow Lehr's and Park's demonstration (7) to recover Klein-Gordon's equation.

The second way (which we will choose instead since it throws some interesting new light on the physics of the problem) is so complete the relativistic generalization of de la Peña's work [3].

In order to integrate (12*a*) and (12*b*) we define the quantities

$$D_{\text{q}} = D_{\text{d}} + \varepsilon D_{\text{s}}\,, \quad \boldsymbol{v}_{\text{q}} = \boldsymbol{v}_{\text{d}} + \varepsilon \boldsymbol{u}_{\text{s}} \quad \text{and} \quad \boldsymbol{F}_{\text{q}} = \boldsymbol{F}^{+} + \varepsilon \boldsymbol{F}^{-} \tag{13}$$

with $\varepsilon = \pm i$.

Relations (12*a*) and (12*b*) can thus be combined into the complex eqs. (14) *i.e.* $m D_{\text{q}} \boldsymbol{v}_{\text{q}} = \boldsymbol{F}_{\text{q}}$ which can be integrated if one assumes that $\boldsymbol{F}_{\text{q}}$ is just the general Lorentz force applied to our fluid of spinless charged particles *i.e.*

$$(F_{\text{q}})_\mu = (e/c)(\partial_\mu A_\mu - \partial_\mu A_\mu)(v_{\text{q}})_\mu$$

with $\boldsymbol{\nabla} \cdot \boldsymbol{A} = \partial_\mu A_\mu = 0\,.$

Indeed if we then write the relation (15) *i.e.* $\boldsymbol{v}_q = \varepsilon D\cdot\nabla S_q - (e/mc)\boldsymbol{A}$, where $D = \hbar/2m$, ∇ and $\boldsymbol{A}$ denoting the four-vectors ∂_μ and A_μ and $S_q = \text{const}$ representing the surfaces orthogonal to the four velocity $\boldsymbol{v}_q$. If we then utilize the Taylor developments (8) and substitute (15) and (16) into (14) we obtain the general relation

$$\nabla(2\varepsilon mD\dot{S}_q + \tfrac{1}{2}m\boldsymbol{v}_q\cdot\boldsymbol{v}_q + \varepsilon mD\nabla\cdot\boldsymbol{v}_q) = 0\,, \tag{16}$$

which admits as first integral eq. (17) *i.e.*

$$-2\varepsilon m\dot{S}_q = 2\varepsilon^2 mD^2[\nabla S_q\nabla S_q + \nabla\cdot\nabla S_q] - 2\varepsilon D(e/c)\boldsymbol{A}\cdot\nabla S_q - \varepsilon D(e/c)\nabla\boldsymbol{A} + (e^2/2mc^2)\boldsymbol{A}\cdot\boldsymbol{A}\,.$$

Introducing further the wave function $\varphi(\boldsymbol{r},\tau) = \exp[\varepsilon mc^2\tau/2\hbar]\psi(\boldsymbol{r})$ *i.e.*

$$\varphi(\boldsymbol{r},\tau) = \exp[\varepsilon mc^2\tau/2\hbar]\,\varrho^{\frac{1}{2}}(\boldsymbol{r})\exp[\varepsilon S_q(\boldsymbol{r})]\,,$$

we obtain from (7) the usual relativistic generalization of the Schrödinger equation, *i.e.*

$$2mD\varepsilon\dot{\varphi} = (1/m)[2mD\varepsilon\nabla - (e/c)\boldsymbol{A}]^2\varphi\,, \tag{18}$$

which reduces to the Klein-Gordon equation

$$(\partial_\mu - \varepsilon(e/c)A_\mu)^2\psi - (m^2c^2/\hbar^2)\psi = 0\,. \tag{19}$$

Relation (19) yields [15] the relations

$$\mathrm{d}\varrho/\mathrm{d}\tau = \dot{\varrho} = 0 \qquad \text{and} \qquad \mathrm{d}(M\boldsymbol{v}_d)/\mathrm{d}\tau = -\nabla(Mc^2) \tag{20}$$

with

$$M^2 = \{m^2 - (\hbar^2/c^2)(\Box R/R)\}\,, \qquad \psi^*\psi = R^2 \qquad \text{and} \qquad \varrho = (M/m)R^2\,.$$

* * *

The author wants to express his thanks to Profs. L. DE BROGLIE, D. BOHM and M. FLATO for long and helpful past discussions stressing the possible importance of Einstein's views on Brownian motion in the interpretation of QM. He is especially grateful to Prof. LUIS DE LA PEÑA-AUERBACH and A. M. CETTO not only for crucial suggestions but also for help in the preparation of this work. Without this help, it would not have been completed.