A Reformulation and a Possible Modification of Quantum Mechanics and the EPR Paradox.

G. C. GHIRARDI, A. RIMINI and T. WEBER

Istituto di Fisica Teorica dell'Università - Trieste
Istituto Nazionale di Fisica Nucleare - Sezione di Trieste

Summary. — A reformulation of quantum mechanics is introduced, which describes the « states » of an ensemble of quantum systems by means of positive real functionals on the Hilbert space of the systems. This reformulation allows us to generalize quantum mechanics in such a way as to induce the transition from second- to first-kind mixtures, which has been suggested to occur by various authors in order to eliminate the EPR paradox. We explicitly build up a dynamical equation for the functionals, which reduces to the Schrödinger equation when the subsystems of a composite quantum system are close together, and gives rise, altering the quantum-mechanical evolution, to a transition to a first-kind mixture when the component subsystems are far apart. This transition is such that, at any time, the predictions concerning measurements of observables referring to one of the subsystems coincide with those which would follow from the pure Schrödinger evolution. The deviations from the standard theory affect, therefore, only the correlations between the subsystems.

1. – Introduction.

The nonseparability of the subsystems of a composite quantum system is at the origin of the conceptual difficulties of quantum mechanics which are evidentiated, for instance, by the so-called EPR paradox. A way to overcome such a deadlock is to assume that standard quantum mechanics holds when the subsystems are close together, but that, when they are far apart, each individual subsystem regains definite physical properties. In the quantum-mechanical language this means that, when the subsystems are far apart, the

second-kind mixture associated to a nonfactorizable pure state of the composite system changes into a first-kind mixture, *i.e.* a statistical mixture of factorized states. Such a possibility has been considered by BOHM and AHARONOV [1] and by other authors [2], and has also been the object of some experimental investigations [3]. The fact that there exists a definite difference between the quantum predictions given by a pure nonfactorizable state and those deriving from any mixture of factorized states, as implied by Bell's inequality [4,2], has made it possible to discuss for a long time on the above-mentioned hypothesis, without any investigation of its formal implications and without any attempt to build a dynamical model for it. In a recent paper [5] it has been proved that the usual description of ensembles of quantum systems based on the density operator formalism does not allow one to describe a transition of the considered type. In fact, such a transition would necessarily lead from (different) statistical ensembles corresponding to the same density operator to statistical ensembles corresponding to different density operators. Therefore, to describe such a transition, it is necessary to deal directly with ensembles and to take into account their detailed composition. In turn, this means to ascribe a weight to each state of the Hilbert space of the considered system and to describe the time evolution of the system by the evolution of such weights. In other words, an ensemble is characterized by a numerically valued real and positive functional, defined on the rays of the Hilbert space of the system, and the time evolution is a time-dependent mapping of functionals into functionals.

The main purpose of this paper is to reformulate quantum mechanics in terms of functionals on Hilbert rays, and to show how it is possible to modify the law of time evolution in such a way as to include the description of the above-considered transition from second- to first-kind mixtures. In sect. **2** and **3** we develop the formalism of density functionals and we express in terms of it the Schrödinger evolution. In sect. **4** we explicitly determine the density functionals for some particular cases which will be useful in what follows. Section **5** is devoted to deriving the general condition that a density functional must satisfy in order to describe a mixture of factorized states. Section **6** deals with the problem of determining density functionals for composed systems by starting from those for the component subsystems. In sect. **7** we define a general class of mappings among density functionals for composite systems leading to density functionals corresponding to a first-kind mixture. Using the results of the preceding sections, we derive in sect. **8** the dynamical equation gen-

[1]   D. BOHM and Y. AHARONOV: *Phys. Rev.*, **108**, 1070 (1957).
[2]   J. M. JAUCH: *Rendiconti S.I.F.*, Course IL (New York, N. Y., and London, 1971).
[3]   L. R. KASDAY, J. D. ULLMAN and C. S. WU: *Nuovo Cimento*, **25** B, 633 (1975), and references quoted therein.
[4]   J. S. BELL: *Physics*, **4**, 195 (1964).
[5]   G. C. GHIRARDI, A. RIMINI and T. WEBER: *Nuovo Cimento*, **31** B, 177 (1976).

eralizing the Schrödinger equation and accounting for the evolution which leads, under the proper conditions, to a first-kind mixture. In sect. **9** we make some conclusive remarks and we point out some problems which deserve further investigation. We have confined in the appendix the density functional description of the reduction of the wave packet due to measurements.

2. – The density functional formalism.

As stated in the introduction, we will describe the «state» of a quantum system by means of a functional defined on the rays of the Hilbert space of the system, the value of the functional representing the weight with which the state associated to the ray enters the ensemble. In practice, we shall use functionals $p(\psi)$, defined on the vectors ψ of the Hilbert space, such that

$$(2.1) \qquad\qquad p(c\psi) = p(\psi), \qquad\qquad c \text{ complex} \neq 0.$$

Owing to its meaning, $p(\psi)$ must be a real, nonnegative numerical function, which we shall call density functional. The density functional $p(\psi)$ may be different from zero on a single state (pure case), on a finite or countable set of states, or even on a continuous set of states. The latter case is a description of a mixture for which the only information we have about the states of the systems of the ensemble is that these states lie in a certain region of the Hilbert space. For example, it is often considered [2] the so-called isotropic mixture of spin-$\frac{1}{2}$ particles, in which the particles are polarized along uniformly distributed directions in the physical space.

In order to normalize a continuous density functional $p(\psi)$, or to compare the (statistical) weights corresponding to different regions in the space of states, it is necessary to define the integral of $p(\psi)$ over a region of such a space. To this purpose, we limit ourselves to the case in which the space of the state vectors of the system is finite dimensional. This is the case, e.g., when only the spin degrees of freedom are considered. The integral over a region of the (complex N-dimensional) Hilbert space $\mathscr{H}$, even though it makes reference to a specific orthonormal basis in the space, must be invariant with respect to unitary transformations of the basis. We start by considering the element of volume

$$(2.2) \qquad\qquad \mathrm{d}\mathscr{H} = \mathrm{d}x_1\, \mathrm{d}y_1\, \mathrm{d}x_2\, \mathrm{d}y_2 \ldots \mathrm{d}x_N\, \mathrm{d}y_N,$$

where

$$(2.3) \qquad\qquad x_k + iy_k = z_k = \langle e_k | \psi \rangle,$$

$e_1, e_2, \ldots, e_N$ being a complete orthonormal basis in $\mathscr{H}$. The invariance of $\mathrm{d}\mathscr{H}$ under unitary transformations follows from the fact that any unitary trans-

formation in a complex N-dimensional linear vector space induces an orthogonal transformation in the corresponding real $2N$-dimensional linear vector space. We shall use polar co-ordinates in each (x_k, y_k)-plane, so that

(2.4)
$$d\mathscr{H} = \varrho_1 \, d\varrho_1 \, d\theta_1 \, \varrho_2 \, d\varrho_2 \, d\theta_2 \dots \varrho_N \, d\varrho_N \, d\theta_N \,,$$

where

(2.5)
$$\varrho_k \exp [i\theta_k] = z_k \,.$$

The requirement (2.1) implies that $p(\psi)$ does not depend on the square norm $\sum_{k=1}^{N} \varrho_k^2$. Furthermore, in the space of the angles θ_k, $p(\psi)$ is a constant along the lines orthogonal to the hyperplane $\sum_{k=1}^{N} \theta_k = 0$. Finally, $p(\psi)$ must take the same value on those of the above-said lines for which any of the angles θ_k is changed by an integer multiple of 2π. It follows that, if we consider a hypercube having edges parallel to the θ-axes of length 2π, all rays of the Hilbert space cross the hypercube, and the sum of the lenghts of the portions inside the hypercube of the lines corresponding to the same ray is $2\pi\sqrt{N}$. To integrate over the rays, we shall use the element of volume

(2.6)
$$d\psi = \delta \left(\sum_{k=1}^{N} \varrho_k^2 - 1 \right) d\mathscr{H} \,,$$

and, for $f(\psi)$ satisfying (2.1), we shall write

(2.7)
$$\int f(\psi) \, d\psi = \int_0^\infty \varrho_1 \, d\varrho_1 \dots \int_0^\infty \varrho_N \, d\varrho_N \int_0^{2\pi} d\theta_1 \dots \int_0^{2\pi} d\theta_N \, \delta \left(\sum_{k=1}^{N} \varrho_k^2 - 1 \right) f(\varrho_1, \dots, \varrho_N, \theta_1 \dots, \theta_N) \,.$$

The δ-function in (2.6) and (2.7) restricts the integration to the normalized ψ's. The invariance of the integral (2.7) under a change of the Hilbert-space basis follows from the invariance of $d\mathscr{H}$ and $\sum_{k=1}^{N} \varrho_k^2$ and from the fact that all the above-described hypercubes are equivalent. We normalize the density functional by requiring

(2.8)
$$\int p(\psi) \, d\psi = 1 \,.$$

The expression for the mean value of an observable A is now obviously given by

(2.9)
$$\langle A \rangle = \int d\psi \, p(\psi) \langle \psi | A | \psi \rangle \,,$$

and the density matrix for the ensemble characterized by $p(\psi)$ is

(2.10)
$$\varrho = \int d\psi \, p(\psi) |\psi\rangle \langle \psi| \,.$$

It is trivial to verify that

$$(2.11) \qquad \langle A \rangle = \mathrm{Tr}\,(A\varrho)\,.$$

3. – The Schrödinger equation.

To express the time evolution given by the Schrödinger equation in the language of the density functionals, one needs to define the Hilbert-space gradient of a functional. We denote such an operation by the symbol grad and define it through

$$(3.1) \qquad \langle e_k | \mathrm{grad}\, p \rangle = \frac{1}{2}\left(\frac{\partial p}{\partial x_k} + i \frac{\partial p}{\partial y_k}\right).$$

It is easy to show that the components of grad p defined by (3.1) actually transform, under unitary transformations of the basis, like the components of a vector.

The Schrödinger evolution of $p(\psi, t)$ is characterized by the fact that $p(\psi, t)$ must take at time t the same value on the state $\exp[-iHt]\varphi$ that $p(\psi, 0)$ had on the state φ. Then we write

$$(3.2) \qquad p(\exp[-iHt]\,\psi, t) = p(\psi, 0)\,,$$

which implies

$$(3.3) \qquad \frac{\mathrm{d}}{\mathrm{d}t} p(\exp[-iHt]\psi, t) = 0\,.$$

Putting

$$(3.4) \qquad \langle e_k | \exp[-iHt]\,\psi \rangle = \langle e_k | \psi(t) \rangle = z_k(t) = x_k(t) + i y_k(t)\,,$$

we have

$$(3.5) \qquad \frac{\partial p}{\partial t} + \sum_{k=1}^{N}\left(\frac{\partial p}{\partial x_k}\dot{x}_k + \frac{\partial p}{\partial y_k}\dot{y}_k\right) = 0\,.$$

If we take into account that p is real and express $\partial p/\partial x_k$ and $\partial p/\partial y_k$ in terms of the components of grad p, eq. (3.5) becomes

$$(3.6) \qquad \frac{\partial p}{\partial t} + \langle \mathrm{grad}\, p | \dot{\psi} \rangle + \langle \dot{\psi} | \mathrm{grad}\, p \rangle = 0\,.$$

Using $\dot{\psi} = -iH\psi$, we get for $p(\psi, t)$ the evolution equation

$$(3.7) \qquad \frac{\partial p}{\partial t} = 2\,\mathrm{Im}\,\langle \psi | H\,\mathrm{grad}\, p \rangle\,,$$

which is equivalent to the Schrödinger equation. If we write symbolically

$$(3.8) \qquad\qquad \frac{\partial p}{\partial t} = Sp \; ,$$

it is easily verified that the mapping among functionals denoted by S is linear with respect to the field of real numbers.

Equation (3.7), being equivalent to the Schrödinger equation, does not change the structure of the set of states of $\mathscr{H}$ on which $p(\psi)$ is different from zero; in particular, a $p(\psi, 0)$, which corresponds to a pure state, evolves into a $p(\psi, t)$, which again describes a pure state. This equation, however, can be generalized, by adding new terms at the right-hand side, to describe a transition from a $p(\psi)$ which is different from zero only on one ray of $\mathscr{H}$ to one which takes nonzero values over several rays. The obtained formulation of the standard quantum evolution is, therefore, particularly suitable for generalizations which can describe the transition from second- to first-kind mixtures which we are interested in.

4. – Examples of density functionals.

In this section we build explicitly the density functionals for some particular cases which shall be useful in what follows.

Let us start with the density functional $p_{\bar{\psi}}(\psi)$ corresponding to the pure state $\bar{\psi}$. Using the variables ϱ_k and θ_k defined by (2.3) and (2.5), we introduce the ray variables

$$(4.1a) \qquad\qquad \sigma_k = \varrho_k \Big/ \sqrt{\sum_{j=1}^{N} \varrho_j^2} \, , \qquad\qquad k = 2, 3, ..., N,$$

$$(4.1b) \qquad\qquad \eta_k = \theta_k - \theta_1 \, , \qquad\qquad k = 2, 3, ..., N,$$

and analogous variables $\bar{\sigma}_k$, $\bar{\eta}_k$ for the state $\bar{\psi}$. Then the functional $p_{\bar{\psi}}(\psi)$ can be written as

$$(4.2) \qquad\qquad p_{\bar{\psi}}(\psi) = \frac{2^N}{2\pi} \prod_{k=2}^{N} [\delta(\sigma_k^2 - \bar{\sigma}_k^2) \, \delta(\eta_k - \bar{\eta}_k)] \, ,$$

where

$$(4.3) \qquad\qquad \delta(\eta_k - \bar{\eta}_k) = \sum_{j=-\infty}^{+\infty} \delta(\eta_k - \bar{\eta}_k - 2j\pi)$$

is a periodic δ-function having the property that its integral over any interval of length 2π is 1. Equation (4.2) gives the normalized density functional corresponding to the pure state described by the vector $\bar{\psi}$. From (4.2) one sees

that $p_{\bar{\psi}}(\psi)$ is a ray function for the dependence both on ψ and on $\bar{\psi}$. Moreover, $p_{\bar{\psi}}(\psi)$ has the following formal property of a δ-function for ray functionals:

$$(4.4) \qquad \int d\bar{\psi} f(\bar{\psi}) p_{\bar{\psi}}(\psi) = f(\psi), \qquad \text{provided} \qquad f(c\psi) = f(\psi),$$

which shows that any real positive ray functional can be expressed as a linear combination of the basic density functionals $p_{\bar{\psi}}(\psi)$.

Another density functional, which will be useful in what follows and will be denoted by $p_{\mathcal{M}}(\psi)$, is the one corresponding to a uniform mixture of the states of a given m-dimensional linear manifold $\mathcal{M}$. By uniform mixture we denote the mixture in which all vectors of $\mathcal{M}$ appear with equal weight. Let us introduce a basis $\{e_k\}$, $k=1,...,N$, in $\mathscr{H}$, whose first m elements span the linear manifold $\mathcal{M}$. According to the above characterization of the uniform mixture we can write

$$(4.5) \qquad p_{\mathcal{M}}(\psi) = \frac{(m-1)!}{\pi^m} \int d\bar{\psi} \, p_{\bar{\psi}}(\psi) \prod_{k=m+1}^{N} \left[\frac{2}{\pi} \delta(\bar{\sigma}_k^2)\right] = \frac{(m-1)!}{\pi^m} \prod_{k=m+1}^{N} \left[\frac{2}{\pi} \delta(\sigma_k^2)\right].$$

In calculating the normalization constant, use has been made of the fact that

$$(4.6) \qquad \int_0^\infty \varrho \, d\varrho \, \delta(\varrho^2) = \tfrac{1}{4}.$$

In eq. (4.5) the δ-functions appearing under the integral sign guarantee that no state which does not belong to $\mathcal{M}$ contributes to $p_{\mathcal{M}}(\psi)$. An equivalent way of constructing $p_{\mathcal{M}}(\psi)$ would have been to integrate $p_{\bar{\psi}}(\psi)$ over the linear manifold $\mathcal{M}$. In so doing one would have obtained essentially the same result (4.5) with only the formal complication of an irrelevant dependence on the angles θ, associated to vanishing ϱ_i's. It is immediately seen that $p_{\mathcal{M}}(c\psi) = p_{\mathcal{M}}(\psi)$ (c complex $\neq 0$). It is also immediate to check that the density operator $\varrho_{\mathcal{M}}$ corresponding to the density functional (4.5) is

$$(4.7) \qquad \varrho_{\mathcal{M}} = \frac{1}{m} \sum_{k=1}^{m} |e_k\rangle \langle e_k|,$$

i.e. it is a multiple of the projection operator on the considered manifold $\mathcal{M}$.

5. – Factorized states and mixtures.

In this section we shall consider composite quantum systems, i.e. systems associated to state vectors belonging to a Hilbert space $\mathscr{H}^{(1,2)}$, which is the direct product of the Hilbert spaces $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ associated with the com-

ponent subsystems. In this case, as discussed in the introduction, the fac-
torized states play a very important role, since they are the only states for
which individual properties can be attributed to the component subsystems.
It is then useful to characterize within $\mathscr{H}^{(1,2)}$ the set of factorized states and
to derive the condition that a density functional $p(\psi)$, with $\psi \in \mathscr{H}^{(1,2)}$, must
satisfy in order that it corresponds to a mixture of factorized states.

Here we briefly recall at heorem due to VON NEUMANN ([8]), which will be useful
in what follows. Let $\mathscr{H}^{(1,2)} = \mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)}$ and consider a normalized state
$\psi \in \mathscr{H}^{(1,2)}$. Von Neumann's theorem states that one can write

$$(5.1) \qquad |\psi\rangle = \sum_j \sqrt{\lambda_j} \, |\varphi_j(1)\rangle |\chi_j(2)\rangle \,,$$

where $\{|\varphi_j(1)\rangle\}$ and $\{|\chi_j(2)\rangle\}$ are orthonormal sets in $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$, respec-
tively, and the λ_j are positive real numbers. The expansion (5.1) is unique,
provided the eigenvalues λ_j are all different among themselves. Considering
the density operator associated with ψ

$$(5.2) \qquad \varrho = |\psi\rangle\langle\psi| \,,$$

and defining

$$(5.3a) \qquad \varrho^{(1)} = \mathrm{Tr}^{(2)}\,\varrho \,,$$

$$(5.3b) \qquad \varrho^{(2)} = \mathrm{Tr}^{(1)}\,\varrho \,,$$

one can see that $\varrho^{(1)}$ and $\varrho^{(2)}$ have the same set of nonzero eigenvalues λ_j, the
corresponding eigenvectors being $|\varphi_j(1)\rangle$ and $|\chi_j(2)\rangle$, respectively. In (5.3a, b)
$\mathrm{Tr}^{(2)}$ and $\mathrm{Tr}^{(1)}$ denote partial tracing over $\mathscr{H}^{(2)}$ and $\mathscr{H}^{(1)}$, respectively.

By means of the above theorem it is then easy to prove that the neces-
sary and sufficient condition for ψ to be factorized can be written as

$$(5.4) \qquad \mathrm{Tr}^{(1)}\big[(\mathrm{Tr}^{(2)}|\psi\rangle\langle\psi|)^2\big] = 1 \,.$$

The necessity is obvious. To prove the sufficiency we observe that (5.4), owing
to (5.1), can be written

$$(5.5) \qquad \sum_j \lambda_j^2 = 1 \,.$$

Since $\mathrm{Tr}\,\varrho = \sum_j \lambda_j = 1$, eq. (5.5) can be satisfied if and only if there is only
one eigenvalue different from zero, and it equals 1. From (5.1) there follows

([8]) J. VON NEUMANN: *Mathematical Foundations of Quantum Mechanics* (Princeton
N. J., 1955), p. 429.

then that ψ is factorized. In the case in which ψ is not normalized, condition (5.4) reads

$$(5.6) \qquad \mathrm{Tr}^{(1)}\big[(\mathrm{Tr}^{(2)}|\psi\rangle\langle\psi|)^2\big] = \|\psi\|^4 .$$

Let us express condition (5.6) in an arbitrary factorized basis of $\mathcal{H}^{(1,2)}$. If we write

$$(5.7) \qquad |\psi\rangle = \sum_{ij} z_{ij}|e_i^{(1)}\rangle|e_j^{(2)}\rangle ,$$

eq. (5.6) becomes

$$(5.8) \qquad \sum_{ijkl} z_{ik}z_{il}^*z_{jl}z_{jk}^* = \sum_{ijkl} |z_{ik}|^2 |z_{jl}|^2 .$$

Because of the identity

$$(5.9) \qquad \sum_{ijkl} z_{ik}z_{il}^*z_{jl}z_{jk}^* = \sum_{ijkl} |z_{ik}|^2 |z_{jl}|^2 - \tfrac{1}{2} \sum_{ijkl} |z_{ik}z_{jl} - z_{il}z_{jk}|^2 ,$$

eq. (5.8) is equivalent to

$$(5.10) \qquad z_{ik}z_{jl} = z_{il}z_{jk} .$$

Suppose that the Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ be n- and m-dimensional, respectively. The $n \times m$ relations (5.10), which are necessary and sufficient for the factorizability of the state (5.7), are not all independent. It is easily proved that only $(n-1) \times (m-1)$ among the $n \times m$ relations (5.10) are independent. If we arrange the z_{ij}'s in a rectangular matrix, the independent relations can be obtained, for instance, by imposing the vanishing of all the 2×2 determinants built with 4 contiguous elements of the matrix z_{ij}, or, provided $z_{11} \neq 0$, built with the elements $z_{11}, z_{1j}, z_{i1}, z_{ij}$. If one expresses condition (5.6) as

$$(5.11) \qquad F(\psi) = 0 ,$$

the condition that a density functional $p(\psi)$ corresponds to a mixture of factorized states can be written as

$$(5.12) \qquad F(\psi)p(\psi) = 0 .$$

If one wants to modify quantum mechanics to eliminate the disturbing features connected with the EPR paradox, one has to require that, when the component subsystems are far apart from each other, the system must be described by density functionals satisfying (5.12). Condition (5.12) amounts to the statement that, in order to be a mixture of factorized states, the density functional must contain δ-factors expressing that (5.10) is satisfied.

Using, instead of the z_{ij}'s, the variables σ_{ij} and η_{ij} defined by relations analogous to (4.1a) and (4.1b), we can express (5.10) by requiring that $p(\psi)$ contains the following product of δ-factors:

$$(5.13) \qquad \prod_{\substack{ij \\ i \neq 1 \\ j \neq 1}} [\delta(\sigma_{ij}^2 \sigma_{11}^2 - \sigma_{i1}^2 \sigma_{1j}^2)\, \delta(\eta_{ij} - \eta_{i1} - \eta_{1j})]\,,$$

where σ_{11} is not an independent variable, but is defined by

$$(5.14) \qquad \sigma_{11} = 1 - \sum_{ij \neq 1,1} \sigma_{ij}^2\,.$$

6. – Ensembles of composite quantum systems.

In this section we shall construct the density functional for an ensemble of composite quantum systems $s = s_1 + s_2$ from the density functionals $p^{(1)}(\psi_1)$ and $p^{(2)}(\psi_2)$ for the subsystems s_1 and s_2, respectively, when particular requirements about the way of associating states of s_1 and s_2 are given. The Hilbert space on which $p(\psi)$ is defined is the direct product of the Hilbert spaces $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$ of s_1 and s_2, which are assumed to be n- and m-dimensional, respectively.

To begin with, we can consider the case in which both s_1 and s_2 are in the pure states $\bar{\psi}_1$ and $\bar{\psi}_2$, respectively. The composite system is then described by the density functional (see eq. (4.2))

$$(6.1) \qquad p_{\bar{\psi}_1 \bar{\psi}_2}(\psi) = \frac{2^{nm}}{2\pi} \prod_{ij \neq 1,1} [\delta(\sigma_{ij}^2 - \bar{\sigma}_{ij}^2)\, \delta(\eta_{ij} - \bar{\eta}_{ij})]\,,$$

where the variables $\bar{\sigma}_{ij}$ and $\bar{\eta}_{ij}$ are defined in the usual way in terms of the variables

$$(6.2) \qquad \bar{\varrho}_{ij} = \bar{\varrho}_i \bar{\varrho}_j\,, \qquad \bar{\theta}_{ij} = \bar{\theta}_i + \bar{\theta}_j\,.$$

By means of a rather cumbersome calculation one can prove that eq. (6.1) can also be written as

$$(6.3) \qquad p_{\bar{\psi}_1 \bar{\psi}_2}(\psi) = g(\psi_1,\, \psi_2) \prod_{\substack{ij \\ i \neq 1 \\ j \neq 1}} [\delta(\sigma_{ij}^2 \sigma_{11}^2 - \sigma_{i1}^2 \sigma_{1j}^2)\, \delta(\eta_{ij} - \eta_{i1} - \eta_{1j})]\, p_{\bar{\psi}_1}^{(1)}(\psi_1)\, p_{\bar{\psi}_2}^{(2)}(\psi_2)\,,$$

$p_{\bar{\psi}_1}^{(1)}(\psi_1)$ and $p_{\bar{\psi}_2}^{(2)}(\psi_2)$ being the pure-state density functionals given by eq. (4.2) for systems s_1 and s_2, respectively. The function $g(\psi_1,\, \psi_2)$ turns out to be

$$(6.4) \qquad g(\psi_1,\, \psi_2) = 2^{(n-1)(m-1)}\, \pi\, \frac{(\sigma_{11}^2)^{(m-1)(n-1)}}{(\sigma_1^{(1)2})^{m-1}\, (\sigma_1^{(2)2})^{n-1}}\,,$$

where σ_{11}^2 is given by eq. (5.14) and

$$(6.5) \qquad \sigma_1^{(\alpha)2} = 1 - \sum_{i \neq 1} \sigma_i^{(\alpha)2}, \qquad \alpha = 1, 2.$$

When the systems s_1 and s_2 are not in pure states, but are described through arbitrary density functionals $p^{(1)}(\psi_1)$ and $p^{(2)}(\psi_2)$, we can write

$$(6.6) \qquad p(\psi) = g(\psi_1, \psi_2) \prod_{\substack{ij \\ i \neq 1 \\ j \neq 1}} [\delta(\sigma_{ij}^2 \sigma_{11}^2 - \sigma_{i1}^2 \sigma_{1j}^2) \, \delta(\eta_{ij} - \eta_{i1} - \eta_{1j})] \, p^{(1)}(\psi_1) p^{(2)}(\psi_2) \, .$$

To understand the meaning of eq. (6.6), let us consider two regions V_1 and V_2 in the Hilbert spaces $\mathscr{H}^{(1)}$ and $\mathscr{H}^{(2)}$, respectively, and a region V in $\mathscr{H}^{(1,2)} = \mathscr{H}^{(1)} \otimes \mathscr{H}^{(2)}$ such that the only factorized vectors of V are the vectors $\psi_1 \psi_2$ with $\psi_1 \in V_1$, $\psi_2 \in V_2$. It can be proved that

$$(6.7) \qquad \int_V p(\psi) \, \mathrm{d}\psi = \int_{V_1} p^{(1)}(\psi_1) \, \mathrm{d}\psi_1 \cdot \int_{V_2} p^{(2)}(\psi_2) \, \mathrm{d}\psi_2 \, .$$

This equation shows that the functional $p(\psi)$ defined by eq. (6.6) describes a mixture of systems $s = s_1 + s_2$, such that the system s is always in a factorized state and that to each state of the sybsystem s_α are associated for s_β $(\alpha, \beta = 1, 2; \alpha \neq \beta)$ all states of $\mathscr{H}^{(\beta)}$ with weights $p^{(\beta)}(\psi_\beta)$. We shall call this mixture the factorized mixture associated to $p^{(1)}(\psi_1)$ and $p^{(2)}(\psi_2)$. It can be checked that the corresponding density operator is the direct product of the density operators corresponding to $p^{(1)}(\psi_1)$ and $p^{(2)}(\psi_2)$:

$$(6.8) \qquad \int p(\psi) |\psi\rangle \langle\psi| \mathrm{d}\psi = \int p^{(1)}(\psi_1) |\psi_1\rangle \langle\psi_1| \mathrm{d}\psi_1 \otimes \int p^{(2)}(\psi_2) |\psi_2\rangle \langle\psi_2| \mathrm{d}\psi_2 \, .$$

7. – Mappings from second- to first-kind mixtures.

Now we are in a position of studying some specific mappings of density functionals which correspond to physically interesting processes. One could, for example, consider the mapping on $p(\psi)$ induced by a measurement of an observable L of the system. Since such a mapping can also be described within the density ope ator formalism, we have confined its study in the language of density functionals to the appendix. In this section we shall consider mappings which account for a transition from a second- to a first-kind mixture satisfying suitable requirements. Such mappings will then be considered, in the next section, as elementary processes whose iteration leads to the explicit construction of an equation describing the mechanism considered by BOHM and AHARONOV [2] and discussed in the introduction.

Let $p(\psi)$ be an arbitrary density functional in the Hilbert space $\mathcal{H}^{(1,2)}$ of a composite quantum system $s = s_1 + s_2$, and let $p^{\#}(\psi)$ be the image of $p(\psi)$ under a mapping B which leads, for any $p(\psi)$, to a mixture of factorized states. Since this type of mappings will be used to account for the transition considered by BOHM and AHARONOV, one has to require that the results of all possible measurements of observables of each subsytem coincide for the two ensembles, which implies

$$(7.1) \qquad\qquad \mathrm{Tr}^{(\alpha)} \varrho = \mathrm{Tr}^{(\alpha)} \varrho^{\#} , \qquad\qquad \alpha = 1, 2,$$

ϱ and $\varrho^{\#}$ being the density operators corresponding to $p(\psi)$ and $p^{\#}(\psi)$, respectively. We start by considering the case in which $p(\psi)$ corresponds to a pure state $\bar{\psi}$ for the composite system. Making use of the von Neumann expansion (5.1) for $\bar{\psi}$, we write

$$(5.1) \qquad\qquad |\bar{\psi}\rangle = \sum_{k=1}^{l} \sqrt{\lambda_k} |\varphi_k(1)\rangle |\chi_k(2)\rangle , \qquad\qquad \sum_{k=1}^{l} \lambda_k = 1 .$$

For the sake of definiteness we suppose that the λ_k's have been ordered according to

$$(7.2) \qquad\qquad \lambda_k > \lambda_{k+1} .$$

Let us consider the density functional $p_{\varphi_k \chi_j}(\psi)$ corresponding to the pure state $\varphi_k(1) \chi_j(2)$. A possible choice for $p_{\bar{\psi}}^{\#}(\psi)$ could be

$$(7.3) \qquad\qquad p_{\bar{\psi}}^{\#}(\psi) = \sum_{kj} \lambda_k \lambda_j p_{\varphi_k \chi_j}(\psi) .$$

It is easily seen that

$$(7.4) \qquad\qquad \varrho_{\bar{\psi}}^{\#} = \mathrm{Tr}^{(2)} |\bar{\psi}\rangle\langle\bar{\psi}| \otimes \mathrm{Tr}^{(1)} |\bar{\psi}\rangle\langle\bar{\psi}| ,$$

so that (7.1) is satisfied. However, (7.3) is undefined when two λ_k's become equal, since (5.1) does not uniquely determine the corresponding $\varphi_k(1)$, $\chi_k(2)$. To overcome this difficulty, one could, in the degenerate case, replace the unidentified pure states $\varphi_k(1)$, $\chi_k(2)$ by the uniform mixture on the degenerate manifolds. Such a procedure can be easily developed, but then one would have a sudden change in $p_{\bar{\psi}}^{\#}(\psi)$ when two λ_k's become equal, so that the mapping $p_{\bar{\psi}}(\psi) \to p_{\bar{\psi}}^{\#}(\psi)$ would not be continuous. Then we proceed in the following way. Let $p_{k,}(\psi)$ be the factorized mixture associated to the uniform mixtures for the subsystems s_1 and s_2, built on the linear manifolds spanned by the vectors $\varphi_1(1), \varphi_2(1), ..., \varphi_k(1)$ and $\chi_1(2), \chi_2(2), ..., \chi_j(2)$, respectively. The density func-

tional $p_{kj}(\psi)$ is not defined when $\lambda_k = \lambda_{k+1}$ or $\lambda_j = \lambda_{j+1}$. Then we take

$$(7.5) \qquad p_{\bar{\psi}}^{\#}(\psi) = \sum_{kj} k(\lambda_k - \lambda_{k+1}) j(\lambda_j - \lambda_{j+1}) p_{kj}(\psi) .$$

It is easily checked that $p_{\bar{\psi}}^{\#}(\psi)$ satisfies the normalization condition (2.8). Note that the undefined $p_{kj}(\psi)$ do not appear in (7.5). If we take into account eqs. (4.7) and (6.8), the density operator corresponding to $p_{kj}(\psi)$ turns out to be

$$(7.6) \qquad \varrho_{kj} = \frac{1}{k} \sum_{i=1}^{k} |\varphi_i(1)\rangle \langle \varphi_i(1)| \otimes \frac{1}{j} \sum_{r=1}^{j} |\chi_r(2)\rangle \langle \chi_r(2)| .$$

Therefore, the density operator corresponding to $p_{\bar{\psi}}^{\#}(\psi)$ turns out to be

$$(7.7) \qquad \varrho_{\bar{\psi}}^{\#} = \sum_{kj=1}^{l} k(\lambda_k - \lambda_{k+1}) j(\lambda_j - \lambda_{j+1}) \varrho_{kj} = \mathrm{Tr}^{(2)} |\bar{\psi}\rangle \langle \bar{\psi}| \otimes \mathrm{Tr}^{(1)} |\bar{\psi}\rangle \langle \bar{\psi}| ,$$

so that (7.1) is satisfied. The mapping

$$(7.8) \qquad p_{\bar{\psi}}^{\#}(\psi) = B p_{\bar{\psi}}(\psi)$$

is continuous. In fact, it can be proved that for any given $\varepsilon > 0$ there exists an $\eta > 0$ such that $\|\bar{\psi} - \bar{\psi}'\| < \eta$ implies $|p_{\bar{\psi}}^{\#}(\psi) - p_{\bar{\psi}'}^{\#}(\psi)| < \varepsilon$. We note that, when $\bar{\psi}$ is factorized, $p_{\bar{\psi}}^{\#}(\psi) = p_{\bar{\psi}}(\psi)$.

Up to now we have only defined the mapping for density functionals corresponding to pure states. Now we extend the mapping to all functionals by linearity, i.e., given an arbitrary density functional $p(\psi)$, owing to (4.4) we define

$$(7.9) \qquad p^{\#}(\psi) = B p(\psi) = \int \mathrm{d}\bar{\psi}\, p(\bar{\psi}) p_{\bar{\psi}}^{\#}(\psi) .$$

Any extension of the mapping (7.8) other than (7.9) would violate the obvious physical requirement that the transition transforms the union of two statistical ensembles into the union of the transformed ensembles. It is easily seen that the density operator $\varrho^{\#}$ associated to $p^{\#}(\psi)$ is given by

$$(7.10) \qquad \varrho^{\#} = \int \mathrm{d}\bar{\psi}\, p(\bar{\psi})\, \mathrm{Tr}^{(2)} |\bar{\psi}\rangle \langle \bar{\psi}| \otimes \mathrm{Tr}^{(1)} |\bar{\psi}\rangle \langle \bar{\psi}| .$$

We note that $\varrho^{\#}$ does not coincide with the density operator

$$(7.11) \qquad \varrho' = \mathrm{Tr}^{(2)} \varrho \otimes \mathrm{Tr}^{(1)} \varrho .$$

To understand the physical meaning of (7.10), one has to consider the correlations between the subsystems s_1 and s_2 of s. The correspondence $\varrho \to \varrho'$

given by (7.11) would imply that all correlations existing between s_1 and s_2, when the ensemble is associated to the density operator ϱ, disappear for ϱ'. On the contrary, the correspondence given by (7.10) entails that only the correlations implied by each state $\bar{\psi}$ of the ensemble are destroyed, while memory remains of the fact that the couples of subsystems s_1 and s_2 were associated in certain definite ways in the original ensemble. For example, let us consider an initial ensemble E of systems s of two nucleons s_1 and s_2 in the singlet spin state, E being composed of two subensembles E' and E'' such that s_1 and s_2 are both neutrons in E' and both protons in E''. Then both density operators (7.10) and (7.11) correspond to a final ensemble in which the spin correlations inherent to the singlet state are destroyed. However, (7.10) implies that whenever s_1 is a neutron (proton), s_2 is a neutron (proton) too, while (7.11) would allow systems s composed of a neutron and a proton.

We note that the choice (7.5) for the desired mapping is not the only possible one. Another possibility would be

$$(7.12) \qquad p_{\bar{\psi}}^{\#}(\psi) = \sum_{k} k(\lambda_k - \lambda_{k+1}) p_{kk}(\psi) \,.$$

If (7.12) is adopted, (7.10) is no longer valid, but (7.1), as well as the continuity of the mapping, remains true.

As we shall discuss in the conclusive section, the mapping (7.5), when used to build a dynamical equation, gives rise to unwanted consequences, so that the above-said arbitrariness can be very important.

8. – Generalizations of the Schrödinger equation.

Now we are in a position to generalize the Schrödinger time evolution equation (3.7) along the lines sketched in the introduction. Let us consider the mapping among density functionals

$$(8.1) \qquad\qquad p(\psi) \to Bp(\psi) \,,$$

where B is defined by eqs. (7.9) and (7.5), and suppose that the ensemble we are considering, besides evolving according to the Schrödinger equation, is repeatedly subjected, with mean frequency λ, to random elementary processes, whose action on $p(\psi)$ is described by (8.1). If we denote by S the generator of time translations for the pure Schrödinger evolution, as in (3.8), the change of $p(\psi, t)$ during an infinitesimal time interval dt is given by

$$(8.2) \qquad p(\psi, t + dt) = (1 - \lambda\,dt)[p(\psi, t) + Sp(\psi, t)\,dt] + \lambda\,dt\,Bp(\psi, t) \,,$$

since λdt is the probability of occurrence of an elementary act (8.1) in dt. From (8.2) we immediately get

(8.3)
$$\frac{\partial p(\psi, t)}{\partial t} = Sp(\psi, t) - \lambda(1 - B)p(\psi, t) \,.$$

This equation constitutes a generalization of the Schrödinger equation. Even though we have derived it as describing the occurrence of repeated sudden processes, eq. (8.3) can also be interpreted as describing an inherently continuous process. It is important to remark that eq. (8.3), provided B satisfies eq. (7.1), implies that the density operators for the subsystems s_1 and s_2 (obtained, as usual, by partial tracing) coincide at any time with the density operators which would be obtained by the Schrödinger evolution alone. Therefore, the differences between the evolution given by (8.3) and the standard Schrödinger evolution manifest only in the correlations between the two subsystems s_1 and s_2 and cannot be evidentiated by measurements involving only one of the two subsystems.

To discuss eq. (8.3), it is convenient to define a new functional $\bar{p}(\psi, t)$, according to

(8.4)
$$p(\psi, t) = \exp[St]\bar{p}(\psi, t) \,,$$

so that eq. (8.3) becomes

(8.5)
$$\frac{\partial \bar{p}(\psi, t)}{\partial t} = - \lambda\bar{p}(\psi, t) + \lambda \exp[- St] B \exp[St]\bar{p}(\psi, t) \,.$$

Now we suppose that the Schrödinger evolution operator commutes with B. This certainly happens if the two subsystems s_1 and s_2 do not interact. In fact, in such a case, the von Neumann decomposition (5.1) for the vector $\bar{\psi}(t)$ is

$$|\bar{\psi}(t)\rangle = \sum_{k=1}^{l} \sqrt{\lambda_k} \exp[- iH_1t] |\varphi_k(1)\rangle \exp[- iH_2t] |\chi_k(2)\rangle \,.$$

There follows that, by applying B to $p_{\bar{\psi}(t)}(\psi)$, the only change in eq. (7.5) is that the factorized mixture $p_{k,j}(\psi)$ is now defined on two linear manifolds, which are the time-evolved ones of those at time $t = 0$. On the other hand, by applying the Schrödinger evolution operator to $p_{\bar{\psi}(0)}^{\#}(\psi)$ given by (7.5) amounts simply to letting the two linear manifolds on which $p_{k,j}(\psi)$ is defined evolve with time. Under such conditions, eq. (8.5) becomes

(8.6)
$$\frac{\partial \bar{p}(\psi, t)}{\partial t} = - \lambda(1 - B)\bar{p}(\psi, t) \,.$$

The initial condition for eq. (8.6) is $\bar{p}(\psi, 0) = p(\psi, 0)$. We write $p(\psi, 0)$ as the sum of two terms:

(8.7) $$p(\psi, 0) = p_t(\psi) + p_{nt}(\psi) ,$$

the first one being different from zero only on the factorized states, the second one only on the nonfactorized states. Since B does not affect the factorized states, we have

(8.8) $$B p_t^{*}(\psi) = p_t(\psi) .$$

We try to solve eq. (8.6) by putting

(8.9) $$\bar{p}(\psi, t) = p_t(\psi) + \alpha(t) p_{nt}(\psi) + [1 - \alpha(t)] B p_{nt}(\psi) ,$$

the initial condition for $\alpha(t)$ being

(8.10) $$\alpha(0) = 1 .$$

If we take into account that $B^2 = B$, eq. (8.6) gives

(8.11) $$\frac{d\alpha(t)}{dt} [p_{nt}(\psi) - B p_{nt}(\psi)] = - \lambda \alpha(t) [p_{nt}(\psi) - B p_{nt}(\psi)] .$$

Since the density functionals appearing on both sides of eq. (8.11) cannot be zero, this equation gives

(8.12) $$\frac{d\alpha(t)}{dt} = - \lambda \alpha(t) ,$$

i.e.

(8.13) $$\alpha(t) = \exp [- \lambda t] .$$

Therefore, under the hypothesis that the Schrödinger time evolution operator commutes with the operator B, we have

(8.14) $$p(\psi, t) = \exp [St] p_t(\psi) +$$
$$+ \exp [- \lambda t] \exp [St] p_{nt}(\psi) + (1 - \exp [- \lambda t]) B \exp [St] p_{nt}(\psi) .$$

All terms in eq. (8.14) are subjected to the Schrödinger evolution. In addition, the part of the initial density functional corresponding to nonfactorized states is gradually transformed into a mixture of factorized states.

According to (8.14) an ensemble associated to any initial state ψ is transformed, after a sufficient time, into a mixture of factorized states. This shows

that eq. (8.3) cannot be assumed as a basic dynamical equation for all composite quantum systems, since it would be incompatible with the properties of stable composite quantum systems. Actually, as has already been repeatedly stated, one wants that the transition from second- to first-kind mixtures takes place only when the two subsystems s_1 and s_2 are widely separated in space from each other. We have then to modify the mapping B to take into account this fact (*). To this purpose, let us define the projection operator $P_>$ such that

$$(8.15) \qquad P_> \psi(r_1, r_2) = \begin{cases} 0, & \text{when } |r_1 - r_2| < a, \\ \psi(r_1, r_2), & \text{when } |r_1 - r_2| > a, \end{cases}$$

where r_1, r_2 are the space co-ordinates of subsystems s_1 and s_2, respectively. Now we define a new mapping $B_>$:

$$(8.16) \qquad p^*(\psi) \equiv B_> p(\psi) = \int [\langle \bar{\psi}|P_>|\bar{\psi}\rangle p_{\bar{\psi}}^{\#}(\psi) + \langle \bar{\psi}|P_<|\bar{\psi}\rangle p_{\bar{\psi}}(\psi)] p(\bar{\psi}) \, \mathrm{d}\bar{\psi},$$

where $P_< = 1 - P_>$, $p_{\bar{\psi}}^{\#}(\psi)$ is given by (7.5), and $p_{\bar{\psi}}(\psi)$ is, as usual, the density functional corresponding to the pure state $\bar{\psi}$. It is easy to check that $p^*(\psi)$ satisfies the normalization condition. The mapping $B_>$ has the following property:

$$(8.17) \qquad B_> p(\psi) = \begin{cases} Bp(\psi), & \text{when } p(\psi) \text{ is such that } \langle \psi|P_<|\psi\rangle p(\psi) = 0, \\ p(\psi), & \text{when } p(\psi) \text{ is such that } \langle \psi|P_>|\psi\rangle p(\psi) = 0, \\ p^*(\psi), & \text{otherwise}. \end{cases}$$

If we use $B_>$ in place of B, the evolution equations (8.5) becomes

$$(8.18) \qquad \frac{\partial \bar{p}(\psi, t)}{\partial t} = - \lambda \bar{p}(\psi, t) + \lambda \exp[- St] B_> \exp[St] \bar{p}(\psi, t).$$

It is easily shown that

$$(8.19) \qquad \exp[- St] B_> \exp[St] p(\psi) =$$
$$= \int [\langle \bar{\psi}| \exp[iHt] P_> \exp[- iHt] |\bar{\psi}\rangle p_{\exp[-iHt]\bar{\psi}}^{\#}(\exp[- iHt] \psi) +$$
$$+ \langle \bar{\psi}| \exp[iHt] P_< \exp[- iHt] |\bar{\psi}\rangle p_{\bar{\psi}}(\psi)] p(\bar{\psi}) \, \mathrm{d}\bar{\psi}.$$

(*) The modification of the mapping B which we are going to consider requires to take into account the space co-ordinates of the subsystems. Therefore, in this case the Hilbert space on which the density functionals are defined is infinite dimensional. We do not want to discuss here the mathematical complications which could arise in our formalism due to the occurrence of infinite dimensions. The following developments would therefore require a more accurate mathematical treatment to be made completely rigorous.

Furthermore, for any $p(\psi)$ nonvanishing only for ψ's such that $\exp[-iHt]\psi$ describes a state in which subsystems s_1 and s_2 are outside their range of interaction in the whole interval $(0, t)$, eq. (8.19) becomes

$$(8.20) \quad \exp[-St]B_> \exp[St]p(\psi) = \int [\langle \bar\psi | \exp[iHt]P_> \exp[-iHt] |\bar\psi\rangle p_{\bar\psi}^{\#}(\psi) +$$
$$+ \langle \bar\psi | \exp[iHt]P_< \exp[-iHt] |\bar\psi\rangle p_{\bar\psi}(\psi)] p(\bar\psi) \, d\bar\psi.$$

In order to give an example of the evolution given by eq. (8.18), we consider a decaying state of the composite system $s_1 + s_2$. We assume that the distance a appearing in the definition of $B_>$ is definitely larger than the range r_0 of the interaction between s_1 and s_2. The initial wave function is a normalized non-factorized pure state χ, localized within the range of the interaction. Then, for such a state, the relations

$$(8.21) \quad \begin{cases} \langle \chi | \exp[iHt]P_> \exp[-iHt] |\chi\rangle = 0, \\ \langle \chi | \exp[iHt]P_< \exp[-iHt] |\chi\rangle = 1 \end{cases}$$

hold up to times in which the two fragments are no more interacting, but still within the distance a from each other. Therefore, for such times, $B_>$ is the identity, and the evolution is the Schrödinger one. Now let us assume, for simplicity, that the two fragments, outside the range of interaction, are still well localized in space and are propagating in such a way that the relative wave function crosses the distance a in a time which is much smaller than $1/\lambda$. We can then assume that suddenly, instead of (8.21), the following relations hold:

$$(8.22) \quad \begin{cases} \langle \chi | \exp[iHt]P_> \exp[-iHt] |\chi\rangle = 1, \\ \langle \chi | \exp[iHt]P_< \exp[-iHt] |\chi\rangle = 0. \end{cases}$$

In such a case $B_> = B$, so that the evolution eq. (8.18) reduces to (8.5). The evolution of $p(\psi, t)$ is then given by (8.14), which has already been shown to lead to a mixture of factorized states in a time interval large with respect to $1/\lambda$. We stress again that the density operators for the subsystems s_1 and s_2 coincide, during the whole process, with those which would be obtained by taking into account only the Schrödinger evolution. Finally, we note that the process leading from the initial nonfactorized state to the final mixture of factorized states is quantitatively governed by two parameters, the length a and the time $1/\lambda$.

9. – Remarks and conclusions.

The formalism of the density functionals which we have introduced and discussed in this paper has been shown, in the last section, to allow the introduction of a consistent dynamical equation, generalizing the Schrödinger equation, which accounts for the desired transition from second- to first-kind mixtures, and which eliminates the EPR paradox. Obviously, the generalized equation violates quantum mechanics for large separation of the subsystems. However, the dynamics has been chosen in such a way that only the correlations between the subsystems turn out to be different from those involved by the standard evolution equation. An important remark, however, has to be made. So far we have tacitly assumed that the mapping (7.5), in terms of which B and $B_>$ have been defined, acts in a similar way on the internal and on the translational degrees of freedom of the component subsystems. From the point of view of overcoming the EPR paradox, however, the internal and translational degrees of freedom do not play identical roles. This has already been discussed in ref. (^{1}). The use of (7.5) for both the internal and the translational degrees of freedom can give rise to rather dramatic effects. This is due to the fact that the nature of the considered mapping is such that the correlations between the subsystems are completely destroyed (compare eq. (7.7)). As a consequence, if one considers, for example, a composite system decaying into two fragments in a relative S-wave, the considered mapping would allow the two fragments to be found both in the same space region, instead of being in opposite regions with respect to the centre of mass. Analogous considerations hold for the correlations between the linear momenta of the fragments. While the complete decorrelation among the internal variables can be accepted, since it has not been clearly disproved by experiments, it is hard to believe that the same absence of correlations for the external variables would not have already been seen, if really existing. This shows that some modifications of the mapping (7.5) can be necessary.

Using the formalism of the density functionals, one could, for example, define a mapping leading to mixtures of factorized states of the kind considered in the appendix of ref. (^{1}), avoiding in this way the above-said unpleasant consequences.

We observe also that the hypothesis of a transition from second- to first-kind mixtures necessarily implies some heterodox consequences about conservation laws. To see this, we can, for instance, consider the case of two spin-$\frac{1}{2}$ particles in the singlet state. Since the expectation value of the total spin S^2 in the singlet state is zero, while all factorized states have nonzero projection on the triplet manifold, any mechanism leading to a mixture of factorized states implies that the expectation value of S^2 changes during the process. However, the expectation values of the components S_i of the total spin do

not change during the process. In fact, if we evaluate the density functional $p_{\bar{\psi}_s}^{\#}(\psi)$, where $\bar{\psi}_s$ is the singlet state, according to (7.7), $\varrho_{\bar{\psi}_s}^{\#}$ turns out to be $\frac{1}{4}I$, so that

$$(9.1) \qquad\qquad \langle S_i \rangle = \mathrm{Tr}\,(S_i\,\varrho_{\bar{\psi}_s}^{\#}) = 0\,.$$

The transition from the singlet state to the factorized mixture takes then place in such a way that $\langle S_i \rangle = 0$ at any time. This means that the $S^2 = 1$ states which necessarily appear in the mixture have spin orientations uniformly distributed in space, so that the isotropy of the initial situation is preserved.

Concluding, even though further investigations about the specific structure of the mapping appearing in eq. (8.16) are necessary, we have introduced in this paper a description of quantum states and a dynamical equation which constitute a generalization of quantum mechanics and provide a natural framework to describe processes eliminating the nonseparability of quantum systems which are far apart from each other.

APPENDIX

In this appendix we want to show how one can express the effect of a measurement on a quantum ensemble in terms of the density functional formalism. We have confined this point here since, in this case, the whole process can be expressed also in terms of the density operator language and its consideration has therefore only an illustrative purpose.

Let us suppose that we subject an ensemble of quantum systems to a measurement of a complete set of commuting observables, which will be denoted by L. Let e_i be the eigenvectors of L. Suppose now that, before the measurement, the ensemble is associated to the state vector $\bar{\psi}$, or, which is the same, that it is described by the density functional $p_{\bar{\psi}}(\psi)$ given by (4.2). If we write

$$(A.1) \qquad\qquad |\bar{\psi}\rangle = \sum_j \bar{z}_j |e_j\rangle\,,$$

the effect of the measurement of L (reduction of the wave packet) transforms $p_{\bar{\psi}}(\psi)$ into a new functional, denoted by $p_{\bar{\psi}}^b(\psi)$, which is obviously given by

$$(A.2) \qquad\qquad p_{\bar{\psi}}^b(\psi) = \sum_j |\bar{z}_j|^2 p_{e_j}(\psi)\,,$$

where $p_{e_j}(\psi)$ is the density functional associated to the pure state e_j. Let us denote the mapping (A.2) by M_c, so that we write

$$(A.3) \qquad\qquad p_{\bar{\psi}}^b(\psi) = M_c\,p_{\bar{\psi}}(\psi)\,.$$

In the general case in which, before the measurement, the ensemble is described by the arbitrary density functional $p(\psi)$, we can use (4.4) and the fact that M_c is obviously a linear mapping. Then we have

$$(\text{A.4}) \qquad p^b(\psi) = M_c p(\psi) = \int d\bar{\psi}\, p(\bar{\psi})\, M_c p_{\bar{\psi}}(\psi) = \sum_j p_{e_j}(\psi) \int |\langle e_j|\bar{\psi}\rangle|^2 p(\bar{\psi})\, d\bar{\psi} ,$$

where, in the last step, use has been made of (A.2).

When L is not a complete set of commuting observables, but the measurement is « moral » [7], introducing the projection operators P_j associated to the eigenmanifolds of L, we have, in place of (A.4),

$$(\text{A.5}) \qquad\qquad p^b(\psi) = M_m p(\psi) = \sum_j \int p_{P_j \bar{\psi}}(\psi) \langle \bar{\psi}|P_j|\bar{\psi}\rangle p(\bar{\psi})\, d\bar{\psi} .$$

As already remarked, the effect of the measurement can be expressed also in terms of density operators. We can establish the general property that a mapping $\mathcal{M}$ among density functionals has to satisfy in order that it can also be expressed in terms of density operators. Obviously, such a mapping must transform density functionals corresponding to the same density operators into density functionals which again correspond to the same density operator. Recalling the expression (2.10), we see that, in order that $\mathcal{M}$ has the desired property, the equation

$$(\text{A.6}) \qquad\qquad \int d\psi\, |\psi\rangle\langle\psi|\, [p_1(\psi) - p_2(\psi)] = 0$$

must imply

$$(\text{A.7}) \qquad\qquad \int d\psi\, |\psi\rangle\langle\psi|\, [\mathcal{M} p_1(\psi) - \mathcal{M} p_2(\psi)] = 0 .$$

When L is a complete set of commuting observables, since

$$(\text{A.8}) \qquad M_c p_1(\psi) - M_c p_2(\psi) = \sum_j p_{e_j}(\psi) \langle e_j| \int d\bar{\psi}\, |\bar{\psi}\rangle\langle\bar{\psi}|\, [p_1(\psi) - p_2(\psi)] |e_j\rangle ,$$

equation (A.6) implies $M_c p_1(\psi) - M_c p_2(\psi) = 0$, so that (A.7) is satisfied. When L is not a complete set of commuting observables, but the measurement is « moral », we have

$$(\text{A.9}) \quad \int d\psi\, |\psi\rangle\langle\psi|\, [M_m p_1(\psi) - M_m p_2(\psi)] =$$
$$= \int d\psi\, |\psi\rangle\langle\psi| \sum_j \int d\bar{\psi}\, p_{P_j\bar{\psi}}(\psi) \langle\bar{\psi}|P_j|\bar{\psi}\rangle\, [p_1(\bar{\psi}) - p_2(\bar{\psi})] .$$

By using the fact that $p_{\bar{\psi}}(\psi) = p_\psi(\bar{\psi})$ and equation (4.4), equation (A.9) becomes

$$(\text{A.10}) \quad \int d\psi\, |\psi\rangle\langle\psi|\, [M_m p_1(\psi) - M_m p_2(\psi)] = \sum_j P_j \left[\int d\bar{\psi}\, |\bar{\psi}\rangle\langle\bar{\psi}|\, \{p_1(\bar{\psi}) - p_2(\bar{\psi})\} \right] P_j ,$$

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[7] B. D'ESPAGNAT: *Conceptual Foundations of Quantum Mechanics* (Menlo Park, Cal., 1971).

so that equation (A.6) implies that (A.7) holds. Now, if one writes an equation like (8.3) identifying the mapping B appearing there with M_c or M_m defined in (A.4) or (A.5), one obtains the evolution equation for an ensemble of quantum systems in the presence of a measuring apparatus, which has been derived in a recent paper [8] using the language of the density operator and the quantum-dynamical semigroup formalism.

[8] G. C. GHIRARDI, A. RIMINI and T. WEBER: *Nuovo Cimento*, **30** B, 133 (1975).

● RIASSUNTO

Si introduce una riformulazione della meccanica quantistica nella quale gli « stati » di un *ensemble* di sistemi quantistici sono descritti per mezzo di funzionali reali positivi definiti sullo spazio di Hilbert dei sistemi. Per questa via si giunge ad una generalizzazione della meccanica quantistica che permette di descrivere la transizione da una miscela di seconda specie a miscele di prima specie ipotizzata da vari autori per eliminare il paradosso di Einstein, Podolsky e Rosen. Si ottiene ció costruendo per i funzionali un'equazione dinamica che si riduce all'equazione di Schrödinger quando i sottosistemi del sistema quantistico composto sono vicini e dà luogo alla transizione ad una miscela di prima specie quando essi sono lontani. La transizione è tale che i risultati di qualsiasi misura su uno dei due sottosistemi sono gli stessi che si avrebbero con la pura evoluzione di Schrödinger. Le deviazioni dalla teoria usuale si possono quindi rilevare solo con misure di correlazione tra i sottosistemi.

Новая формулировка и возможная модификация квантовой механики и EPR парадокс.

Резюме (*). — Вводится новая формулировка квантовой механики, которая описывает «состояния» ансамбля квантовых систем с помощью положительных вещественных функционалов в гильбертовом пространстве систем. Эта новая формулировка позволяет обобщить квантовую механику таким образом, чтобы индуцировать переход от смеси второго рода к смеси первого рода, который, в соответствии с предположением различных авторов, должен иметь место для исключения EPR парадокса. Мы в явном виде выводим динамическое уравнение для функционалов, которое сводится к уравнению Шредингера, когда подсистемы составной квантовой системы близки друг к другу, и приводит, изменяя квантовомеханическую эволюцию, к переходу в смесь первого рода, когда составляющие подсистемы далеки друг от друга. Этот переход оказывается таким, что в любое время предсказания, касающиеся измерений наблюдаемых величин, относящихся к одной из подсистем, совпадают с предсказаниями, которые следуют непосредственно из эволюции Шредингера. Следовательно, отклонения от стандартной теории вызваны только корреляциями между подсистемами.

(*) *Переведено редакцией.*