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Some Remarks on General Relativity and the Divergence Problem of Quantum Field Theory (*).

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The subject of the following lecture is nearly related to those of the foregoing lectures by professors HEITLER and DIRAC, but the point of view is different from theirs although not necessarily contradictory. At its best it is only a sketch of a program towards the development of which very little has been done so far.

In his lecture of this morning professor DIRAC could not use strong enough words to tell us how bad the present theory of quantum electrodynamics is. Although there can of course be no disagreement regarding facts, I would use a rather different language and say that the theory is astonishingly good and that this is mainly due to the circumstance that Diracs theory of negative and positive electrons on which it is based is essentially more realistic than the earlier attempts towards a relativistic formulation of quantum dynamics. The astonishingly wide range of the theory was already suggested by WEISSKOPF's result regarding the self energy of the electron with its weak, logarithmic divergence and became apparent through the new, invariant procedure of calculating observable quantities from the theory developed by TOMONAGA, SCHWINGER, FEYNMAN and their followers. With this in view I would tentatively take the standpoint that what is needed to overcome the present difficulties of quantum theory is not a new procedure of quantization but rather a still more realistic field theory fulfilling still stronger claims of invariance than those with respect to Lorentz and gauge transformations.

(*) Originally this lecture was called « Problems related to the big and small numbers of Physics », my intention having been to include some further considerations concerning the world at large. As it was actually delivered, however, the above title is more adequate.

As the first step towards the development of such a program I would regard the fulfilment of the claim of general relativistic invariance, meaning the introduction of the gravitational field quantities into the basic Lagrangian density of quantum electrodynamics adding to this the Lagrangian density of the pure gravitational field.

Now, the combination of quantum theory and gravitation is known to define a small length of subnuclear dimensions. This small length l_0 properly defined, we may use it as a natural unit of length taking $\hbar/l_0 c$ as the mass unit and $\sqrt{\hbar c}$ as the unit of electric charge. The arbitrary factors appearing in the definition of l_0 we shall choose so as to make the total Lagrangian density just mentioned when written in « natural units » as simple as possible, which leads to

$$l_0 = \sqrt{\frac{8\gamma\hbar}{c^3}} \approx 10^{-32}\ \text{cm}\ ,$$

where γ is the ordinary gravitational constant. This may be expressed by adding the convention $2\varkappa = 1$ ($\varkappa = 8\pi\gamma/c^4$ being the Einstein gravitational constant) to the usual conventions $\hbar = 1$, $c = 1$.

As well known, general relativity seems to require a limitation of the amount of mass which may be concentrated within a sphere of given diameter d, this mass being subject to the inequality

$$M < \frac{c^2}{4\gamma} d\ .$$

In fact, for a larger mass value the gravitational field outside of the mass will exhibit the characteristic Schwarzschild singularity. Let us now apply this inequality to a wave packet of diameter d corresponding to a particle or quantum governed by a linear wave equation as long as gravitation is neglected. If the diameter d is small compared with the Compton wave length of the particle the energy will be of the order of magnitude hc/d and the mass inside the sphere in question thus $\sim h/cd$. The inequality in question gives therefore

$$d > \frac{l_0}{\sqrt{2}}\ ,$$

according to which l_0 would appear to be a natural limit to the linear dimensions of the region within which a particle could be confined.

Although not a proof the above consideration would seem to give a certain basis to the belief that the inclusion of gravitation in quantum electrodynamics would rid the theory of the divergence difficulties [1].

Dr. S. DESER (who had already started an attempt to prove the non-singularity of quantum field theory when gravitation is included) and I have lately tried to regard the question from the point of view of the functional integration method developed by Feynman. This method in the form given by MATHEWS and SALAM has recently been applied by B. LAURENT to the gravitational interaction of two Dirac particles, whereby the well known difficulties of quantizing the gravitational field (connected with the subsidiary conditions) would seem to disappear [2]. The expression for the propagator corresponding to that which is decisive for the infinities of ordinary quantum electrodynamics would in the mentioned treatment take the form of a functional integral taken over all possible values in all points of space-time of the quantities defining the four-dimensional metrics of general relativity theory. Since the light cone depends on these quantities, this would imply a kind of average over mathematical poles differently situated. Although still no proof this consideration would again give an indication of the disappearance of the infinities when gravitation is taken into account and would seem to give mathematical expression to an idea mentioned by PAULI, at the Bern relativity congress of 1955 [3].

At this point Dr. DESER and I began to consider more seriously the question of the compatibility of the suggested result with the general well known theorem given by KÄLLÉN and LEHMANN, according to which under very general conditions the infinities in a system of interacting particles could not be weaker than for the corresponding free particles. Now, this theorem is essentially based on the validity of the energy-momentum principle in the usual form, i.e. the validity of the following relation

$$i\,\frac{\partial \varphi}{\partial x^\mu} = [\varphi, P_\mu]\,,$$

where x^1, x^2, x^3, x^4 are space-time co-ordinates, P_1, P_2, P_3, P_4 energy-momentum components and φ any quantity belonging to the system. The proof of the theorem implies that according to the relation just mentioned not only the position of the centre of gravity in space and time of the system but also that of any individual particle belonging to it may be fixed with any desired degree of precision. Now, already from the above considerations it is very improbable that such a fixation is possible when general relativity is taken into account. Moreover, it is seen that the above equation is not consistent with general relativistic invariance. Thus the definition of the vector P_μ implies that at large distances from the system the co-ordinate system should asymptotically go over into an ordinary Lorentz frame, the P_μ being space integrals taken over the whole system. While the P_μ transform as vector components when the outer co-ordinate frame undergoes a Lorentz transformation they are invariants against an arbitrary transformation of the inner co-ordinates only. As is immediately seen these transformation properties are inconsistent with the above relation.

That this must be so is also clear from the fact that the P_μ are operators for a translation of the entire system within the outward Lorentz frame and that such a translation has no simple connection to the curvilinear co-ordinates used for the description of the inner part of the system. In this connection it should also be remembered that in general relativity theory there is no invariant way to separate the energy-momentum belonging to the gravitational field from the « material » energy-momentum of the system.

It is clear that quantum electrodynamics including gravitation is only a very special and idealized case of a realistic quantum theory, which ought to include nucleons and mesons of all possible kinds as well. I shall not enter here on the difficult problem of the proper extension of the point of view of general relativistic invariance so as to include also the fields corresponding to these particles, which is also the problem of finding an adequate formulation of Yukawa's fruitful idea regarding the connection between nuclear forces and mesons. I shall finish this lecture with a small remark concerning the problems of the world at large i.e. the so-called cosmological problems. Here the typist of the first edition of the congress program has expressed my viewpoint somewhat more strongly than I would have done myself by changing the word big in the title of my lecture into bug. Thus I have tried to argue in favour of a program according to which the problems in question have hardly any closer relation to the entire universe than those of ordinary astrophysics, the question being rather to describe the state and evolution of our system of galaxies in a similar way as is attempted for other stellar systems such as the galaxies themselves.

In this way it seems possible to interpret the famous Eddington relations between « cosmological » and atomic quantities by means of the ordinary physical laws, giving indications as to the state of the metagalactic system in a certain phase of its evolution rather than to the existence of hitherto unknown laws of nature.

REFERENCES

[1] A remark by L. LANDAU [*Niels Bohr and the development of physics* (London, 1955), p. 60] seems to show that he has been led to consider a momentum corresponding to l_0 as an automatic cut off limit due to gravitation in quantum electrodynamics.

[2] B. LAURENT: *Nuovo Cimento* (soon to appear). Professor J. WHEELER has kindly told us that the problem of the quantization of the gravitational field has recently been attacked by C. MISNER, who has been able to overcome certain remaining difficulties of uniqueness by a suitable choice of the integration variables.

[3] W. PAULI: *Bern Congress on Relativity Theory*, 1955.