Long-Time Behaviour of a Two-Level System in Interaction with an Electromagnetic Field.

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Summary. — We present an exactly solvable model describing the interaction between a two-level system and the electromagnetic field. For long interaction times we evidence some purely quantum-mechanical effects, such as the destruction of coherence of radiation and the Gaussian envelope of the transition probability.

1. – Introduction.

In this article we discuss an exactly solvable model which describes the interaction between isolated atoms and the radiation field.

For a rarefied gas the interaction between the atoms can be neglected. In this case we show that if only transitions induced by the radiation field between a finite number of atomic levels occur, then after a sufficiently long time the coherence of the initial beam is destroyed. This is a consequence of saturation of the transitions between the atomic levels.

To observe such effects one can imagine the following experiment. An atom placed in a Perot-Fabry interferometer is irradiated by an electromagnetic field. The energy radiated by the atom into the eigenmodes of the Perot-Fabry is approximately equal to the energy radiated isotropically by the same atom in free space. The corrections become very small for highly reflecting mirrors (1). The behaviour of the system is studied by analysing the weak outgoing waves.

(1) A. KASTLER: *Appl. Opt.*, **1**, 17 (1962).

A large number of atoms interacting with a common radiation field can be treated to a good approximation as a species of harmonic oscillator [2,3]. In this case, for linear interaction between the radiation and the oscillator, an initially coherent state remains coherent [4].

For an isolated atom with a finite number of excitable levels, however, this statement remains approximately true for a short interaction time only.

To describe the behaviour of the system for long interaction times a perturbative treatment is no longer adequate; we therefore discuss a model which is a caricature of the real system, but which allows an exact solution. Such a model may give some indications about how coherence is destroyed in a real system.

The model Hamiltonian is presented in Sect. **2**. Its eigenstates and eigenenergies are discussed. In Sect. **3** we discuss the behaviour of atomic observables for various initial conditions of the radiation field. In particular, if the incident mode is initially in a coherent state $|\alpha_{k_1}\rangle$, the Rabi flipping is shown to have a Gaussian envelope which is independent of $|\alpha_{k_1}|^2$ for $|\alpha_{k_1}|^2 p^{-1} > 10$, p being the number of modes of the field. This effect is a purely quantum-mechanical one and cannot be explained by semi-classical arguments. Scaling properties between the monomode and multimode cases are discussed, and conditions under which the statistical properties of the field do not play a prominent role are evidenced. Section **4** is devoted to the study of the evolution of the radiation field. For this purpose we analyse the time behaviour of the photon statistics of the cavity eigenmodes. It is clearly evident that, for long interaction times, the coherence properties of the field are strongly modified.

2. – The model.

The ideal experiment sketched in the Introduction is described by the following model. The atom is taken as a two-level system, ϱ being the energy separation between its ground level $|A\rangle$ and its excited level $|B\rangle$.

If we choose a representation where $|A\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|B\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ and denote the Pauli matrices by I_x, I_y and I_z, the atomic Hamiltonian reads

$$H_{\text{at}} = \tfrac{1}{2}(-\varrho I_z - \varrho)\,. \tag{1}$$

Defining $I_\pm = I_x \pm iI_y$, one has the relations $I_-|A\rangle = |B\rangle$, $I_+|B\rangle = |A\rangle$ and $I_+|A\rangle = I_-|B\rangle = 0$.

([2]) R. H. DICKE: *Phys. Rev.*, **93**, 99 (1954).
([3]) R. BONIFACIO, D. M. KIM and M. O. SCULLY: *Phys. Rev. A*, **1**, 441 (1969).
([4]) R. J. GLAUBER: *Phys. Lett.*, **21**, 650 (1966); *Rendiconti S.I.F.*, Course XLII, edited by R. J. GLAUBER (New York, N. Y., and London, 1969).

We assume the scattering of radiation on the atom to be elastic, and thus the modes of same frequency ω of the electromagnetic field are relevant. The Hamiltonian describing the radiation reads

$$H_F = \sum_{k=k_1}^{k_p} \hbar\omega b_k^\dagger b_k . \tag{2}$$

$b_k^\dagger$ and b_k are Bose creation and annihilation operators for the mode k, where k_1 corresponds to the incident mode, and $k = k_2 \dots k_p$ corresponds to the Perot-Fabry eigenmodes of frequency ω.

We assume the two-level system and the radiation field to be coupled by a dipole interaction. This interaction reads

$$H_{\text{int}} = \sum_k \lambda_k (b_k + b_k^\dagger)(I_+ + I_-) , \tag{3}$$

where λ_k are (real) coupling constants.

The radiation wavelength being assumed to be much larger than the atomic dimensions, the spatial dependence is neglected in H_{int}.

Near the resonance $\hbar\omega = \varrho$ and for small coupling constants ($\lambda \ll \hbar\omega$) the rotating-wave approximation can be performed [5-7], and H_{int} reduces to

$$H_{\text{int}} = \sum_k \lambda_k (b_k I_- + b_k^\dagger I_+) . \tag{4}$$

A further approximation consists in assuming the isotropy of the coupling constant in k-space:

$$\lambda_k \equiv \lambda = \text{const} . \tag{5}$$

This assumption is not needed in order to diagonalize the model Hamiltonian, but it allows us to simplify the notations. The model Hamiltonian we shall consider reads

$$H = \tfrac{1}{2}(-\varrho I_z - \varrho) + \sum_{k=k_1}^{k_p} \hbar\omega b_k^\dagger b_k + \lambda \sum_k (b_k I_- + b_k^\dagger I_+) . \tag{6}$$

2.1. *Eigenstates and eigenenergies of H.* – The eigenstates and eigenenergies of the model Hamiltonian (6) are known [8].

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(5) P. L. Knight and L. Allen: *Phys. Lett.*, **38** A, 99 (1972).
(6) P. L. Knight and L. Allen: *Phys. Rev. A*, **7**, 368 (1973).
(7) D. F. Walls: *Phys. Lett.*, **42** A, 217 (1972).
(8) A. Quattropani: *Phys. Kondens. Materie*, **5**, 318 (1966).

Let us consider the operator

$$A = \lambda \sum_{k=k_1}^{k_p} b_k \,, \tag{7}$$

$p-1$ being the number of modes of frequency ω of the cavity.

Let us denote by $\{|\phi\rangle\}$ the complete set of eigenvectors of the operator $A^\dagger A$.

The expectation value $\langle\phi|A^\dagger A|\phi\rangle$ being nonnegative, there exists a smallest eigenvalue γ_0 of $A^\dagger A$, corresponding to the eigenstate $|\varphi_0\rangle$, which is defined by

$$A|\varphi_0\rangle = 0 \,. \tag{8}$$

An orthogonal set of eigenstates of $A^\dagger A$ can be constructed from $|\phi_0\rangle$:

$$|\phi_n\rangle = (A^\dagger)^n \frac{1}{[n!(p\lambda^2)^n]^{\frac{1}{2}}} |\phi_0\rangle \,. \tag{9}$$

These states $|\phi_n\rangle$ are normalized if $\langle\phi_0|\phi_0\rangle = 1$. The solution of eq. (8) is not unique; there exist p vectors $|\phi_0^l\rangle$, $l = 1, 2, \dots, p$, verifying this equation. One of them is the tensorial product of the vacuum states $|\psi_{0,k}\rangle$ of the modes $\hbar\omega b_k^\dagger b_k$

$$|\phi_0^1\rangle = \prod_{k=k_1}^{k_p} |\psi_{0,k}\rangle \,, \tag{10}$$

and the other solutions $|\phi_0^l\rangle$, $l = 2, \dots, p$, can be generated by the operators

$$B_l^\dagger = \sum_k u_k^l b_k^\dagger \,, \tag{11}$$

$$|\phi_0^l\rangle = B_l^\dagger |\phi_0^1\rangle \,, \tag{12}$$

where the u_k^l are complex numbers.

The states $|\phi_0^l\rangle$ are solutions of (8) if

$$[A, B_l^\dagger] = \lambda \sum_k u_k^l = 0 \,. \tag{13}$$

λ being given, eq. (13) has $p-1$ linearly independent solutions. The complete set of eigenfunctions of $A^\dagger A$ is

$$|\phi_n^{\{\nu_l\}}\rangle = \prod_{l=2}^{p} \frac{(B_l^\dagger)^{\nu_l}}{\sqrt{\nu_l!}} \frac{(A^\dagger)^n}{[n!(p\lambda^2)^n]^{\frac{1}{2}}} |\phi_0^1\rangle \,. \tag{14}$$

The eigenstates of the model Hamiltonian H are obtained from $\{|\phi_n^{\{\nu_l\}}\rangle\}$ and read

$$|\phi_{n,s}^{\{\nu_l\}}\rangle = r_{1,n,s}^{\{\nu_l\}} |A\rangle |\phi_n^{\{\nu_l\}}\rangle + r_{2,n,s}^{\{\nu_l\}} |B\rangle |\phi_{n-1}^{\{\nu_l\}}\rangle \tag{15}$$

with $s = \pm 1$,

$$|r_{1,n,s}^{\{\nu_l\}}|^2 = 1 - |r_{2,n,s}^{\{\nu_l\}}|^2 = \tag{16}$$
$$= \frac{\lambda^2 np}{2\lambda^2 np + \frac{1}{2}(\hbar\omega - \varrho)^2 + s(\hbar\omega - \varrho)[\frac{1}{4}(\hbar\omega - \varrho)^2 + \lambda^2 np]^{\frac{1}{2}}}.$$

The corresponding eigenenergies are

$$\lambda_{n,s}^{\{\nu_l\}} = \frac{1}{2}(\hbar\omega - \varrho) + \hbar\omega\left(n - 1 + \sum_{l=2}^{p} \nu_l\right) + s[\tfrac{1}{4}(\hbar\omega - \varrho)^2 + \lambda^2 np]^{\frac{1}{2}}. \tag{17}$$

The eigenstate corresponding to $n = 0$ is

$$|\psi_{0,1}^{\{\nu_l\}}\rangle = |A\rangle|\phi_0^{\{\nu_l\}}\rangle. \tag{18}$$

At the resonance $\hbar\omega = \varrho$ the eigenstates and eigenenergies simplify considerably:

$$|r_{1,n,s}^{\{\nu\}}|^2 = |r_{2,n,s}^{\{\nu\}}|^2 = \frac{1}{2} \tag{19}$$

and

$$\lambda_{n,s}^{\{\nu\}} = \hbar\omega\left(n - 1 + \sum_{l=2}^{p} \nu_l\right) + s\lambda\sqrt{n}. \tag{20}$$

3. – Atomic observables.

We evaluate the evolution of atomic observables from different initial conditions, assuming at time $t = 0$ no correlation between the atom, the cavity modes and the incident mode.

3'1. *Initial conditions for the two-level system.* – In this article we shall always consider that the atom is initially in its ground state $|A\rangle$. This condition is well fulfilled if the energy separation ϱ between the two atomic states is much larger than kT:

$$\varrho_{at}(0) = |A\rangle\langle A|. \tag{21}$$

3'2. *Initial conditions for the radiation field.* – The radiation field is composed of two parts, namely the incident mode and the cavity eigenmodes. We shall admit that these cavity modes are not excited when the interaction is switched on

$$\varrho_{cavity}(0) = |\{O_{k_i}\}\rangle\langle\{O_{k_i}\}|. \tag{22}$$

Various initial conditions will be used for the incident radiation mode:

i) *Chaotic state.* In the P-representation GLAUBER [9] has shown that the density matrix describing a chaotic field reads [10]

$$\varrho_{k_1}(0) = \frac{1}{\pi\langle n_{k_1}\rangle}\int \exp\left[-|\alpha_{k_1}|^2/\langle n_{k_1}\rangle\right]|\alpha_{k_1}\rangle\langle\alpha_{k_1}|\mathrm{d}^2\alpha_{k_1}\,. \tag{23}$$

In the basis of the eigenstates $\{|n_{k_1}\rangle\}$ of the harmonic oscillator, this density matrix reads

$$\varrho_{k_1}(0) = \sum_{n_{k_1}} \frac{1}{1+\langle n_{k_1}\rangle}\left(\frac{\langle n_{k_1}\rangle}{1+\langle n_{k_1}\rangle}\right)^{n_{k_1}}|n_{k_1}\rangle\langle n_{k_1}|\,. \tag{24}$$

$\langle n_{k_1}\rangle$ is the average number of photons in the incident mode. The radiation emitted by a spectral lamp is often described (in the monomode case) by such a density matrix, which corresponds to a maximum entropy, for a given average of photons.

ii) *Coherent state.* A Glauber coherent state $|\alpha_{k_1}\rangle$ is defined by the equation

$$b_{k_1}|\alpha_{k_1}\rangle = \alpha_{k_1}|\alpha_{k_1}\rangle\,. \tag{25}$$

Such a state describes the radiation emitted by a laser well above threshold or a microwave field with fixed phase.

iii) *Unphased coherent state.* Such a radiation mode is described in P-representation by the density matrix

$$\varrho_{k_1}(0) = \int_0^{2\pi}\mathrm{d}\phi\,\mathscr{L}(\phi)\,|\alpha_{k_1}\exp[i\phi]\rangle\langle\alpha_{k_1}\exp[i\phi]|\,, \tag{26}$$

where $\mathscr{L}(\phi)$ is the phase distribution of the field. For a coherent state with random phase

$$\mathscr{L}(\phi) = (2\pi)^{-1}\,. \tag{27}$$

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[9] R. J. GLAUBER: *Phys. Rev.*, **131**, 2766 (1963).
[10] For convenience, we shall use the P- or the $\{|n\rangle\}$-representation, depending on the case. For a detailed discussion of the P-representation, see ref. [11-13].
[11] R. J. GLAUBER: in *Optique et électronique quantiques, Les Houches, 1964,* edited by B. DE WITT *et al.* (New York, N. Y., 1965).
[12] K. E. CAHILL and R. J. GLAUBER: *Phys. Rev.*, **177**, 1857 (1969).
[13] K. E. CAHILL and R. J. GLAUBER: *Phys. Rev.*, **177**, 1882 (1969).

In $\{|n_{k_1}\rangle\}$-representation this radiation is described by the density matrix

$$\varrho_{k_1}(0) = \exp[-|\alpha_{k_1}|^2] \sum_{n_{k_1}} \frac{|\alpha|^{2n_{k_1}}}{n_{k_1}!} |n_{k_1}\rangle\langle n_{k_1}| . \tag{28}$$

iv) *Eigenstates of the incident mode*

$$\varrho_{k_1}(0) = |n_{k_1}\rangle\langle n_{k_1}| . \tag{29}$$

This density matrix does not correspond to a realizable state of the radiation, but in some cases it can be used instead of the previous, more realistic, initial condition.

Thus the initial condition for the total system reads

$$\varrho(0) = |A\rangle\langle A| \otimes \varrho_{k_1}(0) \otimes |\{O_{k_i}\}\rangle\langle\{O_{k_i}\}| , \qquad i = 2, \ldots, p , \tag{30}$$

where $\varrho_{k_1}(0)$ is one of the previously discussed states.

3·3. *Probability of no transition.* – Starting from the initial condition (30), we calculate the probability of finding the two-level system in its ground state $|A\rangle$ at time t.

This probability depends only on the diagonal terms of the initial density matrix for the incident mode, expressed in $\{|n_{k_1}\rangle\}$-representation ([14]).

We first discuss the case where the initial state of the incident mode is an eigenstate (29) of the harmonic oscillator.

The probability of no transition $P^A_{A,n_{k_1},\{O_{k_i}\}}(t)$ is given by

$$P^A_{A,n_{k_1},\{O_{k_i}\}}(t) = \mathrm{Tr}\,\{(|A\rangle\langle A| \otimes \mathbf{1})\, U(t)\, \varrho(0)\, U^\dagger(t)\} , \tag{31}$$

where $\mathbf{1}$ is the identity operator in the Hilbert space of the electromagnetic field (incident mode and cavity eigenmodes).

$U(t)$ is the evolution operator.

With (15), (31) becomes

$$P^A_{A,n_{k_1},\{O_{k_i}\}}(t) = \sum_{\{\nu_l\},m,s} \langle \phi^{\{\nu_l\}}_m, A| U(t)\varrho(0)|\psi^{\{\nu_l\}}_{m,s}\rangle (r^{\{\nu_l\}}_{1,m,s})^* \exp\left[\frac{it}{\hbar}\lambda^{\{\nu_l\}}_{m,s}\right] . \tag{32}$$

Introducing the completeness relation

$$\sum_{m',\{\nu_l'\},s'} |\psi^{\{\nu_l'\}}_{m',s'}\rangle\langle\psi^{\{\nu_l'\}}_{m',s'}| = \hat{1} , \tag{33}$$

([14]) P. MEYSTRE: Thèse EPF-L (1974), unpublished.

$\hat{1}$ being the identity operator for the complete system atom + radiation, and taking into account the initial condition, we obtain

$$(34)\qquad P^A_{A,n_{k_1},\{O_{k_i}\}}(t) = \sum_{\{\nu_l\},m,s,s'} |\langle\phi^{\{\nu_l\}}_m|n_{k_1},\{O_{k_i}\}\rangle|^2 |r^{\{\nu_l\}}_{1,m,s}|^2 |r^{\{\nu_l\}}_{1,m,s'}|^2 \exp\left[\frac{it}{\hbar}(\lambda^{\{\nu_l\}}_{m,s} - \lambda^{\{\nu_l\}}_{m,s'})\right].$$

Use of (16) and (17) allows us in principle to obtain the exact form of the probability of no transition.

For simplicity, we present explicitly the resonant case $\hbar\omega = \varrho$ only.

With (19) and (20), (34) reads

$$(35)\qquad P^A_{A,n_{k_1},\{O_{k_i}\}}(t) = \frac{1}{2}\sum_{\{\nu_l\},m}\left[1 + \cos 2\lambda\sqrt{mp}\,\frac{t}{\hbar}\right] |\langle\phi^{\{\nu_l\}}_m|n_{k_1},\{O_{k_i}\}\rangle|^2 .$$

The matrix elements $|\langle\phi^{\{\nu_l\}}_m|n_{k_1},\{O_{k_i}\}\rangle|^2$ are given by [14]

$$(36)\qquad |\langle\phi^{\{\nu_l\}}_m|n_{k_1},\{O_{k_i}\}\rangle|^2 = \delta\left(m + \sum_{l=2}^{p}\nu_l - n_{k_1}\right)\frac{n_{k_1}!\,p^{-n_{k_1}}}{m!\prod_{l=2}^{p}\nu_l!}.$$

The Kronecker symbols $\delta(m + \sum\nu_l - n_{k_1})$ yield the energy conservation. If we introduce (36), the probability of no transition (35) reads

$$(37)\qquad P^A_{A,n_{k_1},\{O_{k_i}\}}(t) = \frac{1}{2}\sum_m\left[1 + \cos 2\lambda\sqrt{mp}\,\frac{t}{\hbar}\right]\sum_{\substack{\{\nu_l\}\\ m+\sum\nu_l=n_{k_1}}}\frac{n_{k_1}!\,p^{-n_{k_1}}}{m!\prod_{l=2}^{p}\nu_l!}.$$

Taking into account the multinomial theorem, we have

$$(38)\qquad \Omega \equiv \sum_{\substack{\{\nu_l\}\\ m+\sum\nu_l=n_{k_1}}}\frac{n_{k_1}!\,p^{-n_{k_1}}}{m!\prod_{l=2}^{p}\nu_l!} = \binom{n_{k_1}}{m}p^{-n_{k_1}}(p-1)^{n_{k_1}-m}.$$

Thus the probability of no transition reads

$$(39)\qquad P^A_{A,n_{k_1},\{O_{k_i}\}}(\tau_p) = \frac{1}{2}\sum_{m=0}^{n_{k_1}}\binom{n_{k_1}}{m}(1/p)^m(1-1/p)^{n_{k_1}-m}[1 + \cos 2\sqrt{m}\,\tau_p]$$

with

$$(40)\qquad \tau_p = \lambda\sqrt{p}\,t/\hbar .$$

This result will be discussed later on.

Let us consider now the case of an incident radiation initially in a coherent state (25).

A derivation completely analogous to the preceding one gives for the probability of no transition

(41) $$P^A_{A,\alpha_{k_1},\{O_{k_l}\}}(t)=\frac{1}{2}\sum_{m,\{\nu_l\}}\left[1+\cos 2\lambda\sqrt{mp}\,\frac{t}{\hbar}\right]|\langle\phi_m^{\{\nu_l\}}|\alpha_{k_1},\{O_{k_l}\}\rangle|^2\,.$$

This form differs from (35) only through the matrix element

$$|\langle\phi_m^{\{\nu_l\}}|\alpha_{k_1},\{O_{k_l}\}\rangle|^2\,,$$

which takes the form [14]

(42) $$|\langle\phi_m^{\{\nu_l\}}|\alpha_{k_1},\{O_{k_l}\}\rangle|^2=\exp[-|\alpha_{k_1}|^2]\,\frac{|\alpha_{k_1}|^{2(m+\Sigma\nu_l)}p^{-(m+\Sigma\nu_l)}}{\prod\limits_{l=2}^{p}\nu_l!\,m!}\,.$$

Introducing (42) in (41) we obtain

(43) $$P^A_{A,\alpha_{k_1},\{O_{k_l}\}}(t)=\frac{\exp[-|\alpha_{k_1}|^2/p]}{2}\sum_{m=0}^{\infty}\left[1+\cos 2\lambda\sqrt{mp}\,\frac{t}{\hbar}\right]\frac{|\alpha_{k_1}|^{2m}}{m!\,p^m}\,.$$

Introducing (40) in (43) and defining

(44) $$\tilde{\alpha}_{k_1}=\alpha_{k_1}/\sqrt{p}\,,$$

we obtain

(45) $$P^A_{A,\alpha_{k_1},\{O_{k_l}\}}(\tau_p)=\frac{1}{2}\exp[-|\tilde{\alpha}_{k_1}|^2]\sum_{m=0}^{\infty}\frac{|\tilde{\alpha}_{k_1}|^{2m}}{m!}\,[1+\cos 2\sqrt{m}\,\tau_p]\,.$$

3·4. *Discussion.* – In the monomode case the probabilities of no transition (39) and (45) read respectively

(46) $$P^A_{A,\alpha}(\tau)=\frac{\exp[-|\alpha|^2]}{2}\sum_m\frac{|\alpha|^{2m}}{m!}\,[1+\cos 2\lambda\sqrt{m}\,\tau]\,,$$

(47) $$P^A_{A,n}(\tau)=\tfrac{1}{2}(1+\cos 2\sqrt{n}\,\tau)\,.$$

Contrary to $P^A_{A,n}(\tau)$, $P^A_{A,\alpha}(\tau)$ is *not* periodic in time [15].

The envelopes of $P^A_{A,n}(\tau)$ and $P^A_{A,\alpha}(\tau)$ are plotted in Fig. 1.

For $|\alpha|^2\geqslant 9$, the envelope of $P^A_{A,\alpha}(\tau)$ is independent of the intensity $|\alpha|^2$ of the incident mode.

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[15] A. Faist, E. Geneux, P. Meystre and A. Quattropani: *Helv. Phys. Acta*, **45**, 956 (1972). (In this reference, $\cos(\lambda\sqrt{n}\,t/\hbar)$ should read $\cos(2\lambda\sqrt{n}\,t/\hbar)$.)

Evidently, no damping process was introduced in our model. Accordingly, the occupation probability contains a Poincaré cycle. Therefore, the behaviour of real physical systems can be described by $P^A_{A,\alpha}(\tau)$ as given by (46) only for

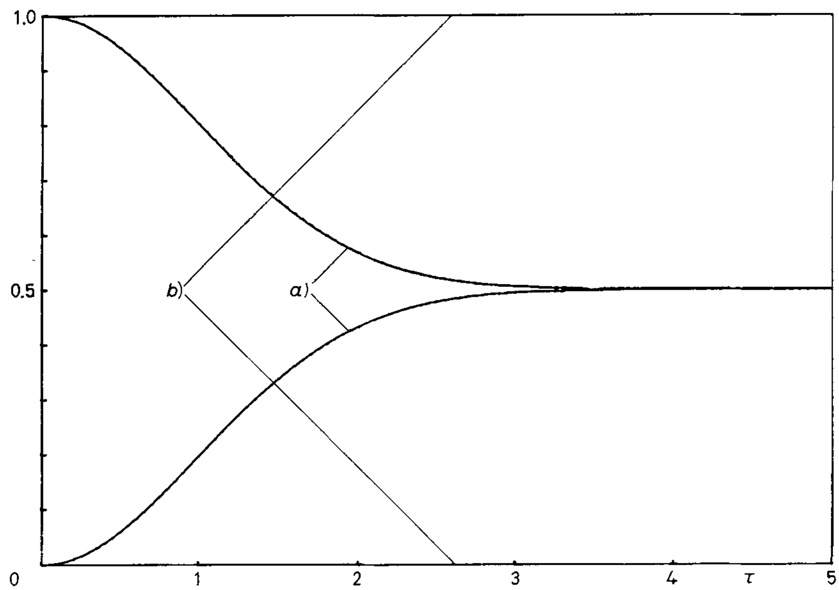

Fig. 1. – Monomode case: envelopes of *a*) $P^A_{A,\alpha}(\tau)$ (for $|\alpha|^2 \gg 10$) and *b*) $P^A_{A,n}(\tau)$ (for $n > 0$). $\tau \equiv \lambda t/\hbar$.

times small compared to their damping times. In the limit $t \ll |\alpha|\hbar/\lambda$, $P^A_{A,\alpha}(\tau)$ reads ([16])

$$P^A_{A,\alpha}(\tau) \simeq \tfrac{1}{2}\{1 + \cos(2|\alpha|\tau)\exp[-\tau^2/2]\}\,. \tag{48}$$

The comparison of expressions (45) and (46) shows that a simple time scaling relates the probability of no transition in the multimode and monomode cases if initially the incident mode is in a coherent state. The envelope of the probability of no transition in the monomode and multimode cases is Gaussian and reaches a quasi-stationary value $\frac{1}{2}$ for

$$\tau_{p,\theta} \equiv \lambda\sqrt{p}\, t_0/\hbar \simeq \pi\,. \tag{49}$$

The scaling property mentioned above, which is also verified for the evolution of the atomic dipole moment, depends drastically on the photon statistics of the incident mode at $t = 0$.

([16]) P. MEYSTRE, A. QUATTROPANI and H. P. BALTES: *Phys. Lett.*, **49** A, 85 (1974).

One easily verifies that *no* time scaling transformation relates (39) to (47).

A useful simplification arises from the comparison of the probability of no transition (39) and (45) evaluated respectively with the initial conditions $\varrho_{k_1}(0) = |n_{k_1}\rangle\langle n_{k_1}|$ and $\varrho_{k_1}(0) = |\alpha_{k_1}\rangle\langle\alpha_{k_1}|$. The time-independent coefficients of (39) and (45) are given respectively by a binomial distribution

$$\mathscr{B}(m) = \binom{n_{k_1}}{m}(1/p)^m(1-1/p)^{n_{k_1}-m} \tag{50}$$

and by a Poisson distribution

$$\mathscr{P}(m) = \exp[-|\tilde{\alpha}_{k_1}|^2]\frac{|\tilde{\alpha}_{k_1}|^{2m}}{m!}\,. \tag{51}$$

In the limit $n_{k_1}\to\infty$, $p\to\infty$ and $n_{k_1}/p = |\tilde{\alpha}_{k_1}|^2$ (finite), $\mathscr{B}(m) - \mathscr{P}(m)\to 0$. Numerically, one shows that $\mathscr{B}(m)$ and $\mathscr{P}(m)$ differ by less than 10 % for $p = 5$ and $n_{k_1} = 5$.

In the above limit the probabilities of no transitions (39) and (45) coincide, although the statistical properties of the incident mode are essentially different. The evolution of the probability of no transition depends mainly on the initially unexcited modes of the Perot-Fabry cavity.

Obviously, this simplification is not possible for the evaluation of observables which depend explicitly on the electric field of the incident mode, as for instance the atomic dipole moment.

3·5. *Atomic dipole moment.* – In the above Subsection we have studied the effect of the statistical nature of radiation on an atomic observable diagonal in the atomic eigenstates.

Let us now analyse a nondiagonal atomic observable. As we shall see, the effects of the statistical nature of the incident mode are more important for such observables.

We consider here the atomic dipole moment $\mathscr{D}$. $\langle B|\mathscr{D}|A\rangle = \mu$ is the atomic-dipole-moment matrix element between the atomic states $|A\rangle$ and $|B\rangle$. Its numerical value μ depends on the considered atom. We assume that the atom has no permanent dipole moment, *i.e.* $\langle A|\mathscr{D}|A\rangle = \langle B|\mathscr{D}|B\rangle = 0$.

We shall not detail the evaluation of $\langle\mathscr{D}\rangle(t) = \operatorname{Tr}\mathscr{D}\varrho(t)$, the calculations being completely analogous to the previous ones:

$$\langle\mathscr{D}\rangle(t) = \sum_{\substack{\{\nu_l\},m,s\\ \{\nu_l'\},m',s'}}\langle\psi^{\{\nu_l\}}_{m,s}|\mathscr{D}|\psi^{\{\nu_l'\}}_{m',s'}\rangle\langle\psi^{\{\nu_l'\}}_{m',s'}|\varrho(0)|\psi^{\{\nu_l\}}_{m,s}\rangle\exp\left[\frac{it}{\hbar}(\lambda^{\{\nu_l\}}_{m,s} - \lambda^{\{\nu_l'\}}_{m',s'})\right]. \tag{52}$$

We first discuss the case where the nondiagonal terms of the density matrix for the incident mode are initially equal to zero in the $\{|n_{k_1}\rangle\}$-representation. Assuming that the atom is initially in its ground state $|A\rangle$, and the cavity modes unexcited, one shows easily that

$$\langle \mathscr{D} \rangle (t) = 0 . \tag{53}$$

This result is not surprising, for the initial incident radiation mode has a vanishing average electric field and no dipole moment can be induced. A nonzero-averaging dipole moment can appear only if the radiation field contains a « minimal coherence » such that the average electric field does not vanish.

Let us now discuss the case of an incident mode initially in a coherent state. At the resonance $\hbar\omega = \varrho$ we obtain ([17])

$$\langle \mathscr{D} \rangle (\tau_p) = \exp[-|\tilde{\alpha}_{k_1}|^2] \sum_{m=0}^{\infty} \frac{|\tilde{\alpha}_{k_1}|^{2m+1}}{\sqrt{m!(m+1)!}} \cdot \cdot \cos(\sqrt{m}\,\tau_p) \sin(\sqrt{m+1}\,\tau_p) \{\exp[i(\omega t - \varphi(\tilde{\alpha}_{k_1}) + \pi/2)] + \text{c.c.}\} , \tag{54}$$

where

$$\tau_p = \lambda \sqrt{p}\, t/\hbar , \tag{40}$$

$$\tilde{\alpha}_{k_1} = |\tilde{\alpha}_{k_1}| \exp[i\varphi(\tilde{\alpha}_{k_1})] . \tag{55}$$

As for the probability of no transition, the monomode ($p = 1$) and multimode cases are related by a simple time scaling.

$\langle \mathscr{D} \rangle (\tau_p)$ and $P^A_{A,\alpha_{k_1},\{0_{k_i}\}}(\tau_p)$ are reported in Fig. 2 for $|\tilde{\alpha}_{k_1}|^2 = |\alpha_{k_1}|^2 p^{-1} = 9$ as a function of $\tau_p = \lambda\sqrt{p}\, t/\hbar$.

The saturation of the atomic transition occurs for $t \geqslant \pi/\lambda\sqrt{p}$ and can be seen from the quasi-stationary value of $P^A_{A,\alpha_{k_1},\{0_{k_i}\}}(\tau_p)$.

The dipole moment still varies for $t \geqslant \pi/\lambda\sqrt{p}$. Such an effect is known from the « coherence resonances » in optical pumping experiments ([18-20]) and may be understood as the time evolution of the relative phase between the two atomic states $|A\rangle$ and $|B\rangle$.

We shall now demonstrate that this effect disappears for $|\tilde{\alpha}_{k_1}|^2 \to \infty$, *i.e.* that the slow variation of $\langle \mathscr{D} \rangle (\tau_p)$ vanishes in this limit.

Let us consider the time-dependent part of $\langle \mathscr{D} \rangle (\tau_p)$ in expression (54)

$$\cos(\sqrt{m}\,\tau_p) \sin(\sqrt{m+1}\,\tau_p) = = \sin[(\sqrt{m+1} - \sqrt{m})\tau_p] + \sin[(\sqrt{m+1} + \sqrt{m})\tau_p] . \tag{56}$$

([17]) P. MEYSTRE, E. GENEUX, A. FAIST and A. QUATTROPANI: *Lett. Nuovo Cimento*, **6**, 287 (1973). (In this reference expression (4) should be corrected $\cos\sqrt{n}\,x \to \cos 2\sqrt{n}\,x$. The scale of the upper part of Fig. 2 should be modified accordingly.)

([18]) E. B. ALEXANDROV, O. B. CONSTANTINOV, B. I. PERELLI and B. A. KHODOVOY: *Sov. Phys. JETP*, **45**, 503 (1963).

([19]) C. J. FAVRE and E. GENEUX: *Phys. Lett.*, **8**, 190 (1964).

([20]) C. COHEN-TANNOUDJI and N. POLONSKY: *Compt. Rend.*, **260**, 5231 (1965).

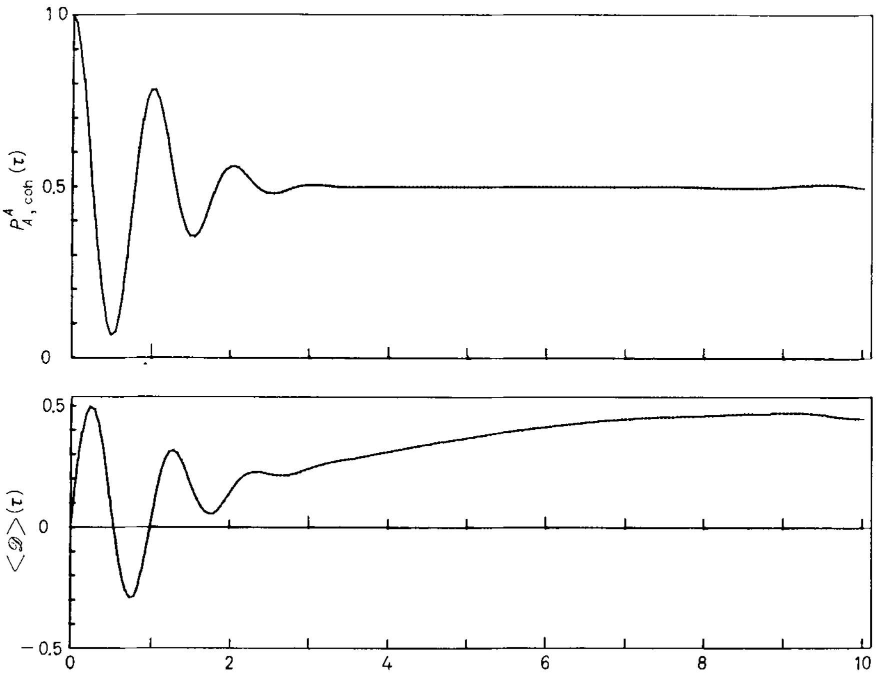

Fig. 2. – Monomode case: comparison of the dipole momentum $\langle\mathscr{D}\rangle(\tau)$ with the probability of no transition $P^{4}_{A,\alpha}(\tau)$. $\tau \equiv \lambda t/\hbar$, $|\alpha|^2 = 9$.

In (54) the oscillating terms are multiplied by a Poisson-like distribution which gives important contributions only for $m \simeq |\tilde{\alpha}_{k_1}|^2$. As in the case of the probability of no transition, the rapidly oscillating term

$$\sin[(\sqrt{m+1}+\sqrt{m})\tau_p]$$

averages to zero for $\tau_p \gg \pi$. For large $|\tilde{\alpha}_{k_1}|^2$, the slowly oscillating term reads

$$\sin\left[(\sqrt{m+1}-\sqrt{m})\tau_p\right] \simeq \sin\left(\tau_p/2\sqrt{m}\right).$$

This term contributes only if the interaction time is of the order of $|\tilde{\alpha}_{k_1}|\hbar/\lambda$, which is generally outside the experimental limits.

4. – Photon statistics.

In this Section we shall analyse the dynamics of the radiation field. The photon statistics $p_{n_{k_1}}(t)$ is defined as the probability of finding n_{k_1} photons in the mode k_i. It is explicitly given by

(57) $$p_{n_{k_i}}(t) = \mathrm{Tr}\,\{|n_{k_i}\rangle\,\langle n_{k_i}|\varrho(t)\}\,.$$

We assume that at time $t = 0$, the incident mode is in a coherent state, when the cavity eigenmodes are unexcited and the atom is in its ground state.

The detailed calculations leading to the explicit form of $p_{n_k}(t)$ are presented in ref. [14].

In the general p-mode case we obtain

(58)
$$\begin{aligned} p_{n_{k_i}}(t) = &\sum_{n_{k_1}\, n_{k_p}}{}' \prod_{k=k_1}^{k_p} (p)^{-2n_k} \frac{|\alpha_{k_1}|^{n_{k_1}}}{n_{k_1}!} \exp[-|\alpha_{k_1}|^2]\cdot \\ &\cdot\left|\prod_{l=2}^{p} \sum_{\nu_{lk}, m_k} \frac{(\nu_l \vdots \nu_{lk})\left(\sum_k n_k - \sum_l (\nu_l \vdots n_k) - \sum_l \nu_{lk}\right) n_k!}{\nu_l!\left(\sum_k n_k - \sum_l \nu_l\right)!}\cdot\right. \\ &\left.\cdot \exp\left[i\varphi_{kl}(\{\nu_{kl}\})\right] \cos\frac{\lambda t}{\hbar}\sqrt{\sum_k n_k - \sum_l \nu_l}\,\right|^2 + \\ &+ \sum_{n_{k_1}\, n_{k_p}}{}' \prod_{k=k_1}^{k_p} (p)^{-2(n_k+\frac{1}{2})} \frac{|\alpha_{k_1}|^{n_{k_1}}}{n_{k_1}!} \exp[-|\alpha_{k_1}|^2]\cdot \\ &\cdot\left|\prod_{l=2}^{p} \sum_{\nu_{lk}, m_k} \frac{(\nu_l \vdots \nu_{lk})\left(\sum_k n_k - \sum_l (\nu_l \vdots n_k) - \sum_l \nu_{lk}\right) n_k!}{\nu_l!\left[\left(\sum_k n_k - \sum_l \nu_l\right)!\left(\sum_k n_k + 1 - \sum_l \nu_l\right)!\right]^{\frac{1}{2}}}\cdot\right. \\ &\left.\cdot \exp\left[i\varphi_{kl}(\{\nu_{kl}\})\right] \sin\frac{\lambda t}{\hbar}\sqrt{\sum_k n_k + 1 - \sum_l \nu_l}\,\right|^2, \end{aligned}$$

where the prime in the sums excludes the summation over n_{k_i} and $(\nu_l \vdots \nu_{lk})$ and $(m \vdots m_k)$ are multinomial coefficients [21]. The $\varphi_{kl}(\{\nu_{kl}\})$ are phase factors arising from the products on $k = k_1, \ldots, k_p$ and $l = 2, \ldots, p$ of the coefficients $(u_k^l)^{\nu_{lk}}$.

These coefficients u_k^l may be calculated explicitly for a given number of modes and are chosen as $u_k^l = p^{-\frac{1}{2}} \exp[i\varphi_{kl}]$.

In the two-mode case ($p = 2$) we have calculated numerically the photon statistics $p_{n_2}(t)$ of the initially unexcited cavity mode [22]. Unfortunately, the capacity of the computer (CDC 6600) does not allow us to evaluate $p_{n_2}(t)$ for $n_2 > 18$.

In Fig. 3 p_{n_2} is plotted as a function of $\tau \equiv \lambda t/\hbar$ for $|\alpha_1|^2 = 9$.

([21]) *Handbook of Mathematical Functions*, edited by H. ABRAMOWITZ and I. A. STEGUN (Washigton, D. C., 1964).

([22]) E. GENEUX, P. MEYSTRE, A. FAIST and A. QUATTROPANI: *Helv. Phys. Acta*, **46**, 457 (1973).

Since a real laser contains an average number of photons $\langle n \rangle \simeq 10^{10}$, this Figure is only of qualitative interest. It shows that the photon statistics $p_{n_2}(t)$ can be fitted neither by a Gaussian nor by a Poisson-like distribution for $t \neq 0$.

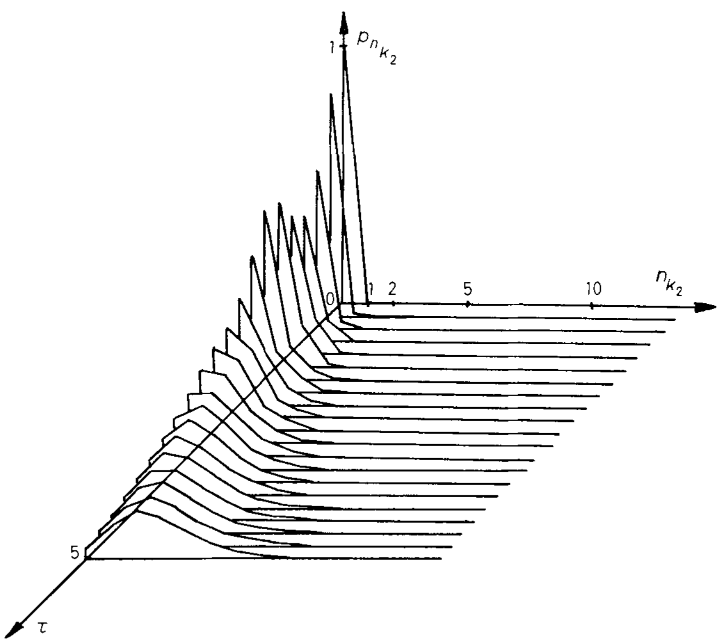

Fig. 3. – Two-mode case: photon statistics of the initially empty mode. $\varrho(0) = |A, \alpha_1, O_2\rangle \langle A, \alpha_1, O_2|$, $\tau \equiv \lambda t/\hbar$, $|\alpha_1|^2 = 9$.

This result is to be compared with those of SCHAEFER and BERNE ([23]) and SCHAEFER and PUSEY ([24]). These authors have evidenced experimentally that the light scattered by a very small number of macromolecules ($\langle N \rangle \simeq 1$) is not Gaussian.

Let us now consider the monomode case. Here, and for initially coherent radiation, the photon statistics are given by ([17])

$$(59) \qquad p_n(t) = \exp\left[-|\alpha|^2\right]\left\{\frac{|\alpha|^{2n}}{n!}\cos^2\left(\lambda\sqrt{n}\,t/\hbar\right) + \frac{|\alpha|^{2n+1}}{(n+1)!}\sin^2\left(\lambda\sqrt{n+1}\,t/\hbar\right)\right\}.$$

For very short times the term $\sin^2(\lambda\sqrt{n+1}\,t/\hbar)$ is small compared to $\cos^2(\lambda\sqrt{n}\,t/\hbar)$, and can be omitted.

Consequently, the photon statistics are very weakly perturbed by the interaction with the two-level system, and remain Poisson-like. For long times,

([23]) D. W. SCHAEFER and B. J. BERNE: *Phys. Rev. Lett.*, **28**, 475 (1972).
([24]) D. W. SCHAEFER and P. N. PUSEY: *Phys. Rev. Lett.*, **29**, 843 (1973).

however, the statistics are drastically modified by the interaction, as illustrated in Fig. 4. The atom acts as a « nonlinear filter » on the coherence properties of the field. An analogous result was obtained by CUMMINGS ([25]), who studied the correlation functions of a harmonic oscillator coupled to a two-level system.

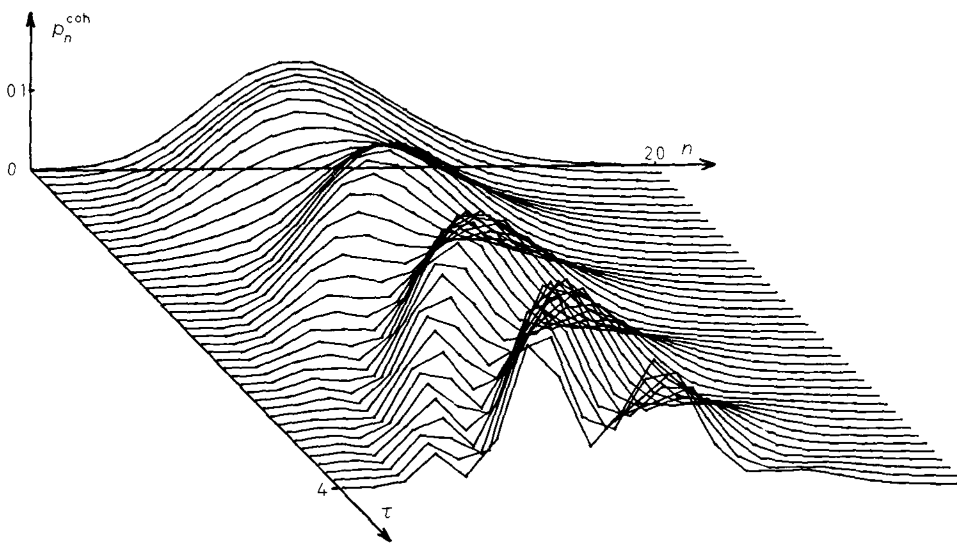

Fig. 4. – Monomode case: photon statistics of an initially coherent state $|\alpha\rangle$. $\tau \equiv \lambda t/\hbar$, $|\alpha|^2 = 9$.

Finally, let us note that the photon statistics for different p (p = number of modes) are not related by a simple scaling, contrary to what occurred for atomic observables.

5. – Conclusion.

The analysis of the long-time behaviour of a two-level system interacting with an electromagnetic field exhibits the importance of the statistical nature of radiation.

A remarkable result is the Gaussian envelope occurring in the probability of no transition of the two-level system interacting with an initially coherent incident mode. We would like to emphasize that this envelope is independent of the intensity $|\alpha_{k_1}|^2$ of the field. This effect is a purely quantum-mechanical one, and cannot be obtained in the frame of a semi-classical description of the system. It brings out the fact that quantum effects are by no means limited to the range of weak intensities, but may well occur even for very intense fields.

([25]) F. W. CUMMINGS: *Phys. Rev.*, **140**, A 1051 (1965).

Another important result is the destruction of coherence of the electromagnetic field through interaction with the two-level system. As noticed in the foregoing Section, the atom acts as a « nonlinear filter » on the coherence properties of light.

The relation between the monomode and multimode cases is quite different, according as we consider atomic or field observables. For an initially coherent incident mode, a simple scaling relates the behaviour of atomic observables for different numbers of modes. However, for field observables there exists no simple scaling. Although it is possible to calculate the dynamics of the photon statistics for any initial conditions and any (countable) number of modes, the analytic form of the result is generally not very clear. Nevertheless, we show that the cavity modes are excited neither chaotically nor coherently through the scattering process. This result is in accordance with those of SCHAEFER *et al.*, who showed that the light scattered by a very small number of centres is not Gaussian.

Finally, we point out that, for some observables, a detailed description of the statistical nature of radiation is not necessary. This is the case for atomic-population observables, as transition probabilities.

* * *

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● RIASSUNTO (*)

Si espone un modello esattamente solubile per descrivere l'interazione fra un sistema a due livelli e il campo magnetico. Si mettono in luce, per lunghi tempi di interazione, certi effetti puramente quantistici, come la distruzione della coerenza di radiazione e l'inviluppo gaussiano della probabilità delle transizioni.

(*) *Traduzione a cura della Redazione.*

Поведение двухуровневой системы, взаимодействующей с электромагнитным полем, при больших временах.

Резюме (*). — Мы предлагаем точно решаемую модель, описывающую взаимодействие между двухуровневой системой и электромагнитным полем. Для больших времен взаимодействия мы подтверждаем некоторые чисто квантовомеханические эффекты, такие как нарушение когерентности изличения и гауссову форму огибающей вероятности перехода.

(*) *Переведено редакцией.*