Table of Integrals and Formulae for Feynman Diagram Calculations.

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Dedicated to the memory
of CRISTIANO DI PIAZZA

1. – Introduction.

In the course of time we have been collecting integrals that, for one reason on another, seem to occur quite often in our calculations in perturbative QCD and QED. Each time we attempt to obtain the analytical result for some physical process, we end up with multidimensional Feynman parameter integrals and the ensuing definite integrals are similar, if not identical, to ones evaluated in earlier calculations. It is, therefore, economical and time saving to compile a list of those integrals which occur most frequently in these calculations.

This paper represents a collection of such integrals, definite and indefinite, together with a collection of formulae which may be of help in the analytical evaluation of Feynamn-parameter integrals. The majority of the integrals contain logarithmic functions and, therefore, often give rise to dilogarithms and to the lowest-order generalized polylogarithms. These highly transcendental functions are not often discussed in the standard reference books for physicists (*e.g.* ERDELYI [1], MAGNUS [2]), and integrals involving these functions are not contained in the most commonly used tables of integrals (*e.g.* GRADSHTEYN and RYZHIK [3], GROBNER and HOFREITER [4]). The primary source of information on polylogarithms is LEWIN [5], while the modern reference to the generalized polylogarithms is KOLBIG [6]. Unfortunately, it is our impression that these references, particularly the latter, are not as widely appreciated as they deserve to be. A useful summary of ref. [6] may be found in the appendix of ref. [7]. Reference [7] and also Levine [8] contain rather extensive tables of definite integrals leading to transcendental constants.

In recent years many physicists, including the authors, have been fortunate enough to be able to use very powerful symbolic manipulation computer programs for our calculations. These programs have made possible many calculations whose length would pose an otherwise insurmountable obstacle. In the present context of analytical evaluation of integrals, MACSYMA [9] is certainly the most powerful and widely available program. MACSYMA is a very useful tool, and can perform a large number of integrals, and yet MACSYMA, at least for the moment, cannot supersede all the familiar tables of integrals. Tables of integrals are still a good companion of the calculating physicist first because not *all* analytically calculable integrals can be evaluated by MACSYMA and, in the second place, because unfortunately MACSYMA and analogous computer codes are not commonly available to the majority of the scientific community.

The tables are organized in the following manner: in sect. **2** we define and present some of the properties of the generalized polylogarithms together with transformation formulae for Li_2, Li_3 and $S_{1,2}$. Section **2** also contains the numerical values of polylogarithms for a few selected values of the argument.

These are useful because they frequently occur in specific calculations and because they can be helpful in numerical checks.

Section **3** is entirely dedicated to definite integrals: we start with simple logarithmic expressions leading to simple answers and end up with the most complicated integrals encountered. For a detailed list of the content of sect. **3** the reader is referred to the index which may be helpful in locating the formulae of interest.

In sect. **4** we present a number of moment integrals.

Section **5** contains a few selected indefinite integrals of logarithmic functions.

Of course, it is inevitable that some particular integral needed will not be found in the tables. Appendix A, therefore, contains the detailed calculation of a few selected integrals: some of the more useful «tricks» for calculating the integrals are presented here. Although none of the tricks are original, still they may not be familiar to all physicists.

In appendix B we present power series expansions for the more common polylogarithms. These power series expansions can be used in writing computer codes for the numerical evaluation of Li_2, Li_3 and $S_{1,2}$.

Finally, appendix C deals with some identities of partial sums occurring in the evaluation of moment integrals. The identities presented in this appendix are only the most commonly encountered, and a more complete table can be found in [10].

These tables have been checked several times and, therefore, should, hopefully, be error free. We would be glad if any misprints or outright errors noticed by the reader were communicated to the authors.

2. – Generalized polylogarithms.

2·1. *Notation and definitions.* – In the following tables we use the notation of Kolbig *et al.* [6]:

$$S_{n,p}(y) = \frac{(-1)^{n+p-1}}{(n-1)!\,p!}\int_0^1 \frac{\ln^{n-1}(x)\ln^p(1-xy)}{x}\,\mathrm{d}x\,. \tag{2.1.1}$$

In particular

$$S_{1,1}(y) = \mathrm{Li}_2(y) = -\int_0^1 \frac{\ln(1-xy)}{x}\,\mathrm{d}x = -\int_0^y \frac{\ln(1-x)}{x}\,\mathrm{d}x\,, \tag{2.1.2}$$

$$S_{2,1}(y) = \mathrm{Li}_3(y) = \int_0^1 \frac{\ln(x)\ln(1-xy)}{x}\,\mathrm{d}x = \int_0^y \frac{\mathrm{Li}_2(x)}{x}\,\mathrm{d}x\,, \tag{2.1.3}$$

(2.1.4) $$S_{n-1,1}(y) = \mathrm{Li}_n(y) = \sum_{s=1}^{\infty} \frac{y^s}{s^n},$$

(2.1.5) $$S_{1,2}(y) = \frac{1}{2}\int_0^1 \frac{\ln^2(1-xy)}{x}\,\mathrm{d}x = \frac{1}{2}\int_0^y \frac{\ln^2(1-x)}{x}\,\mathrm{d}x = \sum_{s=2}^{\infty} \frac{y^s}{s^2}\sum_{r=1}^{s-1}\frac{1}{r}.$$

The above two power series are not suitable for direct numerical evaluation. A more appropriate method for numerical evaluation is presented in appendix B.

$S_{n,p}(y)$, defined only for positive integers n and p, is real for real $y \leqslant 1$. From the definition of $S_{n,p}$ we can find its derivative and integral:

(2.1.6) $$\frac{\mathrm{d}}{\mathrm{d}y} S_{n,p}(y) = \frac{1}{y} S_{n-1,p}(y),$$

(2.1.7) $$\int_0^y \frac{S_{n,p}(x)}{x}\,\mathrm{d}x = S_{n+1,p}(y).$$

In particular, for the functions we consider in these tables, we have

(2.1.6a) $$\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{Li}_2(y) = -\frac{\ln(1-y)}{y},$$

(2.1.6b) $$\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{Li}_3(y) = \frac{\mathrm{Li}_2(y)}{y},$$

(2.1.6c) $$\frac{\mathrm{d}}{\mathrm{d}y} S_{1,2}(y) = \frac{\ln^2(1-y)}{2y}.$$

It has been shown (see, for example, KOLBIG [6]) that $S_{n,p}$'s of argument y, $1-y$ and $1/y$ can be expressed in terms of each other.

(2.1.7) $$S_{n,p}(1-y) = \sum_{s=0}^{n-1}\frac{\ln^s(1-y)}{s!}\left[-\sum_{r=0}^{p-1}\frac{(-1)^r}{r!}\ln^r(y)\,S_{p-r,n-s}(y) + S_{n-s,p}(1)\right] + \frac{(-1)^p}{n!\,p!}\ln^n(1-y)\ln^p(y),$$

(2.1.8) $$S_{n,p}\left(-\frac{1}{y}\right) = (-1)^n\sum_{s=0}^{p-1}(-1)^s\sum_{r=0}^{s}\frac{(-1)^r}{r!}\ln^r(y)\binom{n+s-r-1}{s-r}\cdot S_{n+s-r,p-s}(-y) + (-1)^p\left[\sum_{r=0}^{n-1}\frac{(-1)^r}{r!}\ln^r(y)\,C_{n-r,p} + \frac{(-1)^{n+p}}{(n+p)!}\ln^{n+p}(y)\right],$$

where

(2.1.9) $$C_{n,p} = (-1)^{n+1}\sum_{r=1}^{p-1}(-1)^{p-r}S_{n+r,p-r}(-1)\binom{n+r-1}{r} + (-1)^p[1-(-1)^n]\,S_{n,p}(-1).$$

Repeated applications of these formulae yield connections between $S_{n,p}$ with arguments from the set

$$\{y,\ 1-y,\ 1/y,\ 1/(1-y),\ y/(1-y),\ (1-y)/y\}\,.$$

Specific examples for $\mathrm{Li}_2(y)$, $\mathrm{Li}_3(y)$ and $S_{1,2}(y)$ are given in the next section.

If we use the customary definition of the logarithm in the complex plane, *i.e.*

$$\ln(y+i\varepsilon)=\ln|y|+i\pi T(-y)\,, \tag{2.1.10}$$

where $T(y)=0$ if $y<0$ and $T(y)=1$ for $y>0$, we can find the imaginary part of the $S_{n,p}$'s. For example

$$\mathrm{Im}\{\mathrm{Li}_2(y+i\varepsilon)\}=\pi T(y-1)\ln(y)\,, \tag{2.1.11}$$

$$\mathrm{Im}\{\mathrm{Li}_3(y+i\varepsilon)\}=\pi T(y-1)\frac{\ln^2(y)}{2}\,, \tag{2.1.12}$$

$$\mathrm{Im}\{S_{1,2}(y+i\varepsilon)\}=\pi T(y-1)\left[\mathrm{zeta}\,(2)-\mathrm{Li}_2\left(\frac{1}{y}\right)-\frac{1}{2}\ln^2(y)\right]. \tag{2.1.13}$$

The imaginary part of $S_{n,p}$ (for n and p arbitrary) can be found in the paper by KOLBIG *et al.* [6].

2·2. *Relations between polylogarithms of different arguments.*

$$\mathrm{Li}_2(1-y)=-\mathrm{Li}_2(y)-\ln(y)\ln(1-y)+\mathrm{zeta}\,(2)\,, \tag{2.2.1}$$

$$\mathrm{Li}_2\left(\frac{1}{y}\right)=-\mathrm{Li}_2(y)-\frac{1}{2}\ln^2(-y)-\mathrm{zeta}\,(2)\,, \tag{2.2.2}$$

$$\mathrm{Li}_2\left(\frac{1}{1-y}\right)=\mathrm{Li}_2(y)+\ln(1-y)\ln(-y)-\frac{1}{2}\ln^2(1-y)+\mathrm{zeta}\,(2)\,, \tag{2.2.3}$$

$$\mathrm{Li}_2\left(-\frac{1-y}{y}\right)=\mathrm{Li}_2(y)+\ln(y)\ln(1-y)-\frac{1}{2}\ln^2(y)-\mathrm{zeta}\,(2)\,, \tag{2.2.4}$$

$$\mathrm{Li}_2\left(-\frac{y}{1-y}\right)=-\mathrm{Li}_2(y)-\frac{1}{2}\ln^2(1-y)\,, \tag{2.2.5}$$

$$\mathrm{Li}_2\,[y(1-y)]+\mathrm{Li}_2\left(-\frac{y^2}{1-y}\right)-\mathrm{Li}_2(-y^3)=-3\mathrm{Li}_2(-y)-2\mathrm{Li}_2(y)-\frac{1}{2}\ln^2(1-y)\,, \tag{2.2.6}$$

$$\mathrm{Li}_2\left(\frac{1-y}{1+y}\right)-\mathrm{Li}_2\left(-\frac{1-y}{1+y}\right)=\mathrm{Li}_2(-y)-\mathrm{Li}_2(y)+\frac{3}{2}\,\mathrm{zeta}\,(2)+\ln(y)\ln\left(\frac{1+y}{1-y}\right), \tag{2.2.7}$$

(2.2.8) $$\mathrm{Li}_2(y^2) = 2[\mathrm{Li}_2(y) + \mathrm{Li}_2(-y)] ,$$

(2.2.9) $$\mathrm{Li}_3(1-y) = -S_{1,2}(y) - \ln(1-y)\,\mathrm{Li}_2(y) - \frac{1}{2}\ln(y)\ln^2(1-y) + \mathrm{zeta}\,(2)\ln(1-y) + \mathrm{zeta}\,(3) ,$$

(2.2.10) $$\mathrm{Li}_3\left(\frac{1}{y}\right) = \mathrm{Li}_3(y) + \frac{1}{6}\ln^3(-y) + \mathrm{zeta}\,(2)\ln(-y) ,$$

(2.2.11) $$\mathrm{Li}_3\left(\frac{1}{1-y}\right) = -S_{1,2}(y) - \frac{1}{2}\ln^2(1-y)\ln(-y) + \frac{1}{6}\ln^3(1-y) - \ln(1-y)\,\mathrm{Li}_2(y) + \mathrm{zeta}\,(3) - \mathrm{zeta}\,(2)\ln(1-y) ,$$

(2.2.12) $$\mathrm{Li}_3\left(-\frac{y}{1-y}\right) = S_{1,2}(y) - \mathrm{Li}_3(y) + \ln(1-y)\,\mathrm{Li}_2(y) + \frac{1}{6}\ln^3(1-y) ,$$

(2.2.13) $$\mathrm{Li}_3\left(-\frac{1-y}{y}\right) = S_{1,2}(y) - \mathrm{Li}_3(y) + \ln(1-y)\,\mathrm{Li}_2(y) + \frac{1}{6}\ln^3(y) + \mathrm{zeta}(2)\ln\left(\frac{y}{1-y}\right) + \frac{1}{2}\ln(y)\ln(1-y)\ln\left(\frac{1-y}{y}\right) ,$$

(2.2.14) $$S_{1,2}(1-y) = -\mathrm{Li}_3(y) + \ln(y)\,\mathrm{Li}_2(y) + \tfrac{1}{2}\ln(1-y)\ln^2(y) + \mathrm{zeta}(3) ,$$

(2.2.15) $$S_{1,2}\left(\frac{1}{y}\right) = \mathrm{Li}_3(y) - S_{1,2}(y) - \frac{1}{6}\ln^3(-y) - \ln(-y)\,\mathrm{Li}_2(y) + \mathrm{zeta}(3) ,$$

(2.2.16) $$S_{1,2}\left(\frac{1}{1-y}\right) = \mathrm{Li}_3(y) - S_{1,2}(y) - \ln(-y)\,\mathrm{Li}_2(y) + \mathrm{zeta}(3) + \frac{1}{2}\ln(1-y)\ln(-y)\ln\left(\frac{1-y}{-y}\right) - \frac{1}{6}\ln^3(1-y) ,$$

(2.2.17) $$S_{1,2}\left(-\frac{1-y}{y}\right) = -\mathrm{Li}_3(y) + \ln(y)\,\mathrm{Li}_2(y) + \frac{1}{2}\ln^2(y)\ln(1-y) - \frac{1}{6}\ln^3(y) + \mathrm{zeta}(3) ,$$

(2.2.18) $$S_{1,2}\left(-\frac{y}{1-y}\right) = S_{1,2}(y) - \frac{1}{6}\ln^3(1-y) .$$

2'3. *The zeta-function and values of polylogarithms for selected arguments.*

(2.3.1) $$\mathrm{zeta}(n) = \sum_{k=1}^{\infty}\frac{1}{k^n} ,$$

(2.3.2) $$\mathrm{zeta}(2) = \frac{\pi^2}{6} = 1.644\,934\,066\,848\,23\,\ldots ,$$

(2.3.3) $$\mathrm{zeta}(3) = 1.202\,056\,903\,159\,59\,\ldots ,$$

(2.3.4) $\mathrm{zeta}(4) = \dfrac{\pi^4}{90} = 1.082\,323\,233\,711\,14\ \ldots\,,$

(2.3.5) $\mathrm{zeta}(5) = 1.036\,927\,755\,143\,37\ \ldots\,,$

(2.3.6) $S_{n,p}(0) = 0\,,$

(2.3.7) $\mathrm{Li}_n(1) = \mathrm{zeta}(n)\,,$

(2.3.8) $\mathrm{Li}_n(-1) = -\left\{1 - \dfrac{1}{2^{n-1}}\right\}\mathrm{zeta}(n)\,.$

For values of $S_{n,p}(1)$ see KOLBIG [11]

(2.3.9) $\mathrm{Li}_2(1) = \mathrm{zeta}(2) = 1.644\,934\,066\,848\,23\ \ldots\,,$

(2.3.10) $\mathrm{Li}_2(-1) = -\frac{1}{2}\,\mathrm{zeta}(2) = -0.822\,467\,033\,424\,11\ \ldots\,,$

(2.3.11) $\mathrm{Li}_2(\frac{1}{2}) = \frac{1}{2}\{\mathrm{zeta}(2) - \ln^2 2\} = 0.582\,240\,526\,465\,01\ \ldots\,,$

(2.3.12) $\mathrm{Li}_2(-\frac{1}{2}) = -0.448\,414\,206\,923\,65\ \ldots\,,$

(2.3.13) $\mathrm{Li}_2(\frac{1}{3}) = 0.366\,213\,229\,977\,06\ \ldots\,,$

(2.3.14) $\mathrm{Li}_2(\frac{2}{3}) = 0.833\,271\,886\,477\,39\ \ldots\,,$

(2.3.15) $\mathrm{Li}_2(-2) = -1.436\,746\,366\,883\,68\ \ldots\,,$

(2.3.16) $\mathrm{Li}_2(2) = \frac{3}{2}\mathrm{zeta}(2) + \ln(2)\ln(-1)\,,$ see convention eq. (2.1.11)

(2.3.17) $\mathrm{Li}_3(1) = \mathrm{zeta}(3) = 1.202\,056\,903\,130\,49\ \ldots\,,$

(2.3.18) $\mathrm{Li}_3(-1) = -\frac{3}{4}\mathrm{zeta}(3) = -0.901\,542\,677\,369\,79\ \ldots\,,$

(2.3.19) $\mathrm{Li}_3(\frac{1}{2}) = \frac{7}{8}\mathrm{zeta}(3) - \frac{1}{2}\,\mathrm{zeta}(2)\ln(2) + \frac{1}{6}\ln^3(2) =$
$= 0.537\,213\,193\,608\,04\ \ldots\,,$

(2.3.20) $\mathrm{Li}_3(-\frac{1}{2}) = -0.472\,597\,844\,658\,88\ \ldots\,,$

(2.3.21) $\mathrm{Li}_3(\frac{1}{3}) = 0.348\,827\,861\,154\,83\ \ldots\,,$

(2.3.22) $\mathrm{Li}_3(\frac{2}{3}) = 0.738\,060\,644\,830\,84\ \ldots\,,$

(2.3.23) $\mathrm{Li}_3(-2) = -1.668\,283\,363\,966\,51\ \ldots\,,$

(2.3.24) $S_{1,2}(1) = \mathrm{zeta}(3) = 1.202\,056\,903\,130\,49\ \ldots\,,$

(2.3.25) $S_{1,2}(-1) = \frac{1}{8}\,\mathrm{zeta}(3) = 0.150\,257\,112\,894\,95\ \ldots\,,$

(2.3.26) $S_{1,2}(\frac{1}{2}) = \frac{1}{8}\,\text{zeta}(3) - \frac{1}{6}\ln^3(2) = 0.094\,753\,004\,230\,13\,\ldots\,,$

(2.3.27) $S_{1,2}(-\frac{1}{2}) = 0.046\,936\,455\,382\,06\,\ldots\,,$

(2.3.28) $S_{1,2}(\frac{1}{3}) = 0.035\,826\,579\,380\,65\,\ldots\,,$

(2.3.29) $S_{1,2}(\frac{2}{3}) = 0.206\,214\,841\,840\,66\,\ldots\,,$

(2.3.30) $S_{1,2}(-2) = 0.427\,209\,668\,531\,31\,\ldots\,,$

(2.3.31) $\text{Li}_n(y) \to y \qquad (0 < y \ll 1)\,,$

(2.3.32) $S_{1,2}(y) \to \dfrac{y^2}{4} \qquad (0 < y \ll 1)\,,$

(2.3.33) $\text{Re}\,[\text{Li}_2(y)] \to -\frac{1}{2}\ln^2(y) \qquad (y \gg 1)\,,$

(2.3.34) $\text{Re}\,[\text{Li}_3(y)] \to -\frac{1}{6}\ln^3(y) \qquad (y \gg 1)\,,$

(2.3.35) $\text{Re}\,[S_{1,2}(y)] \to \frac{1}{6}\ln^3(y) \qquad (y \gg 1)\,.$

3. – Definite integrals.

3·1. ln(y), ln(1 − y), ln(1 + y) *and powers of y.*

(3.1.1) $$\int_0^1 y^n \ln(y)\,\mathrm{d}y = -\frac{1}{(n+1)^2}\,,$$

(3.1.2) $$\int_0^1 \ln(1-y)\,\mathrm{d}y = -1\,,$$

(3.1.3) $$\int_0^1 y\ln(1-y)\,\mathrm{d}y = -\frac{3}{4}\,,$$

(3.1.4) $$\int_0^1 y^2\ln(1-y)\,\mathrm{d}y = -\frac{11}{18}\,,$$

(3.1.5) $$\int_0^1 y^3\ln(1-y)\,\mathrm{d}y = -\frac{25}{48}\,,$$

(3.1.6) $$\int_0^1 y^4\ln(1-y)\,\mathrm{d}y = -\frac{137}{300} \qquad \text{(see (4.2.4))}\,,$$

(3.1.7) $$\int_0^1 \ln(1+y)\,\mathrm{d}y = 2\ln(2) - 1\,,$$

(3.1.8) $$\int_0^1 y\ln(1+y)\,\mathrm{d}y = \frac{1}{4}\,,$$

(3.1.9) $$\int_0^1 y^2\ln(1+y)\,\mathrm{d}y = \frac{2}{3}\ln(2) - \frac{5}{18}\,,$$

(3.1.10) $$\int_0^1 y^3\ln(1+y)\,\mathrm{d}y = \frac{7}{48}\,,$$

(3.1.11) $$\int_0^1 y^4\ln(1+y)\,\mathrm{d}y = \frac{2}{5}\ln(2) - \frac{47}{300}$$ (see (4.2.7)),

3·2. *Square of* $\ln(y)$, $\ln(1-y)$, $\ln(1+y)$ *and powers of* y.

(3.2.1) $$\int_0^1 y^n\ln^2(y)\,\mathrm{d}y = \frac{2}{(n+1)^3}\,,$$

(3.2.2) $$\int_0^1 \ln^2(1-y)\,\mathrm{d}y = 2\,,$$

(3.2.3) $$\int_0^1 y\ln^2(1-y)\,\mathrm{d}y = \frac{7}{4}\,,$$

(3.2.4) $$\int_0^1 y^2\ln^2(1-y)\,\mathrm{d}y = \frac{85}{54}\,,$$

(3.2.5) $$\int_0^1 y^3\ln^2(1-y)\,\mathrm{d}y = \frac{415}{288}\,,$$

(3.2.6) $$\int_0^1 y^4\ln^2(1-y)\,\mathrm{d}y = \frac{12\,019}{9\,000}$$ (see (4.2.5)),

(3.2.7) $$\int_0^1 \ln^2(1+y)\,\mathrm{d}y = 2\ln^2(2) - 4\ln(2) + 2\,,$$

(3.2.8) $$\int_0^1 y \ln^2(1+y)\,dy = 2\ln(2) - \frac{5}{4},$$

(3.2.9) $$\int_0^1 y^2 \ln^2(1+y)\,dy = \frac{2}{3}\ln^2(2) - \frac{16}{9}\ln(2) + \frac{55}{54},$$

(3.2.10) $$\int_0^1 y^3 \ln^2(1+y)\,dy = \frac{4}{3}\ln(2) - \frac{241}{288},$$

(3.2.11) $$\int_0^1 y^4 \ln^2(1+y)\,dy = \frac{2}{5}\ln^2(2) - \frac{92}{75}\ln(2) + \frac{6589}{9000} \qquad \text{(see (4.2.8))},$$

3'3. $\ln(y)\ln(1-y)$, $\ln(y)\ln(1+y)$, $\ln(1+y)\ln(1-y)$ *and powers of* y.

(3.3.1) $$\int_0^1 \ln(y)\ln(1-y)\,dy = 2 - \text{zeta}(2),$$

(3.3.2) $$\int_0^1 y\ln(y)\ln(1-y)\,dy = 1 - \frac{1}{2}\,\text{zeta}(2),$$

(3.3.3) $$\int_0^1 y^2\ln(y)\ln(1-y)\,dy = \frac{71}{108} - \frac{1}{3}\,\text{zeta}(2),$$

(3.3.4) $$\int_0^1 y^3\ln(y)\ln(1-y)\,dy = \frac{35}{72} - \frac{1}{4}\,\text{zeta}(2),$$

(3.3.5) $$\int_0^1 y^4\ln(y)\ln(1-y)\,dy = \frac{6913}{18000} - \frac{1}{5}\,\text{zeta}(2) \qquad \text{(see (4.2.9))},$$

(3.3.6) $$\int_0^1 \ln(y)\ln(1+y)\,dy = -2\ln(2) - \tfrac{1}{2}\,\text{zeta}(2) + 2,$$

(3.3.7) $$\int_0^1 y\ln(y)\ln(1+y)\,dy = \tfrac{1}{4}\,\text{zeta}(2) - \tfrac{1}{2},$$

(3.3.8) $$\int_0^1 y^2\ln(y)\ln(1+y)\,dy = -\frac{2}{9}\ln(2) - \frac{1}{6}\,\text{zeta}(2) + \frac{41}{108},$$

(3.3.9) $$\int_0^1 y^3 \ln(y) \ln(1+y)\, dy = \frac{1}{8} \text{zeta}(2) - \frac{17}{72},$$

(3.3.10) $$\int_0^1 y^4 \ln(y) \ln(1+y)\, dy = -\frac{2}{25} \ln(2) - \frac{1}{10} \text{zeta}(2) - \frac{3583}{18000}$$

(see (4.2.10)),

(3.3.11) $$\int_0^1 \ln(1-y) \ln(1+y)\, dy = 2 - 2\ln(2) + \ln^2(2) - \text{zeta}(2),$$

(3.3.12) $$\int_0^1 y \ln(1-y) \ln(1+y)\, dy = \tfrac{1}{4} - \ln(2),$$

(3.3.13) $$\int_0^1 y^3 \ln(1-y) \ln(1+y)\, dy = \frac{13}{96} - \frac{2}{3} \ln(2),$$

(3.3.14) $$\int_0^1 y^4 \ln(1-y) \ln(1+y)\, dy = \frac{413}{1125} - \frac{46}{75} \ln(2) + \frac{1}{5} \ln^2(2) - \frac{1}{5} \text{zeta}(2)$$

3·4. *Combinations of* $\ln(y)$, $\ln(1-y)$ *and* $\ln(1+y)$ *and negative powers of* $1+y$.

(3.4.1) $$\int_0^1 \frac{\ln(y)}{1+y}\, dy = -\frac{1}{2} \text{zeta}(2),$$

(3.4.2) $$\int_0^1 \frac{\ln(y)}{(1+y)^2}\, dy = -\ln(2),$$

(3.4.3) $$\int_0^1 \frac{\ln(y)}{(1+y)^3}\, dy = -\frac{1}{4} - \frac{1}{2} \ln(2),$$

(3.4.4) $$\int_0^1 \frac{\ln(y)}{(1+y)^4}\, dy = -\frac{7}{24} - \frac{1}{3} \ln(2),$$

(3.4.5) $$\int_0^1 \frac{\ln(1-y)}{1+y}\, dy = -\frac{1}{2} \text{zeta}(2) + \frac{1}{2} \ln^2(2),$$

$$(3.4.6)\qquad \int_0^1 \frac{\ln(1-y)}{(1+y)^2}\,dy = -\frac{1}{2}\ln(2)\,,$$

$$(3.4.7)\qquad \int_0^1 \frac{\ln(1-y)}{(1+y)^3}\,dy = -\frac{1}{8}-\frac{1}{8}\ln(2)\,,$$

$$(3.4.8)\qquad \int_0^1 \frac{\ln(1-y)}{(1+y)^4}\,dy = -\frac{5}{48}-\frac{1}{24}\ln(2)\,,$$

$$(3.4.9)\qquad \int_0^1 \frac{\ln^2(y)}{1+y}\,dy = \frac{3}{2}\,\mathrm{zeta}(3)\,,$$

$$(3.4.10)\qquad \int_0^1 \frac{\ln^2(y)}{(1+y)^2}\,dy = \mathrm{zeta}(2)\,,$$

$$(3.4.11)\qquad \int_0^1 \frac{\ln^2(y)}{(1+y)^3}\,dy = \frac{1}{2}\,\mathrm{zeta}(2)+\ln(2)\,,$$

$$(3.4.12)\qquad \int_0^1 \frac{\ln^2(y)}{(1+y)^4}\,dy = \frac{1}{3}\,\mathrm{zeta}(2)+\ln(2)+\frac{1}{6}\,,$$

$$(3.4.13)\qquad \int_0^1 \frac{\ln^2(1-y)}{1+y}\,dy = \frac{7}{4}\,\mathrm{zeta}(3)-\ln(2)\,\mathrm{zeta}(2)+\frac{1}{3}\ln^3(2)\,,$$

$$(3.4.14)\qquad \int_0^1 \frac{\ln^2(1-y)}{(1+y)^2}\,dy = \frac{1}{2}\,\mathrm{zeta}(2)-\frac{1}{2}\ln^2(2)\,,$$

$$(3.4.15)\qquad \int_0^1 \frac{\ln^2(1-y)}{(1+y)^3}\,dy = \frac{1}{8}\,\mathrm{zeta}(2)-\frac{1}{8}\ln^2(2)+\frac{1}{4}\ln(2)\,,$$

$$(3.4.16)\qquad \int_0^1 \frac{\ln^2(1-y)}{(1+y)^4}\,dy = \frac{1}{24}\,\mathrm{zeta}(2)-\frac{1}{24}\ln^2(2)+\frac{1}{8}\ln(2)+\frac{1}{24}\,,$$

$$(3.4.17)\qquad \int_0^1 \frac{\ln(y)\ln(1-y)}{1+y}\,dy = \frac{13}{8}\,\mathrm{zeta}(3)-\frac{3}{2}\,\mathrm{zeta}(2)\ln(2)\,,$$

$$(3.4.18)\qquad \int_0^1 \frac{\ln(y)\ln(1-y)}{(1+y)^2}\,dy = \frac{1}{4}\,\mathrm{zeta}(2)-\frac{1}{2}\ln^2(2)\,,$$

$$\int_0^1 \frac{\ln(y)\ln(1-y)}{(1+y)^3}\,dy = -\frac{1}{16}\,\text{zeta}(2) - \frac{1}{4}\ln^2(2) + \frac{1}{2}\ln(2)\,, \tag{3.4.19}$$

$$\int_0^1 \frac{\ln(y)\ln(1-y)}{(1+y)^4}\,dy = -\frac{5}{48}\,\text{zeta}(2) - \frac{1}{6}\ln^2(2) + \frac{3}{8}\ln(2) + \frac{1}{12}\,, \tag{3.4.20}$$

$$\int_0^1 \frac{\ln(y)\ln(1+y)}{1+y}\,dy = -\frac{1}{8}\,\text{zeta}(3)\,, \tag{3.4.21}$$

$$\int_0^1 \frac{\ln(y)\ln(1+y)}{(1+y)^2}\,dy = \frac{1}{2}\,\text{zeta}(2) - \frac{1}{2}\ln^2(2) - \ln(2)\,, \tag{3.4.22}$$

$$\int_0^1 \frac{\ln(y)\ln(1+y)}{(1+y)^3}\,dy = \frac{1}{4}\,\text{zeta}(2) - \frac{1}{4}\ln^2(2) - \frac{3}{8}\,, \tag{3.4.23}$$

$$\int_0^1 \frac{\ln(y)\ln(1+y)}{(1+y)^4}\,dy = \frac{1}{6}\,\text{zeta}(2) - \frac{1}{6}\ln^2(2) + \frac{7}{72}\ln(2) - \frac{47}{144}\,, \tag{3.4.24}$$

$$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{1+y}\,dy = \frac{1}{8}\,\text{zeta}(3) - \frac{1}{2}\,\text{zeta}(2)\ln(2) + \frac{1}{3}\ln^3(2)\,, \tag{3.4.25}$$

$$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{(1+y)^2}\,dy = \frac{1}{4}\,\text{zeta}(2) - \frac{1}{2}\ln(2) - \frac{1}{2}\ln^2(2)\,, \tag{3.4.26}$$

$$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{(1+y)^3}\,dy = -\frac{3}{16} + \frac{1}{16}\,\text{zeta}(2) + \frac{1}{16}\ln(2) - \frac{1}{8}\ln^2(2), \tag{3.4.27}$$

$$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{(1+y)^4}\,dy = \tag{3.4.28}$$

$$= -\frac{31}{288} + \frac{1}{48}\,\text{zeta}(2) + \frac{7}{144}\ln(2) - \frac{1}{24}\ln^2(2)\,.$$

3·5. $\ln(1+y)$ *and negative powers of* $(1-y)$.

$$\int_0^1 \frac{\ln(1+y)-\ln(2)}{1-y}\,dy = -\frac{1}{2}\{\text{zeta}(2) - \ln^2(2)\}\,, \tag{3.5.1}$$

$$(3.5.2)\qquad \int_0^1 \left[\frac{\ln(1+y)-\ln(2)}{(1-y)^2}+\frac{1}{2(1-y)}\right] dy = -\frac{1}{2}[1-\ln(2)]\,,$$

$$(3.5.3)\qquad \int_0^1 \left[\frac{\ln(1+y)-\ln(2)}{(1-y)^3}+\frac{1}{2(1-y)^2}+\frac{1}{8(1-y)}\right] dy = -\frac{5}{16}+\frac{3}{8}\ln(2)\,,$$

$$(3.5.4)\qquad \int_0^1 \left[\frac{\ln(1+y)-\ln(2)}{1(-y)^4}+\frac{1}{2(1-y)^3}+\frac{1}{8(1-y)^2}+\frac{1}{24(1-y)}\right] dy = -\frac{2}{9}+\frac{7}{24}\ln(2)\,.$$

3·6. *Combinations and powers of* $\ln(y)$, $\ln(1-y)$ *and* $\ln(1+y)$ *and negative powers of* y.

$$(3.6.1)\qquad \int_0^1 \frac{\ln(1-y)}{y}\, dy = -\text{zeta}(2)\,,$$

$$(3.6.2)\qquad \int_0^1 \frac{\ln(1-y)+y}{y^2}\, dy = -1\,,$$

$$(3.6.3)\qquad \int_0^1 \left[\frac{\ln(1-y)}{y^3}+\frac{1}{y^2}+\frac{1}{2y}\right] dy = -\frac{3}{4},$$

$$(3.6.4)\qquad \int_0 \left[\frac{\ln(1-y)}{y^4}+\frac{1}{y^3}+\frac{1}{2y^2}+\frac{1}{3y}\right] dy = -\frac{11}{18},$$

$$(3.6.5)\qquad \int_0^1 \frac{\ln(1+y)}{y}\, dy = \frac{1}{2}\,\text{zeta}(2)\,,$$

$$(3.6.6)\qquad \int_0^1 \frac{\ln(1+y)-y}{y^2}\, dy = 1-2\ln(2)\,,$$

$$(3.6.7)\qquad \int_0^1 \left[\frac{\ln(1+y)}{y^3}-\frac{1}{y^2}+\frac{1}{2y}\right] dy = \frac{1}{4}\,,$$

$$(3.6.8)\qquad \int_0^1 \left[\frac{\ln(1+y)}{y^4}-\frac{1}{y^3}+\frac{1}{2y^2}-\frac{1}{3y}\right] dy = \frac{5}{18}-\frac{2}{3}\ln(2)\,,$$

(3.6.9) $$\int_0^1 \frac{\ln^2(1-y)}{y}\,dy = 2\,\mathrm{zeta}(3)\,,$$

(3.6.10) $$\int_0^1 \frac{\ln^2(1-y)}{y^2}\,dy = 2\,\mathrm{zeta}(2)\,,$$

(3.6.11) $$\int_0^1 \left[\frac{\ln^2(1-y)}{y^3} - \frac{1}{y}\right] dy = \frac{3}{2} + \mathrm{zeta}(2)\,,$$ see also next integral,

(3.6.11a) $$\int_0^1 \left[\frac{\ln^2(1-y)}{y^3} + \frac{\ln(1-y)}{y^2}\right] dy = \frac{1}{2} + \mathrm{zeta}(2)\,,$$

see preceeding integral,

(3.6.12) $$\int_0^1 \left[\frac{\ln^2(1-y)}{y^4} - \frac{1}{y} - \frac{1}{y^2}\right] dy = \frac{11}{6} + \frac{2}{3}\,\mathrm{zeta}(2)\,,$$ see below,

(3.6.12a) $$\int_0^1 \left[\frac{\ln^2(1-y)}{y^4} + \frac{\ln(1-y)}{y^3} - \frac{1}{2y}\right] dy = \frac{13}{12} + \frac{2}{3}\,\mathrm{zeta}(2)\,,$$ see below,

(3.6.12b) $$\int_0^1 \left[\frac{\ln^2(1-y)}{y^4} + \frac{\ln(1-y)}{y^3} + \frac{\ln(1-y)}{2y^2}\right] dy = \frac{7}{12} + \frac{2}{3}\,\mathrm{zeta}(2)\,,$$

(3.6.13) $$\int_0^1 \frac{\ln^2(1+y)}{y}\,dy = \frac{1}{4}\,\mathrm{zeta}(3)\,,$$

(3.6.14) $$\int_0^1 \frac{\ln^2(1+y)}{y^2}\,dy = \mathrm{zeta}(2) - 2\ln^2(2)\,,$$

(3.6.15) $$\int_0^1 \left[\frac{\ln^2(1+y)}{y^3} - \frac{1}{y}\right] dy = \frac{3}{2} - \frac{1}{2}\,\mathrm{zeta}(2) - 2\ln(2)\,,$$

(3.6.16) $$\int_0^1 \left[\frac{\ln^2(1+y)}{y^4} - \frac{1}{y^2} + \frac{1}{y}\right] dy = -\frac{1}{2} + \frac{1}{3}\,\mathrm{zeta}(2) + \frac{4}{3}\ln(2) - \frac{2}{3}\ln^2(2)\,,$$

(3.6.17) $$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{y}\,dy = -\frac{5}{8}\,\mathrm{zeta}(3)\,,$$

(3.6.18) $$\int_0^1 \frac{\ln(1+y)\ln(1-y)}{y^2}\,dy = -\frac{1}{2}\,\text{zeta}(2) - \ln^2(2)\,,$$

(3.6.19) $$\int_0^1 \left[\frac{\ln(1+y)\ln(1-y)}{y^3} + \frac{1}{y}\right] dy = -\frac{3}{2} + \frac{1}{4}\,\text{zeta}(2) + \ln(2)\,,$$

(3.6.20) $$\int_0^1 \left[\frac{\ln(1+y)\ln(1-y)}{y^4} + \frac{1}{y^2}\right] dy =$$
$$= -\frac{2}{3} - \frac{1}{6}\,\text{zeta}(2) + \frac{2}{3}\ln(2) - \frac{1}{3}\ln^2(2)\,,$$

(3.6.21) $$\int_0^1 \frac{\ln(y)\ln(1-y)}{y}\,dy = \text{zeta}(3)\,,$$

(3.6.22) $$\int_0^1 \left[\frac{\ln(y)\ln(1-y)}{y^2} + \frac{\ln(y)}{y}\right] dy = -1 + \text{zeta}(2)\,,$$

(3.6.23) $$\int_0^1 \left[\frac{\ln(y)\ln(1-y)}{y^3} + \frac{\ln(y)}{y^2} + \frac{\ln(y)}{2y}\right] dy = -\frac{3}{8} + \frac{1}{2}\,\text{zeta}(2)\,,$$

(3.6.24) $$\int_0^1 \left[\frac{\ln(y)\ln(1-y)}{y^4} + \frac{\ln(y)}{y^3} + \frac{\ln(y)}{2y^2} + \frac{\ln(y)}{3y}\right] dy = -\frac{11}{54} + \frac{1}{3}\,\text{zeta}(2)\,,$$

(3.6.25) $$\int_0^1 \frac{\ln(y)\ln(1+y)}{y}\,dy = -\frac{3}{4}\,\text{zeta}(3)\,,$$

(3.6.26) $$\int_0^1 \left[\frac{\ln(y)\ln(1+y)}{y^2} - \frac{\ln(y)}{y}\right] dy = 1 + \frac{1}{2}\,\text{zeta}(2) - 2\ln(2)\,,$$

(3.6.27) $$\int_0^1 \left[\frac{\ln(y)\ln(1+y)}{y^3} - \frac{\ln(y)}{y^2} + \frac{\ln(y)}{2y}\right] dy = \frac{1}{8} - \frac{1}{4}\,\text{zeta}(2)\,,$$

(3.6.28) $$\int_0^1 \left[\frac{\ln(y)\ln(1+y)}{y^4} - \frac{\ln(y)}{y^3} + \frac{\ln(y)}{2y^2} - \frac{\ln(y)}{3y}\right] dy =$$
$$= \frac{5}{54} + \frac{1}{6}\,\text{zeta}(2) - \frac{2}{9}\ln(2)\,.$$

3·7. *Square of* $\ln(1+y)$ *and negative powers of* $(1-y)$.

(3.7.1) $$\int_0^1 \frac{\ln^2(1+y)-\ln^2(2)}{1-y}\,dy = \frac{1}{4}\,\text{zeta}(3) - \text{zeta}(2)\ln(2) + \frac{2}{3}\ln^3(2)\,,$$

(3.7.2) $$\int_0^1 \left[\frac{\ln^2(1+y)-\ln^2(2)}{(1-y)^2} + \frac{\ln(2)}{1-y}\right] dy = \frac{1}{2}\,\text{zeta}(2) - \ln(2)\,,$$

(3.7.3) $$\int_0^1 \left[\frac{\ln^2(1+y)-\ln^2(2)}{(1-y)^3} + \frac{\ln(2)}{(1-y)^2} + \frac{\ln(2)-1}{4(1-y)}\right] dy = \\ = \frac{3}{8} + \frac{1}{8}\,\text{zeta}(2) - \frac{7}{8}\ln(2) + \frac{1}{4}\ln^2(2)\,.$$

3·8. *Dilogarithms and negative powers of binomials and of* y.

(3.8.1) $$\int_0^1 \frac{\text{Li}_2(y)}{1+y}\,dy = -\frac{5}{8}\,\text{zeta}(3) + \text{zeta}(2)\ln(2)\,,$$

(3.8.2) $$\int_0^1 \frac{\text{Li}_2(y)}{(1+y)^2}\,dy = \frac{1}{2}\ln^2(2)\,,$$

(3.8.3) $$\int_0^1 \frac{\text{Li}_2(y)}{(1+y)^3}\,dy = \frac{1}{8}\,\text{zeta}(2) - \frac{1}{4}\ln(2) + \frac{1}{4}\ln^2(2)\,,$$

(3.8.4) $$\int_0^1 \frac{\text{Li}_2(y)}{(1+y)^4}\,dy = -\frac{1}{24} + \frac{1}{8}\,\text{zeta}(2) - \frac{5}{24}\ln(2) + \frac{1}{6}\ln^2(2)\,,$$

(3.8.5) $$\int_0^1 \frac{\text{Li}_2(-y)}{1+y}\,dy = \frac{1}{4}\,\text{zeta}(3) - \frac{1}{2}\,\text{zeta}(2)\ln(2)\,,$$

(3.8.6) $$\int_0^1 \frac{\text{Li}_2(-y)}{(1+y)^2}\,dy = -\frac{1}{4}\,\text{zeta}(2) + \frac{1}{2}\ln^2(2)\,,$$

(3.8.7) $$\int_0^1 \frac{\text{Li}_2(-y)}{(1+y)^3}\,dy = \frac{1}{4} - \frac{3}{16}\,\text{zeta}(2) - \frac{1}{4}\ln(2) + \frac{1}{4}\ln^2(2)\,,$$

(3.8.8) $$\int_0^1 \frac{\text{Li}_2(-y)}{(1+y)^4}\,dy = \frac{11}{48} - \frac{7}{48}\,\text{zeta}(2) - \frac{5}{24}\ln(2) + \frac{1}{6}\ln^2(2)\,,$$

(3.8.9) $$\int_0^1 \frac{\mathrm{Li}_2(y) - \mathrm{zeta}(2)}{1-y}\, dy = -2\, \mathrm{zeta}(3)\,,$$

(3.8.10) $$\int_0^1 \left[\frac{\mathrm{Li}_2(y) - \mathrm{zeta}(2)}{(1-y)^2} + \frac{1 - \ln(1-y)}{1-y}\right] dy = -1\,,$$

(3.8.11) $$\int_0^1 \frac{1}{1-y}\left[\mathrm{Li}_2(-y) + \frac{1}{2}\,\mathrm{zeta}(2)\right] dy = -\frac{5}{8}\,\mathrm{zeta}(3)\,,$$

(3.8.12) $$\int_0^1 \left\{\frac{1}{(1-y)^2}\left[\mathrm{Li}_2(-y) + \frac{1}{2}\,\mathrm{zeta}(2)\right] - \frac{\ln(2)}{1-y}\right\} dy = \\ = -\frac{3}{2}\,\mathrm{zeta}(2) + \ln(2) + \frac{1}{2}\ln^2(2)\,,$$

(3.8.13) $$\int_0^1 \frac{\mathrm{Li}_2(y)}{y}\, dy = \mathrm{zeta}(3)\,,$$

(3.8.14) $$\int_0^1 \left[\frac{\mathrm{Li}_2(y)}{y^2} - \frac{1}{y}\right] dy = 2 - \mathrm{zeta}(2)\,,$$

(3.8.15) $$\int_0^1 \frac{\mathrm{Li}_2(-y)}{y}\, dy = -\frac{3}{4}\,\mathrm{zeta}(3)\,,$$

(3.8.16) $$\int_0^1 \left[\frac{\mathrm{Li}_2(-y)}{y^2} + \frac{1}{y}\right] dy = -2 + \frac{1}{2}\,\mathrm{zeta}(2) + 2\ln(2)\,.$$

3'9. *Powers of y and negative powers of $(a + by)$.*

(3.9.1) $$\int_0^1 \frac{1}{a+by}\, dy = \frac{1}{b}\ln\left(\frac{a+b}{a}\right),$$

(3.9.2) $$\int_0^1 \frac{y}{a+by}\, dy = \frac{1}{b^2}\left[b - a\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.3) $$\int_0^1 \frac{1-y}{a+by}\, dy = \frac{1}{b^2}\left[-b + (a+b)\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.4) $$\int_0^1 \frac{y^2}{a+by}\, dy = \frac{1}{b^3}\left[\frac{b^2}{2} - ab + a^2\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.5) $$\int_0^1 \frac{y^3}{a+by}\,dy = \frac{1}{b^4}\left[\frac{b^3}{3}-\frac{1}{2}ab^2+a^2b-a^3\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.6) $$\int_0^1 \frac{1}{(a+by)^2}\,dy = \frac{1}{a(a+b)},$$

(3.9.7) $$\int_0^1 \frac{y}{(a+by)^2}\,dy = \frac{1}{b^2}\left[-\frac{b}{a+b}+\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.8) $$\int_0^1 \frac{1-y}{(a+by)^2}\,dy = \frac{1}{b^2}\left[\frac{b}{a}-\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.9) $$\int_0^1 \frac{y^2}{(a+by)^2}\,dy = \frac{1}{b^3}\left[b+\frac{ab}{a+b}-2\,a\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.10) $$\int_0^1 \frac{y^3}{(a+by)^2}\,dy = \frac{1}{b^4}\left[\frac{b^2}{2}-2ab-\frac{a^2b}{a+b}+3a^2\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.11) $$\int_0^1 \frac{1}{(a+by)^3}\,dy = \frac{1}{2b}\left[\frac{1}{a^2}-\frac{1}{(a+b)^2}\right],$$

(3.9.12) $$\int_0^1 \frac{y}{(a+by)^3}\,dy = \frac{1}{2a(a+b)^2},$$

(3.9.13) $$\int_0^1 \frac{1-y}{(a+by)^3}\,dy = \frac{1}{2a^2(a+b)},$$

(3.9.14) $$\int_0^1 \frac{y^2}{(a+by)^3}\,dy = \frac{1}{b^3}\left[-\frac{b}{a+b}-\frac{b^2}{2(a+b)^2}+\ln\left(\frac{a+b}{a}\right)\right],$$

(3.9.15) $$\int_0^1 \frac{y^3}{(a+by)^3}\,dy = \frac{1}{b^4}\left[b+\frac{2ab}{a+b}+\frac{ab^2}{2(a+b)^2}-3a\ln\left(\frac{a+b}{a}\right)\right].$$

3·10. $\ln(a+by)$, $\ln(y)\ln(a+by)$, $\ln(1-y)\ln(a+by)$ *and powers of* y.

(3.10.1) $$\int_0^1 \ln(a+by)\,dy = \frac{a+b}{b}\ln(a+b)-\frac{a}{b}\ln(a)-1,$$

$$(3.10.2)\qquad \int_0^1 y\ln(a+by)\,\mathrm{d}y = \frac{1}{2b^2}\left\{(b^2-a^2)\ln(a+b)+a^2\ln(a)-\frac{b^2}{2}+ab\right\},$$

$$(3.10.3)\qquad \int_0^1 y^2\ln(a+by)\,\mathrm{d}y = \frac{1}{3b^3}\left\{(a^3+b^3)\ln(a+b)-a^3\ln(a)\right\}- -\frac{1}{9}+\frac{a}{6b}-\frac{a^2}{3b^2},$$

$$(3.10.4)\qquad \int_0^1 y^3\ln(a+by)\,\mathrm{d}y = \frac{1}{4b^4}\left\{(b^4-a^4)\ln(a+b)+a^4\ln(a)\right\}- -\frac{1}{16}+\frac{a}{12b}-\frac{a^2}{8b^2}+\frac{a^3}{4b^3},$$

$$(3.10.5)\qquad \int_0^1 \ln(y)\ln(a+by)\,\mathrm{d}y = \frac{1}{b}\left\{a\ln(a)-(a+b)\ln(a+b)+a\,\mathrm{Li}_2\left(-\frac{a}{b}\right)\right\}+2,$$

$$(3.10.6)\qquad \int_0^1 y\ln(y)\ln(a+by)\,\mathrm{d}y = \frac{1}{4b^2}\left\{(a^2-b^2)\ln(a+b)-a^2\ln(a)\right\}- -\frac{a^2}{2b^2}\,\mathrm{Li}_2\left(-\frac{b}{a}\right)+\frac{1}{4}-\frac{3a}{4b},$$

$$(3.10.7)\qquad \int_0^1 y^2\ln(y)\ln(a+by)\,\mathrm{d}y = \frac{1}{9b^3}\left\{a^3\ln(a)-(a^3+b^3)\ln(a+b)\right\}+ +\frac{a^3}{3b^3}\,\mathrm{Li}_2\left(-\frac{b}{a}\right)+\frac{2}{27}-\frac{5a}{36b}+\frac{4a^2}{9b^2},$$

$$(3.10.8)\qquad \int_0^1 y^3\ln(y)\ln(a+by)\,\mathrm{d}y = \frac{1}{16b^4}\left\{(a^4-b^4)\ln(a+b)-a^4\ln(a)\right\}- -\frac{a^4}{4b^4}\,\mathrm{Li}_2\left(-\frac{b}{a}\right)+\frac{1}{32}-\frac{7a}{144b}+\frac{3a^2}{32b^2}-\frac{5a^3}{16b^3},$$

$$(3.10.9)\qquad \int_0^1 \ln(1-y)\ln(a+by)\,\mathrm{d}y = 2+\frac{1}{b}\left\{a\ln(a)- -(a+b)\left[\ln(a+b)+\mathrm{Li}_2\left(\frac{b}{a+b}\right)\right]\right\},$$

$$(3.10.10)\qquad \int_0^1 y\ln(1-y)\ln(a+by)\,\mathrm{d}y = \frac{1}{4b^2}\left\{a(2b-a)\ln(a)+ +(a+b)(a-3b)\ln(a+b)\right\}+\frac{(a^2-b^2)}{2b^2}\,\mathrm{Li}_2\left(\frac{b}{a+b}\right)+1-\frac{3a}{4b},$$

(3.10.11) $$\int_0^1 y^2 \ln(1-y)\ln(a+by)\,dy = \frac{1}{18b^3}\left\{a(2a^2-3ab+6b^2)\ln(a) - (a+b)(11b^2-5ab+2a^2)\ln(a+b)\right\} - \frac{b^3+a^3}{3b^3}\mathrm{Li}_2\left(\frac{b}{a+b}\right) + \frac{71}{108} - \frac{17a}{36b} + \frac{4a^2}{4b^2},$$

(3.10.12) $$\int_0^1 y^3 \ln(1-y)\ln(a+by)\,dy = \frac{1}{48b^4}\left\{a(12b^3-6ab^2+4a^2b - 3a^3)\ln(a) + (a+b)(25b^3-13ab^2+7a^2b-3a^3)\ln(a+b)\right\} + \frac{a^4-b^4}{4b^4}\mathrm{Li}_2\left(\frac{b}{a+b}\right) + \frac{35}{72} - \frac{49a}{144b} + \frac{29a^2}{96b^2} - \frac{5a^3}{16b^3}.$$

3·11. $\ln(1+ay)\ln(1+by)$ *and powers of* y.

(3.11.1) $$\int_0^1 \ln(1+ay)\ln(1+by)\,dy = 2 + \ln(1+a)\ln(1+b) - \frac{1+a}{a}\ln(1+a) - \frac{1+b}{b}\ln(1+b) + \frac{1}{a}\left[\ln(1+a)\ln\left(\frac{a-b}{a}\right) + \mathrm{Li}_2\left(-\frac{b}{a-b}\right) - \mathrm{Li}_2\left(-b\,\frac{1+a}{a-b}\right)\right] + \frac{1}{b}\left[\ln(1+b)\ln\left(\frac{b-a}{b}\right) + \mathrm{Li}_2\left(-\frac{a}{b-a}\right) - \mathrm{Li}_2\left(-a\,\frac{1+b}{b-a}\right)\right] =$$
$$= 2 + \frac{1+a}{a}\ln(1+a)\ln(1+b) - \frac{1+a}{a}\ln(1+a) - \frac{1+b}{b}\ln(1+b) + \frac{a-b}{ab}\left[\ln\left(\frac{b-a}{b}\right)\ln(1+b) + \mathrm{Li}_2\left(-\frac{a}{b-a}\right) - \mathrm{Li}_2\left(-a\,\frac{1+b}{b-a}\right)\right],$$

(3.11.2) $$\int_0^1 y\ln(1+ay)\ln(1+by)\,dy = \frac{1}{2}\left\{\frac{1}{2} - \frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}\right) + \ln(1+a)\ln(1+b) + \frac{1+a}{a}\ln(1+a)\left(\frac{1}{b}+\frac{1-a}{2a}\right) + \frac{1+b}{b}\ln(1+b)\left(\frac{1}{a}+\frac{1-b}{2b}\right) - \frac{1}{a^2}\left[\ln\left(\frac{a-b}{a}\right)\ln(1+a) + \mathrm{Li}_2\left(-\frac{b}{a-b}\right) - \mathrm{Li}_2\left(-b\,\frac{1+a}{a-b}\right)\right] - \frac{1}{b^2}\left[\ln\left(\frac{b-a}{b}\right)\ln(1+b) + \mathrm{Li}_2\left(-\frac{a}{b-a}\right) - \mathrm{Li}_2\left(-a\,\frac{1+b}{b-a}\right)\right]\right\}.$$

3'12. *Combinations and powers of* $\ln(y)$, $\ln(1-y)$ *and* $\ln(1+y)$ *and negative powers of* $(a+by)$.

$$\int_0^1 \frac{\ln(y)}{a+by}\,dy = \frac{1}{b}\,\mathrm{Li}_2\left(-\frac{b}{a}\right), \tag{3.12.1}$$

$$\int_0^1 \frac{\ln(y)}{(a+by)^2}\,dy = -\frac{1}{ab}\ln\left(\frac{a+b}{a}\right), \tag{3.12.2}$$

$$\int_0^1 \frac{y\ln(y)}{(a+by)^2}\,dy = \frac{1}{b^2}\left[\ln\left(\frac{a+b}{a}\right) + \mathrm{Li}_2\left(-\frac{b}{a}\right)\right], \tag{3.12.3}$$

$$\int_0^1 \frac{(1-y)\ln(y)}{(a+by)^2}\,dy = -\frac{1}{b^2}\left[\frac{a+b}{a}\ln\left(\frac{a+b}{a}\right) + \mathrm{Li}_2\left(-\frac{b}{a}\right)\right], \tag{3.12.4}$$

$$\int_0^1 \frac{\ln(y)}{(a+by)^3}\,dy = -\frac{1}{2a^2}\left[\frac{1}{a+b} + \frac{1}{b}\ln\left(\frac{a+b}{a}\right)\right], \tag{3.12.5}$$

$$\int_0^1 \frac{\ln(y)}{(a+by)^4}\,dy = \frac{1}{3a^3b}\left[-\frac{3}{2} + \frac{a}{a+b} + \frac{a^2}{2(a+b)^2} - \ln\left(\frac{a+b}{a}\right)\right], \tag{3.12.6}$$

$$\int_0^1 \frac{\ln(1-y)}{a+by}\,dy = -\frac{1}{b}\,\mathrm{Li}_2\left(\frac{b}{a+b}\right), \tag{3.12.7}$$

$$\int_0^1 \frac{\ln(1-y)}{(a+by)^2}\,dy = -\frac{1}{b(a+b)}\ln\left(\frac{a+b}{a}\right), \tag{3.12.8}$$

$$\int_0^1 \frac{\ln(1-y)}{(a+by)^3}\,dy = -\frac{1}{2(a+b)^2}\left[\frac{1}{a} + \frac{1}{b}\ln\left(\frac{a+b}{a}\right)\right], \tag{3.12.9}$$

$$\int_0^1 \frac{\ln(1-y)}{(a+by)^4}\,dy = -\frac{1}{3(a+b)^3}\left[\frac{2}{a} + \frac{b}{2a^2} + \frac{1}{b}\ln\left(\frac{a+b}{a}\right)\right], \tag{3.12.10}$$

$$\int_0^1 \frac{\ln(1+y)}{a+by}\,dy = \frac{1}{b}\left[\ln\left(\frac{b-a}{b}\right)\ln\left(\frac{a+b}{a}\right) - \mathrm{Li}_2\left(-\frac{a+b}{b-a}\right) + \mathrm{Li}_2\left(-\frac{a}{b-a}\right)\right], \tag{3.12.11}$$

$$(3.12.12)\qquad \int_0^1 \frac{\ln(1+y)}{(a+by)^2}\,\mathrm{d}y = \frac{2\ln(2)}{a^2-b^2} + \frac{1}{b(a-b)}\ln\left(\frac{a}{a+b}\right),$$

$$(3.12.13)\qquad \int_0^1 \frac{\ln(1+y)}{(a+by)^3}\,\mathrm{d}y = \frac{2a\ln(2)}{(a^2-b^2)^2} - \frac{1}{2a(a^2-b^2)} + \frac{1}{2b(a-b)^2}\ln\left(\frac{a}{a+b}\right),$$

$$(3.12.14)\qquad \int_0^1 \frac{\ln(1+y)}{(a+by)^4}\,\mathrm{d}y = \frac{2(b^2+3a^2)\ln(2)}{3(a^2-b^2)^3} - \frac{4a^2+ab-b^2}{6a^2(a^2-b^2)^2} + \frac{1}{3b(a-b)^3}\ln\left(\frac{a}{a+b}\right),$$

$$(3.12.15)\qquad \int_0^1 \frac{\ln^2(y)}{a+by}\,\mathrm{d}y = -\frac{2}{b}\,\mathrm{Li}_3\left(-\frac{b}{a}\right),$$

$$(3.12.16)\qquad \int_0^1 \frac{\ln^2(y)}{(a+by)^2}\,\mathrm{d}y = -\frac{2}{ab}\,\mathrm{Li}_2\left(-\frac{b}{a}\right),$$

$$(3.12.17)\qquad \int_0^1 \frac{\ln^2(y)}{(a+by)^3}\,\mathrm{d}y = \frac{1}{a^2 b}\left[\ln\left(\frac{a+b}{a}\right) - \mathrm{Li}_2\left(-\frac{b}{a}\right)\right],$$

$$(3.12.18)\qquad \int_0^1 \frac{\ln^2(y)}{(a+by)^4}\,\mathrm{d}y = \frac{1}{3a^3 b}\left[\frac{b}{a+b} + 3\ln\left(\frac{a+b}{a}\right) - 2\mathrm{Li}_2\left(-\frac{b}{a}\right)\right],$$

$$(3.12.19)\qquad \int_0^1 \frac{\ln^2(1+y)}{a+by}\,\mathrm{d}y = \frac{1}{b}\left[2\mathrm{Li}_3\left(-\frac{b}{a-b}\right) - 2\,\mathrm{Li}_3\left(-\frac{2b}{a-b}\right) + 2\ln(2)\,\mathrm{Li}_2\left(-\frac{2b}{a-b}\right) + \ln^2(2)\ln\left(\frac{a+b}{a-b}\right)\right],$$

$$(3.12.20)\qquad \int_0^1 \frac{\ln^2(1+y)}{(a+by)^2}\,\mathrm{d}y = \frac{2}{b(a-b)}\left[\mathrm{Li}_2\left(-\frac{b}{a-b}\right) - \mathrm{Li}_2\left(-\frac{2b}{a-b}\right) + \ln(2)\ln\left(\frac{a-b}{a+b}\right)\right] + \frac{2\ln^2(2)}{a^2-b^2},$$

$$(3.12.21)\qquad \int_0^1 \frac{\ln^2(1+y)}{(a+by)^3}\,\mathrm{d}y = \frac{1}{b(a-b)^2}\left[\mathrm{Li}_2\left(-\frac{b}{a-b}\right) - \mathrm{Li}_2\left(-\frac{2b}{a-b}\right) + \ln\left(\frac{a+b}{a}\right) + \ln(2)\ln\left(\frac{a-b}{a+b}\right)\right] + \frac{2}{(a^2-b^2)^2}\left[a\ln^2(2) - (a+b)\ln(2)\right],$$

(3.12.22) $$\int_0^1 \frac{\ln(y)\ln(1-y)}{a+by}\,dy = \frac{1}{b}\left\{\ln\left(\frac{a+b}{a}\right)\left[\mathrm{Li}_2\left(-\frac{b}{a}\right)-\mathrm{zeta}(2)\right]+ + S_{1,2}\left(-\frac{b}{a}\right)-\mathrm{Li}_3\left(-\frac{b}{a}\right)\right\},$$

(3.12.23) $$\int_0^1 \frac{\ln(y)\ln(1-y)}{(a+by)^2}\,dy = \frac{1}{ab}\mathrm{Li}_2\left(\frac{b}{a+b}\right)- -\frac{1}{b(a+b)}\mathrm{Li}_2\left(-\frac{b}{a}\right)-\frac{\mathrm{zeta}(2)}{a(a+b)},$$

(3.12.24) $$\int_0^1 \frac{\ln(y)\ln(1-y)}{(a+by)^3}\,dy = \frac{1}{2a^2b}\mathrm{Li}_2\left(\frac{b}{a+b}\right)-\frac{1}{2b(a+b)^2}\mathrm{Li}_2\left(-\frac{b}{a}\right)- -\frac{\mathrm{zeta}(2)}{2a(a+b)}\left(\frac{1}{a}+\frac{1}{a+b}\right)+\frac{1}{ab(a+b)}\ln\left(\frac{a+b}{a}\right),$$

(3.12.25) $$\int_0^1 \frac{\ln(y)\ln(1+y)}{a+by}\,dy = \frac{1}{b}\left\{\frac{1}{2}\ln\left(\frac{a+b}{a}\right)\left[\ln^2\left(\frac{b-a}{b}\right)-\ln^2(2)\right]+ + \ln(2)\left[\mathrm{Li}_2\left(\frac{a-b}{2a}\right)-\mathrm{Li}_2\left(\frac{1}{2}\right)\right]+\ln\left(\frac{b-a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a}{a-b}\right)- - \mathrm{Li}_2\left(\frac{a+b}{a-b}\right)\right]-\mathrm{Li}_3\left(-\frac{b}{a}\right)+\mathrm{Li}_3\left(\frac{a-b}{2a}\right)-\mathrm{Li}_3\left(\frac{1}{2}\right)- - S_{1,2}\left(\frac{a}{a-b}\right)+S_{1,2}\left(\frac{a+b}{a-b}\right)\right\},$$

(3.12.26) $$\int_0^1 \frac{\ln(y)\ln(1+y)}{(a+by)^2}\,dy = \frac{1}{b}\left\{\frac{1}{b-a}\left[\frac{1}{2}\mathrm{zeta}(2)+\mathrm{Li}_2\left(-\frac{b}{a}\right)\right]+ + \frac{1}{a}\left[\ln\left(\frac{b}{b-a}\right)\ln\left(\frac{a+b}{a}\right)+\mathrm{Li}_2\left(\frac{a+b}{a-b}\right)+ + \frac{1}{2}\mathrm{zeta}(2)-\mathrm{Li}_2\left(\frac{a}{a-b}\right)\right]\right\}.$$

3·13. $\ln(c+ey)$ *and negative powers of* $(a+by)$.

(3.13.1) $$\int_0^1 \frac{\ln(c+ey)}{a+by}\,dy = \frac{1}{b}\left[\ln\left(\frac{bc-ae}{b}\right)\ln\left(\frac{a+b}{a}\right)- - \mathrm{Li}_2\left(e\frac{a+b}{ae-bc}\right)+\mathrm{Li}_2\left(\frac{ae}{ae-bc}\right)\right],$$

(3.13.2) $$\int_0^1 \frac{\ln(c+ey)}{(a+by)^2}\,\mathrm{d}y = \frac{1}{bc-ae}\left[\frac{c}{a}\ln(c) + \frac{e}{b}\ln\left(\frac{a+b}{a}\right) - \frac{c+e}{a+b}\ln(c+e)\right],$$

(3.13.3) $$\int_0^1 \frac{\ln(c+ey)}{(a+by)^3}\,\mathrm{d}y = \frac{1}{2(bc-ae)^2}\left[\frac{bc-2ae}{a^2}\,c\ln(c) - \frac{e^2}{b}\ln\left(\frac{a+b}{a}\right) + \frac{2ae-bc+be}{(a+b)^2}(c+e)\ln(c+e) + \frac{e(bc-ae)}{a(a+b)}\right],$$

(3.13.4) $$\int_0^1 \frac{\ln(c+ey)}{(a+by)^4}\,\mathrm{d}y = \frac{1}{3(bc-ae)^3}\left[\frac{b^2c^2+3a^2e^2-3abce}{a^3}\,c\ln(c) + \frac{e^3}{b}\ln\left(\frac{a+b}{a}\right) - \frac{b^2(c^2-ce+e^2)+3a^2e^2-3abe(c-e)}{(a+b)^3}(c+e)\cdot \ln(c+e) + \frac{e(2a+b)(bc-ae)^2}{2a^2(a+b)^2} - \frac{e^2(bc-ae)}{a(a+b)}\right].$$

3'14. *Integrals containing logarithms and inverse monomials and binomials.*

(3.14.1) $$\int_0^1 \frac{\ln(a+by)-\ln(a)}{y}\,\mathrm{d}y = -\operatorname{Li}_2\left(-\frac{b}{a}\right),$$

(3.14.2) $$\int_0^1 \left[\frac{\ln(a+by)-\ln(a)}{y^2} - \frac{b}{ay}\right]\mathrm{d}y = \frac{1}{a}\left[b-(a+b)\ln\left(\frac{a+b}{a}\right)\right],$$

(3.14.3) $$\int_0^1 \frac{\ln^2(a+by)-\ln^2(a)}{y}\,\mathrm{d}y = 2\ln(a)\operatorname{Li}_2\left(-\frac{b}{a}\right) + 2S_{1,2}\left(-\frac{b}{a}\right),$$

(3.14.4) $$\int_0^1 \left[\frac{\ln^2(a+by)-\ln^2(a)}{y^2} - \frac{2b}{ay}\ln(a)\right]\mathrm{d}y = \frac{1}{a}\left\{-2(a+b)\ln\left(\frac{a+b}{a}\right) + 2b\ln(a) - (a+b)\ln^2\left(\frac{a+b}{a}\right) - 2b\operatorname{Li}_2\left(-\frac{b}{a}\right)\right],$$

(3.14.5) $$\int_0^1 \frac{\ln(a+by)}{a+by}\,\mathrm{d}y = \frac{1}{2b}\ln\left(\frac{a+b}{a}\right)[\ln(a)+\ln(a+b)],$$

(3.14.6) $$\int_0^1 \frac{\ln(a+by)}{(a+by)^2}\,\mathrm{d}y = \frac{1}{a(a+b)} + \frac{\ln(a)}{ab} - \frac{\ln(a+b)}{b(a+b)},$$

$$(3.14.7)\qquad \int_0^1 \frac{\ln(a+by)}{(a+by)^3}\,dy = \frac{1}{4b}\left[\frac{1}{a^2}-\frac{1}{(a+b)^2}\right]+\frac{\ln(a)}{2a^2b}-\frac{\ln(a+b)}{2b(a+b)^2}.$$

3'15. *More complicated integrals containing logarithms and inverse monomials and binomials.*

$$(3.15.1)\qquad \int_0^1 \frac{\ln(1-y)\ln(1-ay)}{y}\,dy = S_{1,2}(a)+\mathrm{Li}_3(a),$$

$$(3.15.2)\qquad \int_0^1 \frac{\ln(y)\ln(1-ay)}{1-y}\,dy = 2S_{1,2}(a)-\mathrm{Li}_3(a)+ \\ +\ln(1-a)[\mathrm{Li}_2(a)-\mathrm{zeta}(2)],$$

$$(3.15.3)\qquad \int_0^1 \frac{\ln(y)\ln(1-y)}{y-1/a}\,dy = S_{1,2}(a)-2\mathrm{Li}_3(a)+ \\ +\ln(1-a)[\mathrm{Li}_2(a)-\mathrm{zeta}(2)],$$

$$(3.15.4)\qquad \int_0^1 \frac{\ln(1+ay)\ln(1+by)}{y}\,dy = S_{1,2}(-a)+S_{1,2}(-b)- \\ -\frac{1}{2}\ln^2\left(\frac{a}{b}\right)\ln(1+b)+\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a}\right)-\mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right)\right]- \\ -S_{1,2}\left(\frac{a-b}{a}\right)+S_{1,2}\left(\frac{a-b}{a(1+b)}\right)-S_{1,2}\left(\frac{b-a}{1+b}\right),$$

$$(3.15.5)\qquad \int_0^1 \frac{\ln^2(1+ay)}{1+cy}\,dy = \frac{1}{c}\left\{\ln^2\left(\frac{c-a}{c}\right)\ln(1+c)+\right. \\ +2\ln\left(\frac{c-a}{c}\right)\left[\mathrm{Li}_2\left(\frac{a}{a-c}\right)-\mathrm{Li}_2\left(a\frac{1+c}{a-c}\right)\right]+ \\ \left.+2S_{1,2}\left(a\frac{1+c}{a-c}\right)-2S_{1,2}\left(\frac{a}{a-c}\right)\right\},$$

$$(3.15.6)\qquad \int_0^1 \frac{1}{1+cy}\ln^2\left(\frac{1+ay}{1+by}\right)dy = \frac{1}{c}\left\{\ln^2\left(\frac{a}{b}\right)\ln(1+b)+\right. \\ +\ln^2\left(\frac{a-c}{b-c}\right)\ln\left(\frac{1+c}{1+b}\right)+2\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right)-\mathrm{Li}_2\left(\frac{a-b}{a}\right)\right]+ \\ +2\ln\left(\frac{a-c}{b-c}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a-c}\right)-\mathrm{Li}_2\left(\frac{(a-b)(1+c)}{(a-c)(1+b)}\right)\right]+$$

$$+2\left[S_{1,2}\left(\frac{a-b}{a}\right)-S_{1,2}\left(\frac{a-b}{a(1+b)}\right)-S_{1,2}\left(\frac{a-b}{a-c}\right)+\right.$$
$$\left.\left.+S_{1,2}\left(\frac{(a-b)(1+c)}{(a-c)(1+b)}\right)\right]\right\},$$

(3.15.7)
$$\int_0^1 \frac{\ln(1+ay)\ln(1+by)}{1+cy}\,dy=\frac{1}{c}\left\{\frac{1}{2}\ln^2\left(\frac{c-a}{c}\right)\ln(1+c)+\right.$$
$$+\frac{1}{2}\ln^2\left(\frac{c-b}{c}\right)\ln(1+c)-\frac{1}{2}\ln^2\left(\frac{a}{b}\right)\ln(1+b)+$$
$$+\frac{1}{2}\ln^2\left(\frac{a-c}{b-c}\right)\ln\left(\frac{1+b}{1+c}\right)+\ln\left(\frac{c-a}{c}\right)\left[\mathrm{Li}_2\left(\frac{a}{a-c}\right)-\mathrm{Li}_2\left(a\frac{1+c}{a-c}\right)\right]+$$
$$+\ln\left(\frac{c-b}{c}\right)\left[\mathrm{Li}_2\left(\frac{b}{b-c}\right)-\mathrm{Li}_2\left(b\frac{1+c}{b-c}\right)\right]+S_{1,2}\left(a\frac{1+c}{a-c}\right)-$$
$$-S_{1,2}\left(\frac{a}{a-c}\right)+S_{1,2}\left(b\frac{1+c}{b-c}\right)-S_{1,2}\left(\frac{b}{b-c}\right)-S_{1,2}\left(\frac{a-b}{a}\right)+$$
$$+S_{1,2}\left(\frac{a-b}{a(1+b)}\right)-S_{1,2}\left(\frac{(a-b)(1+c)}{(a-c)(1+b)}\right)+S_{1,2}\left(\frac{a-b}{a-c}\right)+$$
$$+\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a}\right)-\mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right)\right]+$$
$$\left.+\ln\left(\frac{a-c}{b-c}\right)\left[\mathrm{Li}_2\left(\frac{(a-b)(1+c)}{(a-c)(1+b)}\right)-\mathrm{Li}_2\left(\frac{a-b}{a-c}\right)\right]\right\},$$

(3.15.8)
$$\int_0^1 \frac{\ln(y)\ln(1+ay)}{1+by}\,dy=\frac{1}{b}\left\{-\frac{1}{2}\ln(1+b)\left[\ln^2(1+a)-\right.\right.$$
$$\left.-\ln^2\left(\frac{b-a}{b}\right)\right]+\ln(1+a)\left[\mathrm{Li}_2\left(\frac{a-b}{1+a}\right)-\mathrm{Li}_2\left(\frac{a}{1+a}\right)\right]-$$
$$-\ln\left(\frac{b-a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a(1+b)}{a-b}\right)-\mathrm{Li}_2\left(\frac{a}{a-b}\right)\right]-\mathrm{Li}_3(-b)+$$
$$\left.+\mathrm{Li}_3\left(\frac{a-b}{1+a}\right)-\mathrm{Li}_3\left(\frac{a}{1+a}\right)-S_{1,2}\left(\frac{a}{a-b}\right)+S_{1,2}\left(\frac{a(1+b)}{a-b}\right)\right\}.$$

3·16. *Dilogarithms and negative powers of* $(a+by)$.

(3.16.1)
$$\int_0^1 \frac{\mathrm{Li}_2(y)}{a+by}\,dy=\frac{a}{b}\left[\ln\left(\frac{a+b}{a}\right)+\mathrm{Li}_3\left(-\frac{a}{b}\right)+S_{1,2}\left(-\frac{a}{b}\right)\right],$$

(3.16.2)
$$\int_0^1 \frac{\mathrm{Li}_2(y)}{(a+by)^2}\,dy=\frac{\mathrm{zeta}(2)}{a(a+b)}-\frac{1}{ab}\mathrm{Li}_2\left(\frac{b}{a+b}\right),$$

$$(3.16.3)\qquad \int_0^1 \frac{\mathrm{Li}_2(y)}{(a+by)^3}\,dy = \frac{1}{2a}\left[\frac{1}{b(a+b)}\ln\left(\frac{a}{a+b}\right)+ + \frac{\mathrm{zeta}(2)}{a+b}\left(\frac{1}{a}+\frac{1}{a+b}\right)-\frac{1}{ab}\mathrm{Li}_2\left(\frac{b}{a+b}\right)\right],$$

$$(3.16.4)\qquad \int_0^1 \frac{\mathrm{Li}_2(y)}{(a+by)^4}\,dy = \frac{1}{3a}\left\{-\frac{1}{2a(a+b)^2}-\frac{3a+2b}{2ab(a+b)^2}\ln\left(\frac{a+b}{a}\right)+ + \frac{\mathrm{zeta}(2)}{a+b}\left[\frac{1}{a^2}+\frac{1}{a(a+b)}+\frac{1}{(a+b)^2}\right]-\frac{1}{a^2b}\mathrm{Li}_2\left(\frac{b}{a+b}\right)\right\},$$

$$(3.16.5)\qquad \int_0^1 \frac{\mathrm{Li}_2(-y)}{a+by}\,dy = \frac{1}{b}\left\{\frac{1}{8}\mathrm{zeta}(3)+\frac{1}{2}\mathrm{zeta}(2)\ln\left(\frac{a}{a+b}\right)- -\ln^2\left(\frac{a}{b}\right)\ln\left(\frac{a+b}{a}\right)+\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a}\right)-\mathrm{Li}_2\left(\frac{a-b}{a+b}\right)\right]+ + S_{1,2}\left(-\frac{b}{a}\right)-S_{1,2}\left(\frac{a-b}{a}\right)+S_{1,2}\left(\frac{a-b}{a+b}\right)-S_{1,2}\left(\frac{b-a}{a+b}\right)\right\},$$

$$(3.16.6)\qquad \int_0^1 \frac{\mathrm{Li}_2(-y)}{(a+by)^2}\,dy = -\frac{\mathrm{zeta}(2)}{2a(a+b)}+\frac{1}{ab}\left\{\ln\left(\frac{b-a}{a}\right)\ln\left(\frac{a+b}{a}\right)- -\mathrm{Li}_2\left(\frac{a+b}{a-b}\right)+\mathrm{Li}_2\left(\frac{a}{a-b}\right)\right\}.$$

3·17. *More complicated integrals containing dilogarithms and inverse monomials.*

$$(3.17.1)\qquad \int_0^1 \frac{\mathrm{Li}_2(a+by)}{a+by}\,dy = \frac{1}{b}\left[\mathrm{Li}_3(a+b)-\mathrm{Li}_3(a)\right],$$

$$(3.17.2)\qquad \int_0^1 \frac{\mathrm{Li}_2(-ay)}{1+by}\,dy = \frac{1}{b}\left\{\ln(1+b)\,\mathrm{Li}_2(-a)-\ln^2\left(\frac{a}{b}\right)\ln(1+b)+ + S_{1,2}(-a)+S_{1,2}(-b)+\ln\left(\frac{a}{b}\right)\left[Li_2\left(\frac{a-b}{a}\right)-\mathrm{Li}_2\left(\frac{a-b}{a+ab}\right)\right]- -S_{1,2}\left(\frac{a-b}{a}\right)+S_{1,2}\left(\frac{a-b}{a+ab}\right)-S_{1,2}\left(\frac{b-a}{1+b}\right)\right\},$$

$$(3.17.3)\qquad \int_0^1 \frac{\mathrm{Li}_2(a+by)-\mathrm{Li}_2(a)}{y}\,dy = \ln(1-a)\,\mathrm{Li}_2\left(-\frac{b}{a}\right)+ + \frac{1}{2}\ln\left(\frac{a+b}{b}\right)\left[\ln^2(1-a)-\ln^2\left(\frac{1-a-b}{1-a}\right)\right]+$$

$$+\ln\left(\frac{1-a-b}{1-a}\right)\left[\mathrm{Li}_2\left(\frac{-b}{a(1-a-b)}\right)-\mathrm{Li}_2\left(\frac{-b}{1-a-b}\right)\right]+$$

$$+\ln(1-a)\left[\mathrm{Li}_2\left(\frac{a(a+b)}{b}\right)-\mathrm{Li}_2(a)\right]-\mathrm{Li}_3\left(-\frac{b}{a}\right)+$$

$$+\mathrm{Li}_3\left(\frac{-b}{a(1-a-b)}\right)-\mathrm{Li}_3\left(\frac{-b}{1-a-b}\right)-S_{1,2}(a)+S_{1,2}\left(\frac{a(a+b)}{b}\right),$$

(3.17.4)
$$\int_0^1\frac{\mathrm{Li}_2(b-ay)}{1+cy}\,\mathrm{d}y=\frac{1}{c}\Bigg\{\ln(1+c)\,\mathrm{Li}_2(b-a)+$$

$$+\ln\left(\frac{a+bc}{c}\right)\ln\left(\frac{b-a}{b}\right)+\mathrm{Li}_2\left(c\,\frac{a-b}{a+bc}\right)-\mathrm{Li}_2\left(\frac{-bc}{a+bc}\right)-$$

$$-a\int_0^1\frac{1}{b-ay}\ln(1+cy)\ln\left(1+\frac{a}{1-b}\,y\right)\mathrm{d}y\Bigg\}\ (^*).$$

4. – Moment integrals.

4·1. *Notation.*

(4.1.1)
$$S_1(n)=\sum_{k=1}^{n}\frac{1}{k},$$

(4.1.2)
$$S_2(n)=\sum_{k=1}^{n}\frac{1}{k^2},$$

(4.1.3)
$$S_3(n)=\sum_{k=1}^{n}\frac{1}{k^3}.$$

4·2. *Moments of logarithms.*

(4.2.1)
$$\int_0^1 x^n\ln(x)\,\mathrm{d}x=-\frac{1}{(n+1)^2},$$

(4.2.2)
$$\int_0^1 x^n\ln^2(x)\,\mathrm{d}x=\frac{2}{(n+1)^3},$$

(4.2.3)
$$\int_0^1 x^n\ln^3(x)\,\mathrm{d}x=-\frac{6}{(n+1)^4},$$

(*) See integral (3.15.7).

$$\int_0^1 x^n \ln(1-x)\,dx = -\frac{S_1(n+1)}{n+1}, \tag{4.2.4}$$

$$\int_0^1 x^n \ln^2(1-x)\,dx = \frac{2}{n+1}\left\{S_2(n+1) + \sum_{j=1}^{n} \frac{S_1(j)}{j+1}\right\}, \tag{4.2.5}$$

$$\int_0^1 x^n \ln^3(1-x)\,dx = -\frac{6}{n+1}\left\{S_3(n+1) + \sum_{j=1}^{n} \frac{S_1(j)}{(j+1)^2} + \right.$$
$$\left. + \sum_{j=1}^{n} \frac{S_2(j)}{j+1} + \sum_{k=1}^{n} \frac{1}{k+1} \sum_{j=1}^{k-1} \frac{S_1(j)}{j+1}\right\}, \tag{4.2.6}$$

$$\int_0^1 x^n \ln(1+x)\,dx = \frac{(-1)^n}{n+1}\left[-S_1(n+1) + \frac{1+(-1)^n}{2} S_1\left(\frac{n}{2}\right) + \right.$$
$$\left. + \frac{1-(-1)^n}{2} S_1\left(\frac{n+1}{2}\right)\right] + [1+(-1)^n]\frac{\ln(2)}{n+1}, \tag{4.2.7}$$

$$\int_0^1 x^n \ln^2(1+x)\,dx = 2\frac{(-1)^n}{n+1}\sum_{j=1}^{n+1}\int_0^1 (-1)^j x^{j-1}\ln(1+x)\,dx +$$
$$+ [1+(-1)^n]\frac{\ln^2(2)}{n+1}, \tag{4.2.8}$$

$$\int_0^1 x^n \ln(x)\ln(1-x)\,dx = \frac{S_1(n+1)}{(n+1)^2} - \frac{1}{n+1}[\text{zeta}(2) - S_2(n+1)], \tag{4.2.9}$$

$$\int_0^1 x^n \ln(x)\ln(1+x)\,dx = (-1)^{n+1}\frac{\text{zeta}(2)}{2(n+1)} - \frac{1+(-1)^n}{(n+1)^2}\ln(2) +$$
$$+ \frac{(-1)^{n+1}}{n+1}\left[-S_2(n+1) + \frac{1+(-1)^n}{4} S_2\left(\frac{n}{2}\right) + \right.$$
$$\left. + \frac{1-(-1)^n}{4} S_2\left(\frac{n+1}{2}\right)\right] + \frac{(-1)^{n+1}}{(n+1)^2}\left[-S_1(n+1) + \right.$$
$$\left. + \frac{1+(-1)^n}{2} S_1\left(\frac{n}{2}\right) + \frac{1-(-1)^n}{2} S_1\left(\frac{n+1}{2}\right)\right], \tag{4.2.10}$$

$$\int_0^1 x^n \ln^2(x)\ln(1-x)\,dx = \frac{2}{n+1}\left[\text{zeta}(3) + \frac{\text{zeta}(2)}{n+1} - \right.$$
$$\left. - \frac{S_1(n+1)}{(n+1)^2} - \frac{S_2(n+1)}{n+1} - S_3(n+1)\right], \tag{4.2.11}$$

(4.2.12) $$\int_0^1 x^n \ln(x)\ln^2(1-x)\,\mathrm{d}x = \frac{2}{n+1}\left\{\operatorname{zeta}(3) - \frac{n+2}{n+1} + \right.$$
$$\left. + \operatorname{zeta}(2) S_1(n+1) - \sum_{j=1}^{n} \frac{1}{j+1}\left[\frac{S_1(j+1)}{j+1} + \frac{S_1(j+1)}{n+1} + S_2(j+1)\right]\right\}.$$

4·3. *Moments containing polylogarithms.*

(4.3.1) $$\int_0^1 x^n \operatorname{Li}_2(x)\,\mathrm{d}x = \frac{1}{n+1}\left[\operatorname{zeta}(2) - \frac{S_1(n+1)}{n+1}\right],$$

(4.3.2) $$\int_0^1 x^n \operatorname{Li}_2(-x)\,\mathrm{d}x = \frac{1}{n+1}\left\{-\frac{\operatorname{zeta}(2)}{2} + \frac{1+(-1)^n}{n+1}\log(2) + \right.$$
$$+ \frac{(-1)^n}{n+1}\left[-S_1(n+1) + \frac{1+(-1)^n}{2} S_1\left(\frac{n}{2}\right) + \right.$$
$$\left.\left. + \frac{1-(-1)^n}{2} S_1\left(\frac{n+1}{2}\right)\right]\right\},$$

(4.3.3) $$\int_0^1 x^n \log(x)\operatorname{Li}_2(x)\,\mathrm{d}x = \frac{1}{(n+1)^2}\left[-2\operatorname{zeta}(2) + \right.$$
$$\left. + 2\frac{S_1(n+1)}{n+1} + S_2(n+1)\right],$$

(4.3.4) $$\int_0^1 x^n \ln(1-x)\operatorname{Li}_2(x)\,\mathrm{d}x = \frac{1}{n+1}\left\{1 - 2\operatorname{zeta}(3) - \right.$$
$$- \operatorname{zeta}(2) S_1(n+1) + \frac{2}{n+1}\left[S_2(n+1) + \sum_{j=1}^{n}\frac{S_1(j)}{j+1}\right] +$$
$$\left. + \sum_{j=1}^{n}\frac{S_1(j+1)}{(j+1)^2}\right\},$$

(4.3.5) $$\int_0^1 x^n \operatorname{Li}_3(x)\,\mathrm{d}x = \frac{1}{n+1}\left[\operatorname{zeta}(3) - \frac{\operatorname{zeta}(2)}{n+1} + \frac{S_1(n+1)}{(n+1)^2}\right],$$

(4.3.6) $$\int_0^1 x^n S_{1,2}(x)\,\mathrm{d}x = \frac{\operatorname{zeta}(3)}{n+1} - \frac{1}{(n+1)^2}\left[S_2(n+1) + \sum_{j=1}^{n}\frac{S_1(j)}{j+1}\right].$$

5. – Indefinite integrals.

(5.1) $$\int \ln(a+by)\,\mathrm{d}y = \frac{1}{b}(a+by)[\ln(a+by) - 1],$$

(5.2) $$\int y \ln(a+by)\,dy = \frac{1}{2}\left(y^2 - \frac{a^2}{b^2}\right)\ln(a+by) - \frac{y^2}{4} + \frac{ay}{2b},$$

(5.3) $$\int y^2 \ln(a+by)\,dy = \frac{1}{3}\left(y^3 - \frac{a^3}{b^3}\right)\ln(a+by) - \frac{y^3}{9} + \frac{a}{6b}y^2 - \frac{a^2 y}{3b^2},$$

(5.4) $$\int \frac{\ln(y)}{1+ay}\,dy = \frac{1}{a}\left[\log(y)\log(1+ay) + \mathrm{Li}_2(-ay)\right],$$

(5.5) $$\int \frac{\ln(y)}{(1+ay)^2}\,dy = \frac{1}{a}\left[\ln\left(\frac{y}{1+ay}\right) - \frac{\ln(y)}{1+ay}\right],$$

(5.6) $$\int \frac{\ln(y)}{(1+ay)^3}\,dy = \frac{1}{2a}\left[\ln\left(\frac{y}{1+ay}\right) - \frac{\ln(y)}{(1+ay)^2} + \frac{1}{1+ay}\right],$$

(5.7) $$\int \frac{\ln(y)}{(1+ay)^4}\,dy = \frac{1}{3a}\left[\ln\left(\frac{y}{1+ay}\right) - \frac{\ln(y)}{(1+ay)^3} + \frac{3+5ay+2a^2y^2}{2(1+ay)^3}\right],$$

(5.8) $$\int \frac{\ln^2(y)}{1+ay}\,dy = \frac{1}{a}\left[-\ln^2(y)\ln(1+ay) + 2\mathrm{Li}_3(-ay) - 2\ln(y)\,\mathrm{Li}_2(-ay)\right],$$

(5.9) $$\int \frac{\ln^2(y)}{(1+ay)^2}\,dy = \frac{y\ln^2(y)}{1+ay} - \frac{2}{a}\left[\mathrm{Li}_2(-ay) + \ln(y)\ln(1+ay)\right],$$

(5.10) $$\int \frac{\ln^2(y)}{(1+ay)^3}\,dy = \frac{1}{a}\left[\ln\left(\frac{1+ay}{y}\right) - \ln(y)\ln(1+ay) - \mathrm{Li}_2(-ay) + \frac{\ln(y)}{1+ay} + \frac{(1+ay)^2-1}{2(1+ay)^2}\ln^2(y)\right],$$

(5.11) $$\int \frac{\ln(a+by)}{c+ey}\,dy = \frac{1}{e}\left[\ln\left(\frac{ae-bc}{e}\right)\ln(c+ey) - \mathrm{Li}_2\left(-b\,\frac{c+ey}{ae-bc}\right)\right],$$

(5.12) $$\int \frac{\ln(a+by)}{(c+ey)^2}\,dy = \frac{1}{e}\left[-\frac{\ln(a+by)}{c+ey} + \frac{b}{ae-bc}\ln\left(\frac{c+ey}{a+by}\right)\right],$$

(5.13) $$\int \frac{\ln(a+by)}{(c+ey)^3}\,dy = -\frac{1}{2e}\left[\frac{\ln(a+by)}{(c+ey)^2} + \frac{b^2}{(ae-bc)^2}\ln\left(\frac{c+ey}{a+by}\right) + \frac{b}{(ae-bc)(c+ey)}\right].$$

Appendix A

Evaluation of a few selected integrals.

Consider the following integral:

$$I(x) = \int_a^b f(x, y)\,\mathrm{d}y\,, \tag{A.1}$$

where the limits of integration a, b possibly depend on x. The derivative of I with respect to x is given by Leibniz's formula

$$\frac{\mathrm{d}}{\mathrm{d}x} I(x) = f(x, b)\frac{\mathrm{d}b}{\mathrm{d}x} - f(x, a)\frac{\mathrm{d}a}{\mathrm{d}x} + \int_a^b \left[\frac{\partial}{\partial x} f(x, y)\right] \mathrm{d}y\,. \tag{A.2}$$

This formula can be helpful in evaluating double integrals. For example consider

$$\int_1^1 \mathrm{d}x \int_0^1 \frac{\ln^2(1 - xy)}{x}\,\mathrm{d}y = \int_0^1 \frac{1}{x^2}\,\mathrm{d}x \int_0^x \ln^2(1 - y)\,\mathrm{d}y\,. \tag{A.3}$$

Integration by parts with respect to x and use of Leibniz's formula yields

$$-\int_0^1 \ln^2(1 - y)\,\mathrm{d}y + \int_0^1 \frac{\ln^2(1 - x)}{x}\,\mathrm{d}x = -2 + 2\,\mathrm{zeta}(3)$$

(see integrals (3.2.2) and (3.6.9)).

Some integrals may be evaluated by first differentiating with respect to one of the parameters occurring in the integrand. This may considerably simplify the definite integral, which may become of a known form. The final answer is then obtained by finding the appropriate indefinite integral with respect to the parameter used in the differentiation. Using this technique, however, additive constants may be lost or introduced: to overcome this obstacle it is necessary to evaluate the starting integral for some specific value of the parameter and to correct accordingly the final result. To make this technique more transparent consider the following integral (see integral (3.15.1)):

$$H = \int_0^1 \frac{\ln(1 - x)\ln(1 - ax)}{x}\,\mathrm{d}x\,. \tag{A.4}$$

Differentiating and then integrating with respect to a, we find

$$(A.4a)\qquad H = -\int \mathrm{d}a \int_0^1 \frac{\ln(1-x)}{1-ax}\,\mathrm{d}x + \text{const} = -\int \frac{1}{a}\,\mathrm{Li}_2\left(\frac{-a}{1-a}\right)\mathrm{d}a + \text{const} \qquad (\text{see } (3.12.7))\,.$$

Using the identity (see eq. (2.2.5))

$$\mathrm{Li}_2\left(\frac{-a}{1-a}\right) = -\,\mathrm{Li}_2(a) - \frac{1}{2}\ln^2(1-a)\,,$$

we obtain

$$(A.4b)\qquad H = \mathrm{Li}_3(a) + S_{1,2}(a) + \text{const}\,.$$

Evaluation at $a = 1$ shows that the additive constant of integration is identically equal to zero in this case.

$$(A.4c)\qquad H = \mathrm{Li}_3(a) + S_{1,2}(a)\,.$$

This technique can be quite useful in evaluating integrals with the help of MACSYMA: in particular, the integral just discussed, which in the starting form cannot be evaluated by MACSYMA, becomes computable following the above description.

Integrals containing polylogarithms can often be evaluated using an appropriate integral representation of the polylogarithm and subsequently either integrating by parts (Leibniz's formula may be of help here), or interchanging the order of integration (see BARBIERI [7]). For example, consider

$$(A.5)\qquad Y = \int_0^a \frac{\mathrm{Li}_2(y)}{1-y}\,\mathrm{d}y = -\int_0^a \frac{1}{1-y}\,\mathrm{d}y \int_0^y \frac{\ln(1-x)}{x}\,\mathrm{d}x\,,$$

where we used

$$\mathrm{Li}_2(y) = -\int_0^y \frac{\ln(1-x)}{x}\,\mathrm{d}x\,.$$

Integration by parts using Leibniz's formula gives

$$(A.5a)\qquad Y = \ln(1-a)\int_0^a \frac{\ln(1-x)}{x}\,\mathrm{d}x - \int_0^a \frac{\ln^2(1-y)}{y}\,\mathrm{d}y = -\ln(1-a)\,\mathrm{Li}_2(a) - 2S_{1,2}(a)\,.$$

A number of additional illustrations of useful tricks are contained in appendix A of BARBIERI [7].

Finally, let us consider in detail the evaluation of the following integral (see LEWIN [5]):

$$\text{(A.6}a\text{)} \qquad J = \int_0^1 \frac{1}{x} \ln(1+ax)\ln(1+bx)\,\mathrm{d}x \qquad \text{(see integral (3.15.6))}.$$

To evaluate J, we use (see eq. (2.1.5))

$$\text{(A.7)} \qquad \int_0^1 \frac{1}{x} \ln^2(1+ax)\,\mathrm{d}x = 2S_{12}(-a)\,.$$

Then

$$\text{(A.6}a\text{)} \qquad \int_0^1 \frac{1}{x} \ln^2\left(\frac{1+ax}{1+bx}\right)\mathrm{d}x = 2S_{1,2}(-a) - 2S_{1,2}(-b) - 2\int_0^1 \frac{1}{x}\ln(1+ax)\ln(1+bx)\,\mathrm{d}x\,.$$

On the left-hand side of the above equation we now make the following change of variable:

$$\text{(A.8)} \qquad \frac{1+ax}{1+bx} = y\,.$$

Hence

$$\text{(A.6}b\text{)} \qquad \int_0^1 \frac{1}{x} \ln^2\left(\frac{1+ax}{1+bx}\right)\mathrm{d}x = \int_1^u \ln^2(y)\left[\frac{1}{y-1} + \frac{b}{a-by}\right]\mathrm{d}y\,,$$

where $u = (1+a)/(1+b)$.

Consider now the first term on the right-hand side: the integration is immediate once we perform the change of variable $y = 1 - z$, and with the help of the tables we find

$$\text{(A.6}c\text{)} \qquad \int_1^u \frac{1}{y-1} \ln^2(y)\,\mathrm{d}y = 2S_{1,2}\left(\frac{b-a}{1+b}\right).$$

To evaluate the second term on the right-hand side we let

$$\text{(A.9)} \qquad y = \frac{a}{b}(1-z)\,.$$

The limits of integration are now

$$A = \frac{a-b}{a(1+b)} \qquad \text{and} \qquad B = \frac{a-b}{a}.$$

The integration is straightforward and we find

$$\text{(A.6}d\text{)} \qquad \int_1^u \frac{b}{a-by} \ln^2(y)\,dy = \int_A^B \frac{1}{z}\left[\ln^2\left(\frac{a}{b}\right) + 2\ln\left(\frac{a}{b}\right)\ln(1-z) + \right.$$

$$\left. + \ln^2(1-z)\right] dz = \ln^2\left(\frac{a}{b}\right)\ln(1+b) + 2\left[S_{1,2}\left(\frac{a-b}{a}\right) - \right.$$

$$\left. - S_{1,2}\left(\frac{a-b}{a(1+b)}\right)\right] + 2\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right) - \mathrm{Li}_2\left(\frac{a-b}{a}\right)\right].$$

Thus, collecting all the terms, we find

$$\text{(A.6}e\text{)} \qquad \int_0^1 \frac{1}{x}\ln^2\left(\frac{1+ax}{1+bx}\right) dx = \ln^2\left(\frac{a}{b}\right)\ln(1+b) +$$

$$+ 2\ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right) - \mathrm{Li}_2\left(\frac{a-b}{a}\right)\right] +$$

$$+ 2\left[S_{1,2}\left(\frac{a-b}{a}\right) - S_{1,2}\left(\frac{a-b}{a(1+b)}\right) + S_{1,2}\left(\frac{b-a}{1+b}\right)\right].$$

Thus, finally, we have

$$\text{(A.6}f\text{)} \qquad \int_0^1 \frac{1}{x}\ln(1+ax)\ln(1+bx)\,dx = S_{1,2}(-a) + S_{1,2}(-b) -$$

$$- \frac{1}{2}\ln^2\left(\frac{a}{b}\right)\ln(1+b) + \ln\left(\frac{a}{b}\right)\left[\mathrm{Li}_2\left(\frac{a-b}{a}\right) - \mathrm{Li}_2\left(\frac{a-b}{a(1+b)}\right)\right] -$$

$$- S_{1,2}\left(\frac{a-b}{a}\right) + S_{1,2}\left(\frac{a-b}{a(1+b)}\right) - S_{1,2}\left(\frac{b-a}{1+b}\right).$$

The method shown above is useful for the evaluation of integrals of the form

$$\text{(A.10)} \qquad \frac{1}{e+fx}\ln(a+bx)\ln(c+dx).$$

Appendix B

Power series expansion of the more common polylogarithms.

We want to expand $\mathrm{Li}_2(x)$, $\mathrm{Li}_3(x)$ and $S_{1,2}(x)$ in power series in order to be able to write efficient and fast computer codes capable of evaluating them with precision as high as desired. Consider

(B.1) $$\mathrm{Li}_2(x) := -\int_0^x \frac{\ln(1-y)}{y}\,\mathrm{d}y\,.$$

Let $y = 1 - \exp[-t]$

(B.2) $$\mathrm{Li}_2(x) = \int_0^u \frac{t}{\exp[t]-1}\,\mathrm{d}t\,,$$

where $u = -\ln(1-x)$. The integrand in the above expression is the generating function of the Bernoulli numbers B_n, thus

(B.3) $$\mathrm{Li}_2(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!}\int_0^u t^n\,\mathrm{d}t = \sum_{n=0}^{\infty} B_n \frac{u^{n+1}}{(n+1)!}\,.$$

In a similar fashion starting from

(B.4) $$S_{1,2}(x) = \frac{1}{2}\int_0^x \frac{\ln^2(1-y)}{y}\,\mathrm{d}y$$

and performing the same change of integration variable as above, after a simple integration, we find

(B.5) $$S_{1,2}(x) = -\sum_{n=0}^{\infty} (n+1)B_n \frac{u^{n+2}}{(n+2)!} \qquad [u = -\ln(1-x)]\,.$$

Finally, in the case of Li_3 we have

(B.6) $$\mathrm{Li}_3(x) = \int_0^x \frac{\mathrm{Li}_2(t)}{t}\,\mathrm{d}t\,,$$

thus, using the expansion for $\mathrm{Li}_2(x)$, we obtain

$$\mathrm{Li}_3(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} B_n \int_0^u \frac{1}{t} [-\ln(1-t)]^{n+1} \mathrm{d}t .$$

We now let $t = 1 - \exp[-z]$ and perform the z-integration:

$$\text{(B.7)} \qquad \mathrm{Li}_3(x) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} B_n \sum_{m=0}^{\infty} \frac{1}{(m+n+1)!} B_m [(m+1)_n] u^{m+n+1} ,$$

where $(m+1)_n = (m+1)(m+2) \dots (m+n)$ is the Pochhammer symbol. Thus, finally

$$\text{(B.8)} \qquad \mathrm{Li}_3(x) = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{1}{q!} B_q B_{p-q} [(p-q+1)_q] \frac{u^{p+1}}{(p+1)!} .$$

These series expansions are most useful when $|x| < \frac{1}{2}$. For evaluation with $|x| > \frac{1}{2}$, and in particular $x > 1$, which leads to imaginary parts, the identities of subsect. **2**.2 are used.

APPENDIX C

Identities and power series expansion of sums occurring in moment integrals.

The partial sums occurring in moment integrals (for the definition see subsect. **4**.1), satisfy a number of identities. Some of the simpler and more common identities are shown below. A more complete table can be found in Gonzales-Arroyo [10]

$$\text{(C.1)} \qquad \sum_{j=1}^{n} S_1(j) = (n+1) S_1(n) - n ,$$

$$\text{(C.2)} \qquad \sum_{j=1}^{n} \frac{1}{j} S_1(j) = \frac{1}{2} [S_1(n)]^2 + \frac{1}{2} S_2(n) ,$$

$$\text{(C.3)} \qquad \sum_{j=1}^{n} S_2(j) = (n+1) S_2(n) - S_1(n) ,$$

$$\text{(C.4)} \qquad \sum_{j=1}^{n} \frac{1}{j} S_2(j) = S_1(n) S_2(n) + S_3(n) - \sum_{j=1}^{n} \frac{1}{j^2} S_1(j) ,$$

$$\text{(C.5)} \qquad \sum_{j=1}^{n} \frac{1}{j} S_1^2(j) = \frac{1}{3} S_1^3(n) - \frac{1}{3} S_3(n) + \sum_{j=1}^{n} \frac{1}{j^2} S_1(j) ,$$

(C.6) $$S_1(n) = \ln(n) + \mathbf{C} + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} + \dots,$$

where $\mathbf{C}$ is Euler's constant,

(C.7) $$S_2(n) = \text{zeta}(2) - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3} + \dots,$$

(C.8) $$S_3(n) = \text{zeta}(3) - \frac{1}{2n^2} + \frac{1}{2n^3} - \frac{1}{4n^4} + \dots.$$

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