Dynamics of Spontaneous Chiral Symmetry Breaking and the Continuum Limit in Quantum Electrodynamics (*).

V. A. MIRANSKY

Institute for Theoretical Physics - 252130 *Kiev*-130, *USSR*

(ricevuto il 14 Giugno 1985)

Summary. — The phase diagram in the coupling constant in QED and its connection with the spontaneous chiral symmetry breaking are discussed. The mechanism of such a breaking connected with the collapse phenomenon is considered and a simple physical interpretation of the recent results of the computer simulations in lattice QED is given. The problem of the existence of the nontrivial continuum QED is analysed and, as a result, the following hypothesis is considered: in the Landau-Pomeranchuk-Fradkin « zero-charge » situation (the renormalization constant $Z_3 = 0$) the S-matrix of continuum QED with a fixed bare coupling constant, $\alpha^{(0)} = \alpha_c \sim 1$, is nontrivial. The physical content of such a hypothetical continuum theory is revealed.

PACS. 11.30. – Symmetry and conservation laws.

1. – Introduction.

Recently, the results concerning spontaneous chiral symmetry breaking in quantum electrodynamics have been obtained by the computer simulation methods in lattice noncompact QED ([1]). The aim of the present paper is to represent a simple physical interpretation of these results and to discuss, from

(*) To speed up publication, the author of this paper has agreed to not receive the proofs for correction.

([1]) J. BARTHOLOMEW, J. KOGUT, S. H. SHENKER, J. SLOAN, M. STONE, H. W. WYLD, J. SHIGMETSU and D. K. SINCLAIR: *Nucl. Phys. B*, **230**, 222 (1984).

the viewpoint of this interpretation, the old problem of the existence of the nontrivial continuum limit (ultraviolet cut-off $\Lambda \to \infty$) in QED.

In 1954 GELL-MANN and LOW (²) indicated thet the nontrivial continuum QED can exist only if the bare coupling constant $\alpha^{(0)}$ is determined by an ultraviolet stable zero of the renormalization group β-function,

$$\beta_{\text{QED}} = \alpha^{(0)} \frac{\partial}{\partial \ln \mu} Z_{3\mu} \simeq -\alpha^{(0)} \frac{\partial}{\partial \ln \Lambda} Z_{3\mu}\,, \tag{1}$$

where μ is the renormalization group parameter and $Z_{3\mu}$ is the renormalization constant of the photon propagator. The question of the existence of such a zero became especially important since the papers of Landau and Pomeranchuck (³) and Fradkin (⁴), appeared, where it was argued that in the continuum limit QED is transformed into a free-field theory; more precisely, in the limit $\Lambda \to \infty$ for any value of the bare coupling constant $\alpha^{(0)}$ the vacuum polarization effects lead to a zero value for the running coupling constant $\bar{\alpha}(r)$ at all nonzero distances: $\bar{\alpha}(r) = 0$ at $r > 0$ and $\bar{\alpha}(0) = \alpha^{(0)}$ (the « zero-charge » situation, $Z_{3\mu} = 0$). Later the possibility of the existence of a nontrivial zero of the β-function was studied in the framework of the « finite QED » program (⁵). However, no definite answer has been obtained there. Moreover, recently new arguments have been given in favour of the impossibility of the existence of a nontrivial zero of the β-function (1) (⁶).

The problem is closely connected with the charge renormalization. In the papers (²,⁵) the relation

$$\alpha_\mu = Z_{3\mu} \alpha^{(0)}(\Lambda) \tag{2}$$

(which is equivalent to eq. (1)) was used. This relation is proved to be valid in every order of perturbation theory, hence its validity in the exact theory also seems to be predetermined. However, as has been pointed out in ref. (⁷), at a sufficiently large bare coupling constant the renormalization relations of nonasymptotically free theories can be essentially changed owing to the dynamics of particle mass generation. This conception has been introduced by exploiting the mechanism of the spontaneous chiral symmetry breaking

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(²) M. GELL-MANN and F. LOW: *Phys. Rev.*, **95**, 1300 (1954).
(³) L. D. LANDAU and I. YA. POMERANCHUK: *Dokl. Akad. Nauk SSSR*, **102**, 489 (1955); L. D. LANDAU: in *Niels Bohr and the Development of Physics*, edited by W. PAULI (Pergamon Press, London, 1955).
(⁴) E. S. FRADKIN: *Ž. Èksp. Teor. Fiz.*, **28**, 750 (1955).
(⁵) K. JOHNSON, M. BAKER and R. WILLEY: *Phys. Rev. B*, **136**, 1111 (1964); S. L. ADLER: *Phys. Rev. D*, **5**, 3021 (1972).
(⁶) N. V. KRASNIKOV: *Phys. Lett. B*, **126**, 483 (1983); Y. MATSUBARA, T. SUZUKI and I. YOTSUYANAGI: preprint DPKU-8304, Kanazawa (1983).
(⁷) V. A. MIRANSKY: *Phys. Lett. B*, **91**, 421 (1980); preprint ITP-81-22E, Kiev (1981).

in massless QED suggested earlier in the papers ([8,9]) (for a review see ref. ([10]); note also that recently this mechanism has been discussed in ref. ([11])). The mechanism leads to an additional mass divergence at the supercritical ($\alpha^{(0)}(\Lambda) > > \alpha_c \sim 1$) values of the bare coupling constant (for details see sect. **2**). From the viewpoint of the renormalization group the critical value α_c, separating the massless and the massive phases, is an ultraviolet stable fixed point. However, it is determined not by a zero of the β-function (1) related to the subcritical ($\alpha^{(0)} < \alpha_c$) phase, but by a zero of the β-function of the supercritical phase. The value $\alpha^{(0)} = \alpha_c$ determines a continuum theory.

Note that very recently a similar phenomenon has been also found in $(2 + n^{-1})$-dimensional, $n \geqslant 1$, φ^{4n+2} models ([12]) (at the classical level these models are scale invariant).

Below we will show that the principal results of the computer simulations in lattice QED can be easily understood from the point of view of this dynamical mechanism. Besides, we discuss to what extent the improvement of the quenched approximation, used in ref. ([1]), can influence the results. As a consequence of this discussion, the following, apparently unexpected, possibility is revealed: in the « zero-charge » situation ($Z_{3\mu} = 0$) the S-matrix of continuum QED with a fixed bare coupling constant $\alpha^{(0)} = \alpha_c$ can be nontrivial. The characteristic feature of such a continuum theory is the appearance of the new, induced, Yukawa-type interaction of fermions, antifermions and *composed* pseudoscalar bosons.

The paper is organized in the following way. In sect. **2** we describe the essential feature of the mechanism of the spontaneous chiral symmetry breaking in QED. In sect. **3** we discuss the pecularities of the dynamics of the chiral symmetry breaking in QED connected with subtleties of the transition to the continuum limit. This point is important since anomalies of different kinds can, in principle, essentially influence the dynamics of symmetry breaking. In sect. **4** we interpret the results of the computer simulations in lattice QED from the viewpoint of this particle mass generation mechanism and discuss the possibility of the existence of the nontrivial continuum limit in QED.

2. – The mechanism of spontaneous chiral symmetry breaking in QED.

The mechanism of spontaneous chiral symmetry breaking in QED considered in ref. ([7-10]) is based on the analogy between this phenomenon and

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([8]) P. I. Fomin and V. A. Miransky: *Phys. Lett. B*, **64**, 166 (1976).
([9]) P. I. Fomin, V. P. Gusynin and V. A. Miransky: *Phys. Lett. B*, **78**, 136 (1978).
([10]) P. I. Fomin, V. P. Gusynin, V. A. Miransky and Yu. A. Sitenko: *Riv. Nuovo Cimento*, **6**, No. 5 (1983).
([11]) G. J. Ni: *Nucl. Phys. B*, **211**, 414 (1983).
([12]) W. A. Bardeen, M. Moshe and M. Bander: *Phys. Rev. Lett.*, **52**, 1188 (1984); M. B. Halpern: *Phys. Lett. B*, **137**, 382 (1984).

that of the electron-positron pair creation in the supercritical Coulomb field. As is known [13], when $Z > Z_c \simeq 137$ $(\alpha = Ze^2/4\pi > 1)$, the Dirac operator with the Coulomb potential $V(r) = -Ze^2/4\pi r$ is not defined, and therefore it is necessary to complete its definition by introducing a cut-off at small distance [*].

For example,

$$V(r) = -\frac{Ze^2}{4\pi r} \to V(r) = -\frac{Ze^2}{4\pi}\left[\frac{\theta(r-r_0)}{r} + \frac{\theta(r_0-r)}{r_0}\right]. \tag{3}$$

To exhibit the role of the cut-off parameter $\Lambda = r_0^{-1}$ in this problem, we quote the expression for the energy $\varepsilon^{(n)}$ of the $nS_{\frac{1}{2}}$-levels in the case of the light (when $m \ll |\varepsilon^{(n)}|$) electron [8]:

$$\varepsilon^{(n)} \simeq \varepsilon_0^{(n)} - \frac{m}{2} - i\,\frac{m^2}{50|\varepsilon_0^{(n)}|}, \tag{4}$$

where the energy $\varepsilon_0^{(n)}$ of the massless electron is

$$\begin{cases} \varepsilon_0^{(n)} = a\Lambda(\sin\varphi - i\cos\varphi)\exp\left[\dfrac{-\pi n}{\sqrt{\alpha^2-1}}\right], & n = 1, 2, \ldots, \\ a \simeq 0.4\,, \quad \varphi = -\dfrac{\pi}{2}\operatorname{ctgh}\pi \simeq -\dfrac{\pi}{2}\cdot 1.004\,. \end{cases} \tag{5}$$

According to the conventional interpretation of the levels with $\operatorname{Re}\varepsilon < 0$, the level $\varepsilon^{(n)}$ determines the positron state energy

$$\varepsilon_p^{(n)} = -\varepsilon^{(n)}\,, \qquad \operatorname{Re}\varepsilon_p^{(n)} > 0\,, \quad \operatorname{Im}\varepsilon_p^{(n)} < 0$$

(*) In other words, in the relativistic theory the «fall into the centre» (collapse) phenomenon [13,14] takes place for this potential and such a system has no ground state. The formal (mathematical) reason for this phenomenon is connected with the fact that such a Hamiltonian is a Hermitian but not a self-adjoint operator, and it should be extended (defined completely) to become a self-adjoint one [15]. The physical reason is connected with the fact that the properties of the system depend on the way used to define completely the Hamiltonian at small distances.

(13) YA. B. ZEL'DOVICH and V. S. POPOV: *Usp. Fiz. Nauk*, **105**, 403 (1971); J. RAFELSKI, L. FULCHER and A. KLEIN: *Phys. Rep.*, **38**, 229 (1978).

(14) L. D. LANDAU and E. M. LIFSHITS: *Quantum Mechanics* (Nauka, Moscow, 1974), Chapt. 35.

(15) M. REED and B. SIMON: *Methods of Modern Mathematical Physics* (Academic Press, New York, N. Y., 1975), Vol. 2.

corresponding to the outgoing positron wave (*). In the limit $\Lambda \to \infty$ the energy $\varepsilon^{(n)}$ diverges, which reflects the collapse situation.

In ref. (8) it was hypothesized that a similar phenomenon takes place in QED with a sufficiently large bare coupling constant which results in the spontaneous chiral symmetry breaking (**). This hypothesis was realized (in the ladder approximation) in ref. (9) (for more details see the review (10)). For the sake of completeness the equations for the fermion dynamical mass function $m_d(q^2) = B_d(q^2)/A(q^2)$ (the fermion propagator $S(q) = (-\hat{q}A(q^2) + B_d(q^2))^{-1}$) and for the wave functions of Goldstone bosons are considered in appendix A (***). Here we state the main results (the chiral group is $SU_{L,K} \times SU_{R,K}$, K is the number of the fermion flavours).

For the supercritical values $\alpha^{(0)} > \alpha_c = \pi/3$ the dynamical mass m_d is determined by a relation of the form

$$m_d = \Lambda f(\alpha^{(0)}) , \tag{6}$$

where for the near-critical values $\alpha^{(0)} (\alpha^{(0)} - \alpha_c \ll \alpha_c)$ the function $f(\alpha^{(0)})$ is

$$f(\alpha^{(0)}) \simeq 4 \exp[-\pi/2\gamma] , \qquad \gamma = \frac{1}{2}\sqrt{\frac{\alpha^{(0)}}{\alpha_c} - 1} . \tag{7}$$

The Bethe-Salpeter wave function of the Goldstone bosons in the Euclidean region reads

$$\chi^r = C\lambda^r \gamma_5 \chi(q^2) , \qquad \chi(q^2) = (q^2 + m_d^2)^{-1} F(\tfrac{1}{2} + i\gamma, \tfrac{1}{2} - i\gamma, 2; -q^2/m_d^2) , \tag{8}$$

(*) The appearance of such quasi-stationary levels is interpreted as instability with respect to the spontaneous creation of electron-positron pairs from the vacuum (13). The created electron is coupled to the centre thus shielding the charge of the latter while the positron goes to infinity; the process is repeated until the charge of the centre is reduced to a subcritical value.

(**) The role of the fermion mass in the problem of the supercritical Coulomb field can be seen from eqs. (4) and (5). The imaginary part of the energy $\operatorname{Im} \varepsilon^{(n)}$ decreases (*i.e.* stability of the system increases) with increasing mass. Thus there are in principle two possibilities for the system with the supercritical charge to become stable: to shield spontaneously the charge or to generate spontaneously the fermion mass. In the problem of the Coulomb centre the first possibility can be only realized (which is already due to the formulation of the problem as a one-particle one). It has been suggested in ref. (8) that the second possibility—dynamical generation of the fermion mass—is realized in QED.

(***) The equations are considered in the Landau gauge. This choice is not accidental and is dictated by the following reasons. The correct statement of the problem of spontaneous symmetry breaking in a given approximation is only possible when this approximation is consistent with the Ward identities corresponding to the symmetry investigated. As is shown in appendix A, this selects the Landau gauge in the ladder approximation as the most preferable one.

where λ^r are K^2-1 matrices of the fundamental representation of the SU_K algebra, q is the relative fermion-antifermion momentum, F is a hypergeometric function and the normalization constant C can in principle be determined from the normalization condition for the wave function χ^2.

Let us consider, following ref. (7), the renormalizations in this problem. In the continuum limit $\Lambda \to \infty$, $\alpha^{(0)} = \text{const} > \alpha_c$, the mass m_d diverges. This is connected with the following fact. Using the asymptotic expansion for hypergeometric functions (16) we find that in the limit $q^2 \to \infty$ the function $\chi(q^2)$ takes the form

$$(9)\qquad \chi(q^2) \underset{q^2\to\infty}{\simeq} \frac{1}{q^2}\left(\frac{q^2}{m_d^2}\right)^{-\frac{1}{2}} \left(\frac{\operatorname{ctgh}\pi\gamma}{\pi\gamma(\gamma^2+\frac{1}{4})}\right)^{\frac{1}{2}} \sin\left(\gamma\ln\frac{q^2}{m_d^2} - \operatorname{arctg}2\gamma + \Sigma(\gamma)\right),$$

where

$$\Sigma(\gamma) = \arg\left[\frac{\Gamma(1+2i\gamma)}{\Gamma^2(\frac{1}{2}+i\gamma)}\right],$$

Γ is the Euler gamma-function. Therefore, in the continuum limit ($\Lambda = \infty$) and for any finite value of m_d the wave function $\chi(q^2)$ has an infinite number of zeroes. This is a typical manifestation of the collapse (« fall into the centre ») phenomenon (14) when the energy of the ground state is not bounded from below and therefore the energy (mass) gap is infinite.

To remove this divergence, a renormalization must be performed. Performing a renormalization means making the bare parameters (in our case $\alpha^{(0)}$, there is no mass term in the chiral invariant Lagrangian) depend on Λ in such a way that the physical parameters (the mass m_d in our case) remain finite in the limit $\Lambda \to \infty$. As it follows from eqs. (6) and (7), in the limit $\Lambda \to 0$ the mass m_d remains finite if the coupling constant is fixed:

$$(10)\qquad \alpha^{(0)}(\Lambda) = \alpha_c + \frac{\pi^2\alpha_c}{\ln^2(4\Lambda/m_d)} \xrightarrow[\Lambda\to\infty]{} \alpha_c = \frac{\pi}{3}.$$

From the viewpoint of the renormalization group the critical value $\alpha_c = \pi/3$, separating the massless and the massive phases, is an ultraviolet stable fixed point. The appearance of such a point in the ladder approximation is caused by the dynamics which cannot be obtained in perturbation theory.

We also emphasize that the origin of the mass divergence (6) is different from that of the loop divergences of perturbation theory. The latter are due to processes in which the particle number is not conserved, while divergence (6) is connected with a singular character (at small distances) of the exchange

(16) H. Bateman and A. Erdélyi: *Higher Transcendental Functions*, Vol. 1 (McGraw-Hill, New York, N. Y., 1953).

interaction which conserves the particle number. Since such divergences occur already in quantum mechanics (see, for example, eq. (5)), it is natural to name such divergences as quantum-mechanical ones.

We also indicate the two following characteristic facts:

1) Since in the ladder approximation the renormalization constant of the photon propagator $Z_{3\mu}$ is equal to unity, the renormalization (10) destroys the renormalization relation (2).

2) In the continuum limit (10) the wave function

$$\chi(q^2) = (q^2 + m_d^2)^{-1} F\left(\frac{1}{2}, \frac{1}{2}, 2; -\frac{q^2}{m_d^2}\right) \underset{q^2\to\infty}{\simeq} \frac{2}{\pi q^2}\left(\frac{q^2}{m_d^2}\right)^{-\frac{1}{2}} \ln\frac{q^2}{m_d^2}, \tag{11}$$

and therefore the renormalization (10) changes the form of the wave function, in particular, the oscillations disappear (compare with eq. (9)). We remind that for the standard renormalizations of perturbation theory the following relations between renormalized ($G^{(\mu)}$) and nonrenormalized ($G^{(\Lambda)}$) Green's functions take place:

$$G^{(\mu)}(\{q\}, d_\mu) = Z\left(\frac{\Lambda}{\mu}, \alpha_\mu\right) G^{(\Lambda)}(\{q\}, \alpha^{(0)}(\Lambda)) + \text{small power corrections}\,.$$

Therefore, in this case, up to small power corrections μ/Λ, q/Λ, etc., the form of renormalized and nonrenormalized Green's functions as the functions of the momenta $\{q\}$ is the same. The violation of this property by the renormalization (10) is easily understood: removing the cut-off and making the mass m_d finite we get rid of the collapse and, as a result, of its manifestation, oscillations.

In the considered approximation the phase diagram of the chiral invariant QED is as follows: for all subcritical values $\alpha^{(0)} < \alpha_c = \pi/3$ the β-function equals zero (there are no ultraviolet divergences), and all these values $\alpha^{(0)}$ form the line of the fixed points; in the massive phase with $\alpha^{(0)}(\Lambda) > \alpha_c$ the additional renormalization of the charge takes place which leads to the ultraviolet stable fixed point $\alpha^{(0)} = \alpha_c$. The form of the phase diagram of the exact theory depends also on other divergences. In ref. [7] the following hypothesis has been put forward: for sufficiently small values of the bare coupling constant, due to the argument of ref. [3,4], the « zero-charge » situation takes place in QED (*i.e.* the continuum limit QED degenerates in a free theory), however, in the supercritical phases in the continuum limit a nontrivial field theory arises at the fixed value $\alpha^{(0)} = \alpha_c$.

We shall discuss the physical content of such a continuum theory in sect. 4. Here we would like to note the following. It is essential that the vacuum rearrangement considered above is connected with the « fall into the supercritical Coulomb centre » phenomenon, the existence of which in relativistic quantum mechanics follows directly from the uncertainty principle (*). Moreover, the similar, collapse, phenomenon at supercritical values of coupling constant takes place in some two-dimensional models (7)—in the lattice massless Thirring model (17) and in the sine-Gordon model (18) (this model is considered in appendix B). All these facts make sure that this phenomenon is not an accidental artifact of the ladder approximation. We shall return to this point in sect. **4**, but before, in the next section, we shall discuss peculiarities of the dynamics of the chiral symmetry breaking in QED connected with subtleties of the transition to the continuum limit.

3. – Continuum limit and the character of the chiral-symmetry breakdown in QED.

In field theories anomalies can essentially influence the dynamics of symmetry breaking. The known example is the Adler-Bell-Jackiw (ABJ) anomaly of the singlet axial-vector current (19). Still before the discovery of this anomaly it was known (from the analysis of the Johnson-Baker-Willey solution (5) in QED) that the vanishing of the bare mass does not ensure the conservation of the axial-vector currents in continuum theories (5,20) (in the literature this circumstance is sometimes called the Johnson-Pagels anomaly). In the present section we will show that this anomaly, unlike the ABJ one, can be removed if one uses a suitable transition to the continuum limit in the equations of QED. We shall also show that in this way the old Goldstein problem (21) can be solved.

Let us consider QED with K fermion flavours. In the continuum limit the K^2-1 anomaly-free axial-vector currents

$$J^r_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\lambda^r\psi\,, \qquad r = 1, 2, \ldots, K^2-1\,,$$

satisfy the equation

$$\partial^\mu J^r_{5\mu} = \lim_{\Lambda\to\infty} m^{(0)}(\Lambda)(\bar{\psi}\gamma_5\lambda^r\psi)_\Lambda\,, \tag{12}$$

(*) Indeed, in relativistic theory the kinetic energy $E_k = (q^2+m^2)^{\frac{1}{2}} - m \underset{q\to\infty}{\simeq} q$. Therefore, the energy $E = m + E_k - \alpha/r \underset{r\to 0}{\simeq} 1 - \alpha/r$ (due to the uncertainty principle, the momentum $q \sim 1/r$), and the collapse phenomenon happens at $\alpha > 1$.

(17) B. M. McCoy and T. T. Wu: *Phys. Lett.*, **87**, 50 (1979).

(18) S. Coleman: *Phys. Rev. D*, **11**, 2088 (1975).

(19) S. L. Adler: *Phys. Rev.*, **177**, 2426 (1969); J. S. Bell and R. Jackiw: *Nuovo Cimento A*, **60**, 47 (1969).

(20) Th. A. J. Maris, G. Jacob and B. Liberman: *Nuovo Cimento A*, **52**, 116 (1967); H. Pagels: *Phys. Rev. D*, **7**, 3689 (1973).

(21) S. Goldstein: *Phys. Rev.*, **91**, 1516 (1953).

where $m^{(0)}(\Lambda)$ is the bare fermion mass. It is essential that the operator $(\bar{\psi}\gamma_5\lambda^r\psi)_\Lambda$ is composite and depends on the cut-off parameter:

$$(\bar{\psi}\gamma_5\lambda^r\psi)_\Lambda = Z_{m\mu}^{-1}(\bar{\psi}\gamma_5\lambda^r\psi)_\mu\,, \tag{13}$$

where μ is a subtraction point and, as is well known ([22]), the renormalization constant $Z_{m\mu}$ coincides with the renormalization constant of the operator $(\bar{\psi}\psi)_\Lambda$ $((\bar{\psi}\psi)_\Lambda = Z_{m\mu}^{-1}(\bar{\psi}\psi)_\mu)$. From eqs. (12) and (13) we obtain the condition which ensures the conservation of the anomaly-free axial-vector currents in the continuum theory:

$$\lim_{\Lambda\to\infty} m^{(0)}(\Lambda)Z_{m\mu}^{-1} = 0\,. \tag{14}$$

It can be shown (see appendix C) that in the ladder approximation the renormalization constant

$$Z_{m\mu} \simeq (\mu^2/\Lambda^2)^{\frac{1}{2}-\gamma'}, \qquad \gamma' = i\gamma = \frac{1}{2}\left(1 - \frac{3\alpha^{(0)}}{\pi}\right)^{\frac{1}{2}} \tag{15}$$

at subcritical values of the coupling constant, $\alpha^{(0)} < \alpha_c = \pi/3$, and

$$Z_{m\mu} \simeq \mu/\Lambda \tag{16}$$

at supercritical values of $\alpha^{(0)}$. From here and condition (14) we see that the vanishing of the bare mass in the continuum limit, $m^{(0)} = \lim_{\Lambda\to\infty} m^{(0)}(\Lambda) = 0$, does not ensure the conservation of the axial-vector currents. To ensure this, one must require a rapid enough (as $o(Z_{m\mu})$) decrease of the bare mass at $\Lambda \to \infty$. In particular, this condition is satisfied if one chooses $m^{(0)}(\Lambda) = 0$, *i.e.* if the Lagrangian of the theory with cut-off is already chosen to be chiral invariant and the continuum theory is considered as the limit of this one (just such a way was used in sect. **2**).

Let us demonstrate this conclusion studying directly the equation for the fermion mass function (see eq. (C.4) of appendix C):

$$m(q^2) = m^{(0)}(\Lambda) + \frac{3\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2} dk^2 \frac{m(k^2)}{m^2 + k^2}\left[\frac{k^2}{q^2}\theta(q^2 - k^2) + \theta(k^2 - q^2)\right] \tag{17}$$

(when $m^{(0)}(\Lambda) = 0$, the mass m coincides with the dynamical mass m_d). The solution of this equation has the same form as that one of the equation with $m^{(0)}(\Lambda) = 0$ $(m(q^2) = CF(\frac{1}{2} + \gamma', \frac{1}{2} - \gamma', 2; -q^2/m^2)$, where the normalization constant $C = \xi m$, ξ is a numerical constant), however it satisfies a somewhat

([22]) S. L. Adler and W. A. Bardeen: *Phys. Rev. D*, **4**, 3045 (1971).

different boundary condition at $q^2 = \Lambda^2$ (see appendix C):

$$\left(q^2 \frac{\mathrm{d}m(q^2)}{\mathrm{d}q^2} + m(q^2)\right)\Big|_{q^2=\Lambda} = m^{(0)}(\Lambda) . \tag{18}$$

Using the asymptotic expansion for the hypergeometric function at $m^2/\Lambda^2 \ll 1$ ([16]), we find from eq. (18) the following relations:

$$m\xi \frac{\Gamma(2\gamma')}{\Gamma^2(\frac{1}{2}+\gamma')} = \left(\frac{\Lambda}{m}\right)^{1-2\gamma'} m^{(0)}(\Lambda) \tag{19}$$

at the subcritical values $\alpha^{(0)} < \pi/3$, and

$$m\xi \left(\frac{\operatorname{ctgh} \pi\gamma}{\pi\gamma}\right)^{\frac{1}{2}} \sin\left[\gamma \ln \frac{\Lambda^2}{m^2} + \Sigma(\gamma)\right] = \frac{\Lambda}{m} m^{(0)}(\Lambda) , \tag{20}$$

$\Sigma(\gamma) = \arg\left(\Gamma(1+2i\gamma)/\Gamma^2(\frac{1}{2}+i\gamma)\right)$, at the supercritical values of $\alpha^{(0)}$.

Taking into account condition (14) and eq. (15), we find from eq. (19) that there are no solutions corresponding to the spontaneous chiral symmetry breaking at subcritical values $\alpha^{(0)} < \pi/3$. On the other hand, from eqs. (14) and (16) we find that in the limit $\Lambda \to \infty$ the relation (20) is reduced to that one with $m^{(0)}(\Lambda) = 0$ and therefore we return to the picture with the ultraviolet stable fixed point $\alpha_c = \pi/3$ discussed in sect. **2**.

Note that since for all values $\alpha^{(0)}$ the function $m(q^2) = m\xi F(\frac{1}{2}+\gamma', \frac{1}{2}-\gamma', 2; q^2/m^2)$ satisfies the boundary condition (18) (and therefore eq. (17)) at $\Lambda = \infty$ and $m^{(0)}(\Lambda)|_{\Lambda=\infty} = 0$, not all solutions of the equation without cut-off and with zero fermion bare mass corresponding to the spontaneous chiral symmetry breaking. This point is sufficiently general and is characteristic for the problem of dynamical symmetry breaking in continuum theories (*). In particular, it is closely connected with the so-called Goldstein problem ([21]). In the paper ([21]) the BS equation (in the ladder approximation) for the massless pseudoscalar fermion-antifermion bound states (*i.e.* in fact, for Goldstone bosons) was investigated in QED. With the substitution

$$\chi(q^2) \to (q^2 + m^2)^{-1} m(q^2)$$

this equation coincides with eq. (17) in the limit $\Lambda \to \infty$ and $m^{(0)}(\Lambda)|_{\Lambda=\infty} = 0$.

(*) Note that taking into account condition (14) in the case of quantum chromodynamics it is possible to determine uniquely the ultraviolet asymptotics of the dynamical quark mass function directly from the equations for Green's functions without using the assumption of the validity of operator product expansion ([23]).

([23]) V. A. MIRANSKY: *Yad. Fiz.*, **38**, 468 (1983).

Formally, at *all* values of $\alpha^{(0)}$ the function

$$\chi(q^2) = (q^2 + m^2)^{-1}\dot{F}(\tfrac{1}{2} + \gamma', \tfrac{1}{2} - \gamma', 2; -q^2/m^2)$$

is the solution of this equation. A paradoxical situation occurs: the *massless* parapositronium exists at arbitrary small coupling in QED. However, if the subtleties of the transition to the limit $\Lambda \to \infty$ are taken into account, one shall come to the picture with the ultraviolet stable fixed point considered in sect. **2**.

4. – The phase diagram in coupling constant and continuum limit in QED.

In this section we discuss the results of the computer simulations in noncompact lattice QED [1] from the viewpoint of the dynamical picture considered above. Moreover, we consider the form of the QED phase diagram in coupling constant and discuss the possibility of the existence of the nontrivial continuum limit in this theory.

The principal results of the computer simulations [1] realized in the quenched approximation are as follows:

1) In massless QED the order parameter $\langle 0|(\bar{\psi}\psi)_\Lambda|0\rangle$ is nonzero (*i.e.* spontaneous chiral symmetry breaking takes place) for all coupling constant $\alpha^{(0)}$ greater than the critical value $\alpha_c \simeq 0.3$. The value $\langle 0|(\bar{\psi}\psi)_\Lambda|0\rangle$ is sensitive to the short-distance dynamics of QED.

2) Computer simulations on asymmetric lattices do not reveal any significant temperature dependence in the chiral-symmetry-breaking dynamics.

The first point qualitatively agrees with the dynamical picture of the spontaneous chiral symmetry in QED corresponding to the collapse (« fall into the supercritical Coulomb centre ») phenomenon. Since the value of the critical coupling constant (opposite to critical indices) is strongly influenced by the form of the ultraviolet regularization, the direct comparison of the lattice critical coupling and the critical coupling of the ladder approximation theory with cut-off in momentum space does not allow one to estimate the significance of nonladder diagrams. However, the qualitative agreement of the results of the computer simulations with those of the ladder approximation evidences that the ladder approximation reproduces the characteristic features of the dynamics of the spontaneous chiral symmetry breaking in QED and therefore this approximation can be used as a plausible model for the study of this phenomenon.

For what concerns the second point, it can be easily understood if one takes into account that the value of the critical temperature T_c, at which a symmetry is restored, is ordinarily expressed through those distances, r, where the dynamics of symmetry breaking is on the whole formed, $T_c \sim r^{-1}$. In the collapse situation such distances are small, $r \sim \Lambda^{-1}$, and therefore only at very large values, $T \sim \Lambda$, temperature can influence the dynamics of the spontaneous chiral symmetry breaking.

Let us discuss now the problem of the existence of the continuum limit in QED. First of all we will show the validity of the following general statement; if in massless QED with ultraviolet cut-off the second-order phase transition, connected with spontaneous chiral symmetry breaking, takes place at the value of the bare coupling constant $\alpha^{(0)} = \alpha_c > 0$, then the continuum QED with the fixed value of the bare coupling constant $\alpha^{(0)} = \alpha_c$ has a nontrivial S-matrix.

By the assumption, at the supercritical values $\alpha^{(0)} > \alpha_c$ the fermion dynamical mass m_d appears. Since Λ is the only dimensional parameter of the theory, this mass has the form

$$m_d = \Lambda f(\alpha^{(0)}) , \tag{21}$$

where f is some function. The equation

$$f(\alpha^{(0)}) = 0 \tag{22}$$

has a positive root which coincides with the critical value α_c separating the massless and the massive phase. In the continuum limit, $m_d/\Lambda = f(\alpha^{(0)})_{\Lambda\to\infty} \to 0$, this value of the coupling constant determines the continuum theory with the nontrivial S-matrix: the Bethe-Salpeter wave function of the pseudoscalar Goldstone bosons (corresponding to the spontaneous chiral symmetry breaking) determines the effective interaction vertex of a goldstonion with a fermion and an antifermion, and therefore there must be the pole, corresponding to the Goldstone boson, in the S-matrix of the fermion-antifermion scattering. The appearance of a sufficiently small bare fermion mass (PCAC situation) should not significantly influence this picture.

Thus in order to prove the existence of a nontrivial continuum QED, it is sufficient to show that the spontaneous chiral symmetry breaking takes place at large values $\alpha^{(0)}$ in the theory *with cut-off*.

The computer simulations in ref. ([1]) have been realized in the quenched approximation. To determine the phase diagram in coupling constant of the exact theory, it is necessary to know to what extent the improvement of the quenched approximation, and, first of all, the inclusion of the vacuum polarization effects, can influence the results. Below we will argue that the spontaneous chiral symmetry breaking is realized in QED with sufficiently large coupling $\alpha^{(0)}$ even in the case in which the Landau-Pomeranchuk-Fradkin « zero-charge » picture ([3,4]) for the vacuum polarization effects takes place there.

The analysis of the ladder and of the quenched approximations indicates that the dynamics of the spontaneous chiral symmetry breaking in QED with cut-off should be formed in the region in which the running coupling constant $\bar{\alpha}(r) \geqslant \alpha_c \sim 1$. On the other hand, due to the arguments of Landau and Pomeranchuk ([3]) the relation

$$\bar{\alpha}(r) = \frac{\alpha^{(0)}}{1 + (2K\alpha^{(0)}/3\pi)\ln(\Lambda r)}, \tag{23}$$

where K is the number of fermion flavours, remains qualitatively correct at large values of $\alpha^{(0)}$ too. Due to this relation, at any sufficiently large bare coupling constant $\alpha^{(0)}$ the vacuum polarization effects reduce the value of the running coupling $\bar{\alpha}(r)$ to a value of the order of unity already at the distance $r = \varrho/\Lambda$, where the parameter ϱ is larger than unity (also, due to eq. (23), it is not too large, $\varrho < 10$ at $K = 3$). Therefore, these effects can be imitated by introducing the infra-red cut-off $\delta = \Lambda/\varrho$. Since the qualitative picture of the spontaneous chiral symmetry breaking in QED is similar in the ladder and in the quenched approximations, one can suppose that the role of the infra-red cut-off is already correctly represented by the equations of the ladder approximation. The coupling constant in these equations should considered as some averaged value of the running coupling $\bar{\alpha}(r)$ in the region $1/\Lambda < r < \varrho/\Lambda$.

As has already been noted, in the ladder approximation the Bethe-Salpeter equation with the infra-red cut-off δ for Goldstone bosons was studied (for other purposes) in ref. ([24]). From this paper it follows that the value of the critical coupling α_c is determined from the relation (compare with eq. (21))

$$\gamma_c \ln\left(\frac{\Lambda}{\delta}\right) + \operatorname{arctg} 2\gamma_c = \frac{\pi}{2}, \qquad \gamma_c = \frac{1}{2}\left(\frac{3\alpha_c}{\pi} - 1\right)^{\frac{1}{2}}. \tag{24}$$

For our purpose it is important that at $\delta = \Lambda/\varrho$ the value $\alpha_c(\varrho)$ determined by eq. (24) remains finite for any value $\varrho > 1$ ($\alpha_c(\varrho) \to \infty$ when $\varrho \to 1$). Besides, since at $\delta = \Lambda/\varrho$ the parameter Λ disappears from relation (24), then (despite the switching off the interaction at all nonzero, $r > 0$, distances in the limit $\Lambda \to \infty$: $\lim_{\Lambda\to\infty} \Delta r = \lim_{\Lambda\to\infty} (\varrho - 1)\Lambda^{-1} = 0$) the spontaneous chiral symmetry breaking takes place at $\alpha^{(0)} \geqslant \alpha_c(\varrho)$ in the continuum limit too. However, and it is important, the dynamical mass

$$m_d = \Lambda f(\alpha^{(0)}, \varrho)$$

remains finite in this limit only at the fixed value $\alpha^{(0)} = \alpha_c(\varrho)$ (compare with eq. (6)).

([24]) V. A. Miransky, V. P. Gusynin and Yu. A. Sitenko: *Phys. Lett. B*, **100**, 157 (1981).

Thus this analysis leads us to the following hypothesis. In the « zero-charge » situation continuum QED can be nontrivial theory: the residual δ-like ($\bar{\alpha}(0) = \alpha_c$ and $\bar{\alpha}(r) = 0$ at $r > 0$) potential of the fermion-antifermion interaction is able to form the composed Goldstone boson and, as a consequence, the induced fermion-antifermion-boson interaction vertex appears.

In such a situation the phase diagram in coupling constant takes the following form: there is only one trivial infra-red stable fixed point, $\alpha_\mu \equiv \equiv \bar{\alpha}(\mu^{-1}) = 0$, in the subcritical phase with $\alpha^{(0)} < \alpha_c$ and, therefore, at all these values $\alpha^{(0)}$ a free theory arises in the continuum limit (the conventional « zero-charge » picture ([3,4])); in the supercritical phase there is the ultraviolet stable fixed point, $\alpha^{(0)} = \alpha_c$, determining the interacting continuum theory with the Yukawa-type coupling of fermions, antifermions and pseudoscalar *composed* Goldstone bosons. The appearance of a sufficiently small bare fermion mass (PCAC situation) should not significantly influence the phase diagram (*).

It would be interesting to examine this dynamical picture by computer simulation methods.

5. – Conclusions.

Thus the collapse phenomenon in quantum field theory may essentially influence (and in some two-dimensional models (see appendix B) does influence) the renormalization structure, *i.e.* the structure of the phase diagram of a theory.

The analysis of QED with the chiral invariant Lagrangian indicates the existence of the critical coupling constant $\alpha_c \sim 1$ separating the massless and the massive phases. This critical coupling is an analogue of the critical coupling constant $Z_c e^2/4\pi \simeq 1$ in the problem of the Dirac equation with the Coulomb potential. From the renormalization group viewpoint the value α_c is an ultraviolet stable fixed point. We have argued that this value determines a nontrivial continuum theory.

It is essential that this dynamical picture allows one to represent a simple physical interpretation of the recent results of computer simulations in QED ([1]). Besides, the hypothesis about the existence (even in the case of the « zero-charge » situation) of nontrivial continuum QED can in principle be examined by the computer simulation methods in the near future.

In conclusion we would like to note the following. At present there exist

(*) Real QED (*i.e.* phenomenological theory with ultraviolet cut-off describing the low-energy interaction of leptons and photon) apparently relates to the subcritical phase. Indeed, there are no candidates for the Goldstone (or « almost » Goldstone) bosons composed of leptons. Besides, if QED is a part of GUT, then the electrodynamical running coupling constant $\bar{\alpha}(r)$ should be small at all distances (for example, in SU_5 theory $\bar{\alpha}(r) \leqslant 0.02$).

problems for which the mechanism of scale symmetry breaking is primarily important (finite supersymmetric theories ([25]), gravitation (for a review see, for example, the general discussion in ref. ([26]))). It would be interesting to examine the possibility of the realization of the scale symmetry breaking mechanism connected with the collapse phenomenon there.

* * *

The author is thankful to P. I. FOMIN, V. P. GUSYNIN and D. V. VOLKOV for useful discussions.

APPENDIX A

In this appendix we consider the equations in the ladder approximation for the dynamical fermion mass function and for the Bethe-Salpeter (BS) wave functions of pseudoscalar Goldstone bosons.

First of all we will show that from the viewpoint of the Ward identities the Landau gauge is the most preferable one in the ladder approximation.

When the spontaneous breakdown of the $SU_{L,K}\times SU_{R,K}$ chiral symmetry takes place in massless QED, the structure of the fermion propagator has the following form:

$$S_{ij}(q) = \delta_{ij}\big(-\not{q}A(q^2) + B_d(q^2)\big)^{-1}, \tag{A.1}$$

$i, j = 1, 2, \ldots, K$. In the approximation with the bare photon propagator and with the bare fermion-antifermion-photon vertex

$$\Gamma_{1\mu;ij} = \gamma_\mu \delta_{ij} \tag{A.2}$$

the Schwinger-Dyson equations for the fermion propagator in a covariant gauge with the gauge parameter d_l takes the form (in the Euclidean region)

$$A(q^2) - 1 = d_l \frac{\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2} \mathrm{d}k^2 \frac{A(k^2)}{k^2A^2(k^2) + B_d^2(k^2)}\left[\frac{k^4}{q^4}\theta(q^2-k^2) + \theta(k^2-q^2)\right], \tag{A.3}$$

$$B_d(q^2) = (3 + d_l)\frac{\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2} \mathrm{d}k^2 \frac{B_d(k^2)}{k^2A^2(k^2) + B_d^2(k^2)}\left[\frac{k^2}{q^2}\theta(q^2-k^2) + \theta(k^2-q^2)\right]. \tag{A.4}$$

([25]) S. MANDELSTAM: *Nucl. Phys. B*, **213**, 149 (1983); L. BRINK, O. LINGREN and B. NILSSON: *Phys. Lett. B*, **123**, 323 (1983); P. HOWE, K. S. STELLE and P. TOWNSEND: *Nucl. Phys. B*, **214**, 519 (1983); S. FUBINI and E. RABINOVICI: preprint TH. 3825-CERN (1984).

([26]) In *Proceedings of the XVIII Solvay Conference on Physics*, edited by L. VAN HOVE, *Phys. Rep.*, **104**, 201 (1984).

In the Landau gauge ($d_l = 0$) the function $A(q^2)$ equals unity. Moreover, in this gauge eqs. (A.3) and (A.4) are not changed if the vertex $\Gamma_{1\mu;ij}$ is chosen in the form

$$\Gamma_{1\mu;ij}(q_2, q_1) = \gamma_\mu \delta_{ij} + \frac{P_\mu}{P^2}\delta_{ij}\varphi\,, \tag{A.5}$$

where $P_\mu = q_{2\mu} - q_{1\mu}$, φ is an arbitrary Lorentz-invariant function. If $\varphi = B(q_1^2) - B(q_2^2)$, the vertex (A.5) satisfies the Ward identity

$$P^\mu \Gamma_{1\mu;ij}(q_2, q_1) = S_{ij}^{-1}(q_1) - S_{ij}^{-1}(q_2)\,, \tag{A.6}$$

which follows from the conservation of the electromagnetic current.

Let us consider now the Ward identity for the vertex $\Gamma_{5\mu}^r$ of the axial-vector current $J_{5\mu}^r = \bar{\psi}\gamma_\mu\gamma_5\lambda^r\psi$, $r = 1, 2, \dots, K^2 - 1$:

$$P^\mu \Gamma_{5\mu}^r(q_2, q_1) = -\gamma_5\lambda^r S^{-1}(q_1) - S^{-1}(q_2)\gamma_5\lambda^r\,. \tag{A.7}$$

Under spontaneous breaking of chiral symmetry the vertex $\Gamma_{5\mu}^r$ has a pole at zero in the variable $P^2 = (q_2 - q_1)^2$; the residue at this pole is expressed through the ES wave function of the Goldstone boson ([27]):

$$\Gamma_{5\mu}^r(P, q)|_{P^2 \simeq 0} \simeq \frac{iP_\mu f}{P^2} S^{-1}(q_2)\chi^r(P, q)S^{-1}(q_1)\,, \tag{A.8}$$

where $q = q_1 + q_2/2$ and the parameter f is determined by the equation

$$\langle 0|J_{5\mu}^{r'}|P, r\rangle = i\delta_{rr'} f P_\mu\,,$$

$|P, r\rangle$ is the state vector of the Goldstone boson and $\chi^r(P, q)$ is the BS wave function. Substituting the expression (A.8) into eq. (A.7) and going over to the limit $P_\mu \to 0$, we obtain the relation

$$\begin{cases} \chi^r(q) \equiv \chi^r(P, q)|_{P=0} = \lambda^r\gamma_5\chi(q^2)\,, \\ \gamma_5\chi(q^2) = 2if^{-1}S(q)\gamma_5 B_d(q^2)S(q)\,. \end{cases} \tag{A.9}$$

In the ladder approximation in the Euclidean region the function

$$\chi(q^2) = 2if^{-1}\frac{B_d(q^2)}{q^2 + m_d^2}\,, \tag{A.10}$$

and the BS equation for $\chi(q^2)$ takes the following form in the Landau gauge:

$$(q^2 + m_d^2)\chi(q^2) = \frac{3\alpha^{(0)}}{4\pi}\int_0^{\Lambda^2} dk^2 \left[\frac{\theta(q^2 - k^2)}{q^2} + \frac{\theta(k^2 - q^2)}{k^2}\right] k^2\chi(k^2)\,. \tag{A.11}$$

([27]) V. DE ALFARO, S. FUBINI, G. FURLAN and C. ROSSETTI: *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973).

Comparing eqs. (A.3), (A.4) and (A.11) and taking into account the relation (A.10), we find that in the Landau gauge the ladder approximation corresponds to the linear version of eq. (A.4) when in the denominator the function B_d^2 is replaced by m_d^2. Equation (A.11) can be solved in the following way. Differentiating it in q^2, we obtain the differential equation

$$\frac{d}{dq^2}\left\{q^4 \frac{d}{dq^2}[(q^2+m_d^2)\chi]\right\} + \frac{3\alpha^{(0)}}{4\pi} q^2\chi = 0 \tag{A.12}$$

and two boundary conditions:

$$\left\{q^4 \frac{d}{dq^2}[(q^2+m_d^2)\chi]\right\}\bigg|_{q^2=0} = 0\,, \tag{A.13}$$

$$\left\{q^2 \frac{d}{dq^2}[(q^2+m_d^2)\chi] + (q^2+m_d^2)\chi\right\}\bigg|_{q^2=\Lambda^2} = 0\,. \tag{A.14}$$

The general solution to eq. (A.12) has the form

$$B_d(q^2) \equiv \frac{f}{2i}(q^2+m_d^2)\chi(q^2) = C_1 B_1(q^2) + C_2 B_2(q^2) \tag{A.15}$$

(see eq. (A.10)), where

$$\begin{cases} B_1 = F(\tfrac{1}{2}+i\gamma, \tfrac{1}{2}-i\gamma, 2; -x^2)\,, \\ B_2 = x^{-1-2i\gamma}F(\tfrac{1}{2}+i\gamma, -\tfrac{1}{2}+i\gamma, 1+2i\gamma; -x^2) + \\ \qquad + x^{-1+2i\gamma}F(\tfrac{1}{2}-i\gamma, -\tfrac{1}{2}-i\gamma, 1-2i\gamma; -x^2) \end{cases} \tag{A.16}$$

$\gamma = \frac{1}{2}(3\alpha^{(0)}/\pi - 1)^{\frac{1}{2}}$, $x^2 = q^2/m_d^2$, F is a hypergeometric function. From eq. (A.13) we find that $C_2/C_1 = 0$ and, therefore,

$$B_d(q^2) = CF(\tfrac{1}{2}+i\gamma, \tfrac{1}{2}-i\gamma, 2; -x^2)\,, \tag{A.17}$$

the normalization parameter C has the form $C = \xi m_d$, where ξ is a numerical constant, and it can in principle be determined from the normalization condition for the Bethe-Salpeter wave function. The second boundary condition (A.14) determines the mass spectrum of the equation. The analytical answer can be obtained in the case of $m_d^2/\Lambda^2 \ll 1$. Using the asymptotic expansion of hypergeometric functions ([16]), in this case we get the following equation from eq. (A.14):

$$\sin\left(\gamma \ln\frac{\Lambda^2}{m_d^2} + \Sigma(\gamma)\right) = 0\,, \tag{A.18}$$

where $\Sigma(\gamma) = \arg[\Gamma(1+2i\gamma)/\Gamma^2(1+i\gamma)]$. Equation (A.18) yields the mass spectrum in the form

$$m_d^{(s)} = \Lambda \exp\left[\frac{-\pi s + \Sigma(\gamma)}{2\gamma}\right] \underset{\gamma\ll 1}{\simeq} 4\Lambda \exp\left[\frac{-\pi s}{2\gamma}\right], \qquad s = 1, 2, \ldots. \tag{A.19}$$

One can show ([10]) that only the maximum value, $m_d^{(1)}$, corresponds to the stable vacuum.

APPENDIX B

In this appendix we show that the phenomenon of the rearrangement of the renormalization structure at supercritical values of coupling constant takes place in the two-dimensional sine-Gordon model.

The Hamiltonian density of the sine-Gordon model has the form

$$H = N_m \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\varphi)^2 - \frac{\varkappa}{g^2}\cos g\varphi\right], \tag{B.1}$$

where π is the canonical momentum, N_m is the symbol of the normal-ordering operation with the contraction function

$$\Delta(x_1, m) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \frac{dk_1}{\sqrt{k_1^2 + m^2}} \exp[-ik_1x_1]. \tag{B.2}$$

One can easily verify that (due to the two-dimensionality of space-time) the normal-ordering operation N_m removes all ultraviolet divergences in any order of perturbation theory. On the other hand, as has been indicated by COLEMAN ([18]), this models has no ground state at supercritical values $g^2 > 8\pi$. We shall show below that this phenomenon appears due to the presence of additional ultraviolet divergences in the supercritical phase with $g^2 > 8\pi$; these divergences are similar to those which arise in the «fall into the centre» (collapse) situation. To remove them, an additional renormalization of the parameters g^2 and $\varkappa$ should be performed.

Let us introduce the cut-off Λ in the contraction function (B.2):

$$\Delta^{(\Lambda)}(x_1, m) = \frac{1}{4\pi}\int_{-\Lambda}^{+\Lambda} \frac{dk_1}{\sqrt{k_1^2 + m^2}} \exp[-ik_1x_1], \tag{B.3}$$

$$\Delta^{(\Lambda)}(0, m) = \frac{1}{2\pi}\ln\frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m}. \tag{B.4}$$

Since the spatial component of momentum, k_1, is bounded, such a cut-off corresponds to the anisotropic lattice in the Euclidean domain (the time axis is continuous). In connection with this we recall that the character of a phase diagram is independent of the lattice form (the property of universality of phase transitions).

Following ref. ([18]), we choose the vacuum appropriate to a free field of mass μ as our trial vacuum state:

$$\varphi^-(x_1, \mu)|0, \mu\rangle = \pi^-(x_1, \mu)|0, \mu\rangle = 0. \tag{B.5}$$

Using the contraction function (B.3), we find

$$(B.6)\qquad H = N_\mu\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\varphi)^2 - \frac{\varkappa}{g^2}\left(\frac{\mu^2}{m^2}\right)^{g^2/8\pi}\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{\Lambda+\sqrt{\Lambda^2+\mu^2}}\right)^{g^2/4\pi}\cos g\varphi\right] + \frac{1}{4\pi}\Lambda(\sqrt{\Lambda^2+\mu^2}-\sqrt{\Lambda^2+m^2}).$$

Therefore, the energy density of the vacuum is equal to

$$(B.7)\qquad \varepsilon_V(\Lambda) = \langle 0,\mu|H|0,\mu\rangle = \frac{1}{4\pi}\Lambda(\sqrt{\Lambda^2+\mu^2}-\sqrt{\Lambda^2+m^2}) - \frac{\varkappa}{g^2}\left(\frac{\mu^2}{m^2}\right)^{g^2/8\pi}\left(\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{\Lambda+\sqrt{\Lambda^2+\mu^2}}\right)^{g^2/4\pi}.$$

When the cut-off is removed, the energy density (B.7) is turned into the expression of ref. [18]:

$$(B.8)\qquad \varepsilon_V(\infty) = \frac{1}{8\pi}(\mu^2-m^2) - \frac{\varkappa}{g^2}\left(\frac{\mu^2}{m^2}\right)^{g^2/8\pi}.$$

At $g^2 > 8\pi$ the last expression is unbounded from below as μ goes to infinity. Therefore, if $g^2 > 8\pi$, the continuum theory has no ground state (collapse). On the other hand, as it follows from eq. (B.7), no collapse happens in the theory with cut-off.

To understand the situation better, let us find the extrema of the energy density $\varepsilon_V(\Lambda)$:

$$(B.9)\qquad \frac{d\varepsilon_V(\Lambda)}{d\mu^2} = \frac{1}{8\pi}(1+\mu^2/\Lambda^2)^{-\frac{1}{2}}\cdot\left[1 - 4^\nu\frac{\varkappa}{m^2}\left(\frac{\mu^2}{m^2}\right)^{\nu-1}(1+\sqrt{1+\mu^2/\Lambda^2})^{-2\nu}\right] = 0\,,\qquad \nu \equiv g^2/8\pi.$$

We obtain from eq. (B.9):

1) $\nu < 1$: there is the absolute minimum at the value

$$(B.10)\qquad \mu^2 \simeq \varkappa\left(\frac{m^2}{\varkappa}\right)^{\nu/\nu-1};$$

2) $\nu > 1$: in this case the value $\mu^2 \simeq \varkappa(m^2/\varkappa)^{\nu/\nu-1}$ becomes the maximum, and the absolute minimum is

$$(B.11)\qquad \mu^2 = 4^\nu\varkappa\left(\frac{\Lambda}{m}\right)^{2\nu} + O(\Lambda^2, \Lambda^{2(\nu-1)}).$$

As it follows from eq. (B.11), there is the additional mass divergence in the supercritical phase with $g^2 > 8\pi$ ($\nu > 1$). Let us show that this divergence

can be removed by performing a renormalization of the parameters $\varkappa$ and g^2.
Equation (B.9) implies that

$$\nu = \ln(\mu\varkappa^{-\frac{1}{2}})/\ln[2\mu m^{-1}(1+\sqrt{1+\mu^2/\Lambda^2})^{-1}]. \tag{B.12}$$

In the limit $\Lambda \to \infty$, $\mu < \infty$ the condition $d^2\varepsilon_V(\Lambda)/d(\mu^2)^2 > 0$ guaranteeing that an extremum is a minimum takes the form

$$\nu - 1 < \frac{\mu^2 \nu}{2\Lambda^2}. \tag{B.13}$$

Therefore, in the supercritical phase ($\nu > 1$) this limit exists provided that the condition

$$0 \leqslant \nu - 1 \simeq \frac{\ln(m\varkappa^{-\frac{1}{2}}) + \mu^2/4\Lambda^2}{\ln(\mu m^{-1}) - \mu^2/4\Lambda^2} < \frac{\mu^2}{2\Lambda^2}\nu \tag{B.14}$$

holds. From here we find that the mass parameter μ remains finite in the continuum limit if the values of $\varkappa$ and g^2 are fixed:

$$\varkappa \to m^2, \qquad \nu = g^2/8\pi \to 1. \tag{B.15}$$

The renormalization (B.15) must be performed along a trajectory in the $(\varkappa, g^2)$-plane on which condition (B.14) is satisfied.

The meaning of relation (B.15) becomes clear if one notes that in the continuum limit $\Lambda \to \infty$ the energy density is independent of μ at $(\varkappa, g^2) = (m^2, 8\pi)$:

$$\varepsilon_V(\infty)|_{(\varkappa,g^2)=(m^2,8\pi)} = -\frac{m^2}{8\pi}. \tag{B.16}$$

Therefore, the parameter μ is arbitrary in this limit at $(\varkappa, g^2) = (m^2, 8\pi)$.

Thus, at $g^2 > 8\pi$ the rearrangement of the vacuum takes place (the expression for the minimum of the energy density (B.10) is replaced by expression (B.11)), which manifests itself in the appearance of the mass divergence that can be removed by an additional renormalization of the parameters $\varkappa$ and g^2. From the point of view of the renormalization group this situation reflects the existence of the ultraviolet stable fixed point $(\varkappa, g^2) = (m^2, 8\pi)$.

Appendix C

In this appendix we calculate in the ladder approximation the renormalization constant $Z_{m\mu}$ of the composed operator $(\bar{\psi}\psi)_\Lambda$:

$$(\bar{\psi}\psi)_\Lambda \simeq Z_{m\mu}^{-1}(\bar{\psi}\psi)_\mu, \tag{C.1}$$

μ is a subtraction point.

For this purpose we use the relation [22]

$$m^{(0)}(\Lambda)(\bar{\psi}\psi)_\Lambda = m^c_\mu(\bar{\psi}\psi)_\mu\,, \tag{C.2}$$

where $m^{(0)}(\Lambda)$ is the bare fermion mass, m^c_μ is the renormalized fermion current mass connected with the subtraction point μ. We emphasize that both these masses lead to the explicit chiral symmetry breaking (eq. (C.2) reflects the fact that the combination $m^c_\mu(\bar{\psi}\psi)_\mu$ is renormalization group invariant, *i.e.* independent of μ). From eq. (C.2) it follows that

$$Z_{m\mu} = m^{(0)}(\Lambda)/m^c_\mu\,. \tag{C.3}$$

To determine $m^{(0)}(\Lambda)/m^c_\mu$ we use the equation for the mass function $m(q^2) = B(q^2)/A(q^2)$. In the ladder approximation in the Landau gauge ($A(q^2)\simeq 1$) this equation, in the case in which $m^{(0)}(\Lambda)\neq 0$, takes the form (compare with eq. (A.4))

$$m(q^2) = m^{(0)}(\Lambda) + \frac{3\alpha^{(0)}}{4\pi}\int\limits_0^{\Lambda^2}\mathrm{d}k^2\,\frac{m(k^2)}{k^2+m^2}\left[\frac{k^2}{q^2}\theta(q^2-k^2)+\theta(k^2-q^2)\right], \tag{C.4}$$

where in the chiral limit, $m^{(0)}(\Lambda)\to 0$, the mass m coincides with the dynamical mass m_d. The analysis of this equation can be developed in complete analogy with that of eq. (A.11). The solution of eq. (C.4) is the function (compare with eq. (A.17))

$$m(q^2) = \xi m F(\tfrac{1}{2}+i\gamma,\,\tfrac{1}{2}-i\gamma,\,2;\;-q^2/m^2) \tag{C.5}$$

($\gamma = \frac{1}{2}(3\alpha^{(0)}/\pi - 1)^{\frac{1}{2}}$, ξ is a numerical constant) satisfying the following boundary condition:

$$\left(q^2\frac{\mathrm{d}m(q^2)}{\mathrm{d}q^2} + m(q^2)\right)\Bigg|_{q^2=\Lambda^2} = m^{(0)}(\Lambda)\,. \tag{C.6}$$

At first let us consider the case of subcritical values $\alpha^{(0)}<\pi/3$. In this case the spontaneous chiral symmetry breaking does not take place and the value $m = m(q^2)|_{q^2=m^2}$ coincides with the current fermion mass corresponding to the subtraction point $\mu = m$. Using the asymptotic expansion of the hypergeometric function at $\Lambda^2/m^2\gg 1$ [16] one finds from eqs. (C.5) and (C.6):

$$Z_m \equiv Z_{m\mu}|_{\mu=m} = m^{(0)}(\Lambda)/m \simeq \frac{\xi\Gamma(2\gamma')}{\Gamma^2(\frac{1}{2}+\gamma')}\left(\frac{m}{\Lambda}\right)^{1-2\gamma'}; \qquad \gamma' = i\gamma. \tag{C.7}$$

From here, in turn, we find that for $\mu\gg m$ the normalization constant

$$Z_{m\mu} \simeq \left(\frac{\mu}{\Lambda}\right)^{1-2\gamma'}. \tag{C.8}$$

Let us consider now the case of supercritical values $\alpha^{(0)}>\pi/3$. In this case the spontaneous chiral symmetry breaking takes place and the value $m = m(q^2)|_{q^2=m^2}$ is equal to $m_d + m^c$, where the current mass $m^c \equiv m^c_\mu|_{\mu=m}$ (for our purpose it is sufficient to consider the PCAC situation when $m^c \ll m_d$). From eqs. (C.5) and (C.6) we find in this case, in the limit $m/\Lambda \ll 1$, the relation

$$\text{(C.9)} \qquad \frac{\xi m^2}{\Lambda}\left(\frac{\operatorname{ctgh}\pi\gamma}{\pi\gamma}\right)^{\frac{1}{2}}\sin\left[2\gamma\ln\frac{\Lambda}{m}+\Sigma(\gamma)\right]=m^{(0)}(\Lambda)\,; \qquad \Sigma(\gamma)=\arg\frac{\Gamma(1+2i\gamma)}{\Gamma^2(\frac{1}{2}+i\gamma)}\,.$$

From here it follows that at $m^c \ll m$ the renormalization constant

$$\text{(C.10)} \qquad Z_m \equiv Z_{m\mu}|_{\mu=m} = m^{(0)}(\Lambda)/m^c \simeq 2\xi\left(\frac{\gamma\operatorname{ctgh}\pi\gamma}{\pi}\right)^{\frac{1}{2}}\frac{m_d}{\Lambda} \underset{\gamma\ll 1}{\simeq} \frac{2\xi m_d}{\pi\Lambda}\,.$$

From here, in turn it follows that at $\mu \gg m_d$ the renormalization constant

$$\text{(C.11)} \qquad Z_{m\mu} \simeq \mu/\Lambda\,.$$

● RIASSUNTO (*)

Si discutono il diagramma di fase nella costante di accoppiamento in QED e la sua connessione con la rottura di simmetria chirale spontanea. Si considera il meccanismo di questa rottura connesso con il fenomeno di collasso e si fornisce una semplice interpretazione fisica dei risultati recenti delle simulazioni con il calcolatore nel QED del reticolo. Si analizza il problema dell'esistenza della QED non banale nel continuo e, come risultato, si considera l'ipotesi seguente: nella situazione « a carica zero » di Landau-Pomeranchuk-Fradkin (la costante di rinormalizzazione $Z_3 = 0$) la matrice S della QED nel continuo con una costante di accoppiamento nuda fissata, $\alpha^{(0)} = \alpha_c \sim 1$ è non banale. Si rivela il contenuto fisico di questa teoria ipotetica nel continuo.

(*) *Traduzione a cura della Redazione.*

Динамика спонтанного нарушения киральной симметрии и непрерывной предел в квантовой электродинамике.

Резюме (*). — Обсуждается фазовая диаграмма по константе связи в квантовой электродинамике и ее связь со спонтанным нарушением киральной симметрии. Рассматривается механизм такого нарушения, связанный с явлением коллапса. Предлагается простая физическая интерпретация недавних результатов моделирования на ЭВМ в рамках квантовой электродинамики на решетке. Анализируется проблема существования нетривиальной непрерывной квантовой электродинамики. Рассматривается следующая гипотеза: в случае « нулевого заряда » Ландау-Померанчука-Фрадкина (постоянная перенормировки $Z_3=0$) S-матрица непрерывной квантовой электродинамики с фиксированной голой постоянной связи, $\alpha^{(0)}=\alpha_c\sim 1$, является нетривиальной. Анализируется физический смысл такой гипотетической непрерывной теории.

(*) *Переведено редакцией.*