Contribution to the Decay Theory of Unstable Quantum Systems. - II.

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Summary. — In the context of a theory of decay proposed by the authors, two theorems, which are necessary in order to guarantee the physical consistency of the formalism, are proved.

In a previous paper on this subject [1] we have given a simple and concise physical prescription for obtaining the wave function describing an unstable system when this system appears as a resonance in a scattering experiment. In the context of the theory of decay obtained in I and restricting ourselves, for simplicity, to potential scattering, we shall prove the following two theorems:

A) The formalism of I allows the description of an unstable state by limiting the considerations only to the isolated resonance responsible for that state.

B) Two resonating phase shifts which differ slightly in the resonance region, while outside that region they may well be far away from each other, give rise for the corresponding unstable states to almost identical decay laws.

(1) L. FONDA and G. C. GHIRARDI: *Nuovo Cimento*, **67** A, 257 (1970). This paper will be referred to as I in what follows.

Both these conditions must be satisfied in order to have a physically consistent formalism. In fact, the first guarantees that we can talk of an unstable system if the proper conditions are satisfied in the resonance energy interval, independently of the dynamics of the process for energies far apart from that region, while the second is necessary since in practice a pure Breit-Wigner resonance never occurs, so that the statement that we have an unstable system is sensible provided we have a sufficiently Breit-Wigner–like behaviour in a proper energy interval.

Let us recall the basic equations of I to be used in what follows. The unstable system is defined through the state vector (*)

$$\psi_{\text{unstable}}(t) = \int c(\underline{k}) \exp[-iE_k t]\, \psi^{(+)}(\underline{k})\, \mathrm{d}^3 k\,, \tag{1}$$

where $\psi^{(+)}$ are the outgoing-wave scattering states. The coefficients $c(\underline{k})$ yield the nondecay probability amplitude

$$A(t) \equiv \left(\psi_{\text{unstable}}(0),\, \psi_{\text{unstable}}(t)\right) = \int |c(\underline{k})|^2 \exp[-iE_k t]\, \mathrm{d}^3 k\,. \tag{2}$$

By means of the localization procedure introduced in Sect. **2** of I, the coefficients $c(\underline{k})$ have the following expression for potential scattering:

$$\begin{cases} c(\underline{k}) \;=\; Y_{lm}(\hat{\underline{k}})\, c_l(k)\,, \\[1ex] c_l(k) = \dfrac{N_l(k)}{f_l(k)}\,, \end{cases} \tag{3}$$

where we assume that the resonance appears in the l-th wave. $f_l(k)$ is the Jost function and $N_l(k)$ is given by

$$N_l(k) = \frac{2}{\pi}\int \frac{g_l(k')}{f_l(-k')}\, I_l(k, k')\, k^l k'^{\,l+2}\, \mathrm{d}k'\,, \tag{4}$$

$g_l(k)$ is the l-th wave energy form factor of the impinging wave packet and $I_l(k, k')$ is given by

$$I_l(k, k') = \int_0^R \varphi_l(k, r)\varphi_l(k', r)\, \mathrm{d}r + \int_R^\infty \varphi_l(k, r)\left[\varphi_l(k', r) - \varphi_l^{\text{out}}(k', r)\right] \mathrm{d}r\,, \tag{5}$$

where $\varphi_l(k, r)$ is the regular solution of the radial Schrödinger equations behaving at the origin as $\varphi_l(k, r) \underset{r\to 0}{\simeq} r^{l+1}/(2l+1)!!$ and $\varphi_l^{\text{out}}(k, r)$ is the free wave to

(*) Natural unita $\hbar = c = 1$ will be used.

which $\varphi_l(k, r)$ converges for $r \to \infty$. Substituting (3) into (2) gives

$$A(t) = \int_0^\infty \frac{|N_l(k)|^2}{|f_l(k)|^2} \exp[-iE_k t] k^2 \, \mathrm{d}k \,. \tag{6}$$

In order to prove Theorem A) we must investigate the detailed shape of $|N_l(k)|^2$. We must show that this function is sharply peaked near the resonance energy and vanishes rapidly outside the resonance region, providing thereby a natural cut-off in energy. In this way an isolated resonance appearing in $f_l(k)$ will describe completely the unstable system thereof obtained. We start by observing that in eq. (4) $g_l(k')$ is of course taken to be sharply peaked near the resonance energy $k' = k_R$ and that $f_l(-k')$ is almost zero at the same point. We can then treat the factor $g_l(k')/f_l(-k')$ as a delta-function $\delta(k' - k_R)$ in the integral on the right-hand side of (4). In this way we get

$$N_l(k) \propto k^l I_l(k, k_R) k_R^{l+2} \,. \tag{7}$$

The problem is now that of studying $I_l(k, k_R)$ for fixed k_R as a function of k. For simplicity we assume that the potential is a square well of range R. All the following considerations can however easily be extended to more general cases. Let us start with S-waves.

Under the above assumption we have

$$\varphi_0(k_R, r) = \varphi_0^{\text{out}}(k_R, r) \qquad \text{for } r > R \,,$$

so that $I_0(k, k_R)$ greatly simplifies:

$$I_0(k, k_R) = \int_0^R \mathrm{d}r \, \varphi_0(k, r) \varphi_0(k_R, r) = \frac{1}{\bar{k}\bar{\bar{k}}} \int_0^R \sin \bar{k}r \sin \bar{\bar{k}}r \, \mathrm{d}r = \tag{8}$$

$$= \frac{R}{2\bar{k}\bar{\bar{k}}} \left[\frac{\sin(\bar{k} - \bar{\bar{k}})R}{(\bar{k} - \bar{\bar{k}})R} - \frac{\sin(\bar{k} + \bar{\bar{k}})R}{(\bar{k} + \bar{\bar{k}})R} \right],$$

where we have defined

$$\bar{k} = \sqrt{k^2 + 2m|V|} \qquad \text{and} \qquad \bar{\bar{k}} = \sqrt{k_R^2 + 2m|V|} \,.$$

We see that $I_0(k, k_R)$, and consequently, through eq. (7), $N_0(k)$, are peaked near k_R and vanish by moving away from the resonance region. The energy interval ΔE in which $N_0(k)$ is appreciably different from zero can easily be seen to be greater than the width Γ of the resonance. In fact, since $\Delta k \simeq 1/R$ and therefore $\Delta E \simeq v/R$, the inequality

$$\Delta E > \Gamma \tag{9}$$

is equivalent to $v/R > \Gamma$, that is

$$v\tau > R . \tag{10}$$

Equation (10) states that the range of the forces should be smaller than the distance travelled by the decay products during one lifetime, a condition that must be satisfied in order that it be meaningful to talk of an unstable system.

The same result is obtained for $l \neq 0$ waves. Actually, since the functions j_l for $l \neq 0$ oscillate more than j_0, the function $N_l(k)$ turns out to be more peaked at k_R in this case. We have therefore completed the proof that $|N_l(k)|^2$ has the desired shape, *i.e.* it is peaked at the resonance energy E_R and vanishes rapidly when moving away from the resonance region, being appreciably different from zero in a region ΔE greater than the width Γ of the resonance. The isolated resonance appearing at E_R is then able to describe completely the corresponding unstable state through eq. (2), *i.e.* we can forget about other resonances or wild energy behaviours occurring outside the considered energy region ΔE. In this way proof of Theorem *A*) is accomplished.

Let us now prove Theorem *B*). We shall show that if we choose two resonating phase shifts almost alike in the region $|E - E_R| < \Delta E$, the corresponding decay laws turn out to be almost the same. This result is not at all obvious since even though only the values of the Jost function in the considered energy interval ΔE will contribute to the integral (6) owing to the cut-off properties of $|N_l(k)|^2$, the Jost function itself depends on the overall energy dependence of the phase shift. It would seem then that even changes in the phase shift $\delta(E)$ far away from the resonance region could affect the final result.

Let us make the following assumptions:

i) We consider a phase shift $\delta_1(E)$ having a sharp resonance at $E = E_R$ with width Γ.

ii) The relevant contribution to the considered integral comes from a region $E_R - \Delta E \leqslant E \leqslant E_R + \Delta E$ with $\Delta E > \Gamma$.

iii) A second phase shift is given:

$$\delta_2(E) = \delta_1(E) + \varepsilon(E) ,$$

where $\varepsilon(E)$ is assumed to be very small in a region

$$E_R - a < E < E_R + a$$

with $a > \Delta E$. $\delta_2(E)$ is then also resonating near $E = E_R$.

In other words, the phase shifts are rather alike in a region surrounding the resonance, but can possibily be very different from each other sufficiently far away from E_R. Let us recall the expression of the Jost function

$$f_l(k)=\prod_n\left(1-\frac{E_n}{E}\right)\exp\left[-\frac{1}{\pi}\int_0^\infty\frac{\delta(E')\,\mathrm{d}E'}{E'-E+i\eta}\right], \tag{11}$$

where the product extends over the bound states. Due to the cut-off introduced by the function $|N_l(k)|^2$, in the functions $(1-E_n/E)$ we can safely put $E=E_R$. Apart from a constant we get then for the moduli of the two Jost functions corresponding to $\delta_1(E)$ and $\delta_2(E)$:

$$|f_l^{(2)}(k)|\simeq|f_l^{(1)}(k)|\exp\left[-\frac{P}{\pi}\int_0^\infty\frac{\varepsilon(E')\,\mathrm{d}E'}{E'-E}\right], \tag{12}$$

where P stands for the « principal value » of the integral. Let us write the integral on the right-hand side of eq. (12) as follows:

$$\frac{P}{\pi}\int_0^\infty\frac{\varepsilon(E')\,\mathrm{d}E'}{E'-E}=\frac{P}{\pi}\int_{E_R-a}^{E_R+a}\frac{\varepsilon(E')\,\mathrm{d}E'}{E'-E}+\frac{1}{\pi}\left[\int_0^{E_R-a}+\int_{E_R+a}^\infty\right]\frac{\varepsilon(E')\,\mathrm{d}E'}{E'-E}. \tag{13}$$

The first integral on the right-hand side of eq. (13) is small because $\varepsilon(E)$ is small in the interval of integration. To be precise, it can immediately be proved that this integral vanishes in the limit as $\varepsilon(E)\to 0$ in the considered energy interval. As for the second integral on the right-hand side, we remark that in the energy interval which is relevant for the integration of eq. (6), *i.e.* for $|E-E_R|<a$, the denominator is a slowly varying function of E so that we can safely put $E=E_R$ there. We get then

$$|f_l^{(2)}(k)|\simeq\mathrm{const}\,|f_l^{(1)}(k)|\,. \tag{14}$$

From eq. (6) we also see that the corresponding decay laws will be practically identical over many lifetimes if hypotheses i)-iii) are satisfied.

We have therefore completed the proofs of theorems *A*) and *B*). In conclusion we can then say that for all physical effects which can be related to the interpretation of an isolated resonance as a formation of an unstable state, the only relevant quantity is the phase shift in a narrow region near the resonance energy, and that small changes of the phase shift in that region induce correspondingly small changes in the decay law of the unstable system.

● RIASSUNTO

Nel contesto di una teoria del decadimento proposta dagli autori, si stabiliscono due teoremi che sono essenziali per garantire la coerenza fisica del formalismo.

Резюме не получено.