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A Class of Inhomogeneous Gödel-Type Models.

M. J. REBOUÇAS (*)

School of Mathematical Sciences, Queen Mary College
Mile End Road, London E1 4NS, *England*

J. TIOMNO

Centro Brasileiro de Pesquisas Físicas, CNPq/CBPF
Rua Dr. Xavier Sigaud 150, 22290 *Rio de Janeiro*, RJ, *Brasil*

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Summary. — A class of inhomogeneous Gödel-type solutions of Einstein's field equations is studied. The source of the geometries is a fluid with heat flow plus a scalar field. The thermodynamic properties of one specific solution are discussed. When the heat flux vanishes, space-time homogeneous solutions, previously found by the authors, result.

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1. – Introduction.

Some years ago GÖDEL (1) (1949) presented a cosmological solution of Einstein's gravitational field equations of the type

$$ds^2 = \big(dt + H(x)\,dy\big)^2 - D^2(x)\,dy^2 - dx^2 - dz^2\,, \tag{1.1}$$

which we call Gödel-type solution.

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(1) K. GÖDEL: *Rev. Mod. Phys.*, **21**, 447 (1949).

Gödel solution is given by

$$H = \exp[mx], \qquad D = \frac{\exp[mx]}{\sqrt{2}} \tag{1.2}$$

with m constant.

Since Gödel's discovery many attempts have been made to find new exact cosmological solutions of Einstein's equation of the Gödel-type as quoted by REBOUÇAS and TIOMNO (2). The class of all space-time homogeneous Gödel-type metrics was considered for the first time by RAYCHAUDHURI and THAKURTA (3), who found the necessary conditions for a Riemannian Gödel-type manifold to be space and time (ST) homogeneous. In a recent paper (2) we have shown that Raychaudhuri-Thakurta conditions are not only necessary but also sufficient. In that paper we also presented the most general irreducible set of Gödel-type ST homogeneous metrics.

A new class of ST homogeneous solutions of Einstein equations is given in that paper by the introduction of an uniform scalar field. The limiting solution with a pure scalar field has no noncausal region which is present in the previous solutions. A whole family of solutions with ST homogeneous metrics and with no noncausal region was found by OLIVEIRA, TEIXEIRA and TIOMNO (4) with a spinning fluid as a source (Einstein-Cartan formulation), thus exhausting the spectrum of the Rebouças-Tiomno (2) classification.

In this paper we discuss a general class of inhomogeneous Gödel-type solutions. The source of these geometries is a scalar field plus a fluid which has not been thermalized, so that there is a heat flux. As in Bradley and Sviestins (5) paper we relate the heat flow to a temperature and discuss some thermodynamics questions for a specific solution. This was not done in previous solutions with heat flow, as in the papers of Novello (6), Novello and Rebouças (7), Ray (8), Rebouças and Lima (9), Bergman (10), Som and Santos (11) and Rebouças (12).

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(2) M. J. REBOUÇAS and J. TIOMNO: *Phys. Rev. D*, **28**, 1251 (1983), and references therein quoted on ST homogeneous Gödel-type solutions.

(3) A. K. RAYCHAUDHURI and S. N. G. THAKURTA: *Phys. Rev. D*, **22**, 802 (1980).

(4) J. D. OLIVEIRA, A. F. DA F. TEIXEIRA and J. TIOMNO: *X International Conference on General Relativity and Gravitation*, contributed papers (Padova, 1983), p. 507.

(5) J. M. BRADLEY and E. SVIESTINS: *Gen. Rel. Grav.*, **16**, 1119 (1984) . Also E. SVIESTINS: *Gen. Rel. Grav.*, in press.

(6) M. NOVELLO: *Phys. Lett. A*, **69**, 309 (1979).

(7) M. NOVELLO and M. J. REBOUÇAS: *Astrophys. J.*, **225**, 719 (1978).

(8) D. RAY: *J. Math. Phys.* (*N. Y.*), **24**, 1797 (1980).

(9) M. J. REBOUÇAS and J. A. S. LIMA: *J. Math. Phys.* (*N. Y.*), **22**, 2699 (1981).

(10) O. BERGMAN: *Phys. Lett. A*, **82**, 283 (1981).

(11) M. M. SOM and N. O. SANTOS: *Phys. Lett. A*, **87**, 89 (1981).

(12) M. J. REBOUÇAS: *Nuovo Cimento B*, **67**, 120 (1982).

2. – Source and field equations.

We consider a class of exact ST nonhomogeneous solutions of Einstein scalar field equation in which the Raychaudhuri-Thakurta conditions are not fulfilled.

The Gödel-type metric may be written as

$$ds^2 = \eta_{AB}\theta^A\theta^B = (\theta^0)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 , \tag{2.1}$$

where the 1-forms $\theta^A = e^A{}_\alpha \, dx^\alpha$ are obviously given by

$$\begin{cases} \theta^0 = dt + H(x)\, dy , & \theta^1 = dx , \\ \theta^2 = D(x)\, dy , & \theta^3 = dz . \end{cases} \tag{2.2}$$

Here and in what follows we shall use the notation of Rebouças and Tiomno ([2]). There it was shown that the scalar field

$$S = az + \alpha , \tag{2.3}$$

where a and α are arbitrary constants, satisfies the source-free scalar field equation, and has the energy-momentum tensor in the tetrad basis:

$$T^{(s)}_{00} = - T^{(s)}_{11} = - T^{(s)}_{22} = T^{(s)}_{33} = \frac{a^2}{2} . \tag{2.4}$$

The material content source of our geometries is a fluid of density ϱ and pressure p with heat flow q_A plus the scalar field S, *viz.*,

$$T_{AB} = \varrho V_A V_B - p h_{AB} + 2 q_{(A} V_{B)} + T^{(s)}_{AB} , \tag{2.5}$$

where

$$h_{AB} = \eta_{AB} - V_A V_B , \qquad V_A = V^A = \delta^A_0 \quad \text{and} \quad q_A V^A = 0 . \tag{2.6}$$

The rotation of the matter relative to the compass of inertia is

$$2\Omega(x) = H'/D \tag{2.7}$$

for the velocity field V^A given by (2.6). Here the prime indicates x-derivative. It has vanishing shear, expansion and acceleration.

The Einstein field equations with cosmological constant in tetrad frame are

$$R_{AB} = K(T_{AB} - \tfrac{1}{2} T\eta_{AB}) - \Lambda\eta_{AB} . \tag{2.8}$$

Equation (2.8), where K and Λ are the gravitational and cosmological constants, for a Gödel-type metrics, has solutions for a perfect fluid with heat flux plus the scalar field S given by

$$K(\varrho - p) = -2(\Lambda + Ka^2)\,, \tag{2.9}$$

$$K(\varrho + p) = 2\Omega^2 - Ka^2 = 4\Omega^2 - m^2\,, \tag{2.10}$$

$$Kq_A = Kq_{(2)}\delta^2_A = \Omega'\delta^2_A \tag{2.11}$$

and

$$m^2(x) \equiv \frac{D''}{D} = Ka^2 + 2\Omega^2\,, \tag{2.12}$$

to which we impose the additional energy conditions $\varrho - p \geqslant 0$ and $\varrho + p \geqslant 0$.

If $\Omega' \neq 0$, these equations define a general class of inhomogeneous Gödel-type metrics. Any such a solution can be found, given a and $\Lambda < 0$ satisfying $\varrho - p \geqslant 0$ and if we choose an arbitrary function $D(x)$ such that $D''/D > 2Ka^2$. Then eq. (2.12) determines $\Omega(x)$, except for a sign, and such that $\varrho + p > 0$ is satisfied. Thus ϱ, p and q_A are found from eqs. (2.9) to (2.11). Finally from eq. (2.7) we find $H(x)$ by quadrature.

For Ω constant the ST homogeneous solutions of the scalar plus Einstein's equations arise which were previously determined by REBOUÇAS and TIOMNO (2).

3. – Fluid temperature and thermodynamical aspects.

We may obtain the fluid temperature distribution from $q_A(x) = e_A{}^\alpha q_\alpha(x)$ if we use the equation due to ECKART (13), *viz.*,

$$q_\alpha = \chi(T, x^i, t)\, h_\alpha{}^\beta(T_{,\beta} + Ta_\beta)\,, \tag{3.1}$$

where $T(x^i, t)$ is the temperature, $\chi > 0$ is the thermal conductivity and $a_\beta = V_{\alpha;\beta}V^\alpha$ is the acceleration, which, as said before, vanishes for our problem. Here , means ordinary derivative and ; covariant derivative.

In tetrad form eq. (3.1) becomes

$$q_A = \chi[T_{|A} - T_{|B}V^BV_A + (V_{A|B} + \gamma_{ABC}V^C)V^BT]\,, \tag{3.2}$$

where $T_{|A} = T_{,\alpha}e^\alpha{}_A$, $V_{A|B} = V_{A,\mu}e_B{}^\mu$ and γ_{ABC} are the Ricci coefficients.

In our case, as referred above, only the terms in $T_{,\mu}$ remain and eq. (3.2)

(13) C. ECKART: *Phys. Rev.*, **58**, 919 (1940).

becomes

$$q_A = -\chi T_{,0} \frac{H}{D} \delta_A^2 + \frac{\chi}{D} T_{,i} \delta_A^i . \tag{3.3}$$

From eqs. (2.11) and (3.3) we find $T = T(t, y)$.

In this paper, for the sake of simplicity, we shall consider only solutions of Einstein's equations of the form

$$D = D_0 \exp\left[\frac{Ka^2}{2} x^2 + m_0 x\right], \tag{3.4}$$

where D_0 and m_0 are arbitrary constants. Thus we find

$$\Omega = \pm \frac{1}{\sqrt{2}} (Ka^2 x + m_0) , \tag{3.5}$$

$$H = \pm \sqrt{2} D , \tag{3.6}$$

$$q_A = \pm \frac{1}{\sqrt{2}} a^2 \delta_A^2 . \tag{3.7}$$

We shall choose the sign minus in eqs. (3.5) to (3.7) for reasons to become clear in what follows.

From Eckart's equation (3.1) we obtain

$$q_A = \chi \delta_A^2 \left(- T_{,0} \sqrt{2} + \frac{1}{D} T_{,2}\right). \tag{3.8}$$

If we assume, for simplicity,

$$\chi = \chi(T) = \frac{\mathrm{d}\lambda}{\mathrm{d}T}, \qquad T = T(t), \tag{3.9}$$

and compare (3.7) with (3.8), we find

$$\frac{1}{\sqrt{2}} a^2 = \sqrt{2} \frac{\mathrm{d}\lambda}{\mathrm{d}t} .$$

Thus

$$\lambda = \lambda_0 + \beta t$$

with

$$\beta = \tfrac{1}{2} a^2 , \qquad \lambda_0 = \text{const} .$$

One solution could be

$$\chi = \chi_0 = \text{const} . \tag{3.10}$$

Then from eq. (3.9) we find

$$T = \frac{a^2}{2\chi_0}(t - t_0)\,. \tag{3.11}$$

Notice that, in spite of both $g_{\mu\nu}$ and S being stationary, T changes with time. One may argue that this is because heat is flowing into the volume element in question. This is not true, however, because the divergence of the heat current q^μ vanishes everywhere. Also the absence of viscosity prevents heat production by friction and no mechanical work is done. Thus the heat must be produced by an exothermic nuclear and/or chemical process, like the heat production in the stars. We have then a burning universe.

However, the burning rate must increase with time to keep the temperature increasing as the combustible material will diminish in time. Thus we shall take $\chi(T) \neq \chi_0$ and T cannot be given by (3.11).

Now we assume the more plausible time dependence of T, which, for constant specific heat, implies constant rate of reaction per reacting particle:

$$T = T_0(1 - \exp[-\gamma t]) = T_0\left(1 - \exp\left[-\frac{\gamma}{\beta}(\lambda - \lambda_0)\right]\right), \tag{3.12}$$

where γ is a positive constant. Thus from eq. (3.9) we have

$$\chi^{-1} = \frac{\mathrm{d}T}{\mathrm{d}\lambda} = \frac{\gamma}{\beta}T_0 \exp\left[-\frac{\gamma}{\beta}(\lambda - \lambda_0)\right] = \frac{\gamma}{\beta}(T_0 - T)\,, \tag{3.13}$$

or

$$T_0 - T \equiv T_0 \exp[-\gamma t] = \frac{\beta}{\gamma\chi} \geqslant 0\,. \tag{3.14}$$

Therefore, the thermal conductivity $\chi(t)$ starts from the finite value $\beta/\gamma T_0$ at $T = t = 0$ and tends to infinite as $t \to \infty$, when $T \to T_0$.

In conclusion, we should like to point out some drawbacks of this cosmological model of a burning universe.

First, for times prior to $t = 0$ the absolute temperature becomes negative. This may be a consequence of the fact that the Eckart's equation (3.1) was used in the linear approximation. If the friction is taken into account, possibly we shall extend the range of positive T to $t = -\infty$.

Second, although the energy condition $\varrho \pm p \geqslant 0$ can be imposed as

$$\varrho \geqslant 0\,, \qquad \varrho \geqslant p\,, \tag{3.15}$$

the condition

$$\varrho > -p \qquad \text{with } p < 0 \tag{3.16}$$

cannot be imposed in the region

$$-\left(\frac{1}{a\sqrt{K}}+\frac{m_0}{Ka^2}\right)<x<\left(\frac{1}{a\sqrt{K}}-\frac{m_0}{Ka^2}\right), \tag{3.17}$$

thus leading to a violation of causality as in this region the velocity of the sound overcomes the velocity of light. Violation of causality in a region of space, however, is known to occur in Gödel-type metrics (1-3) and other cosmological models.

We were not able to obtain other models without these drawbacks. In any case the present model stands for its simplicity and the possibility of a complete analysis.

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● RIASSUNTO (*)

Si studia un gruppo di soluzioni inomogenee del tipo di Gödel delle equazioni di campo di Einstein. La sorgente delle geometrie è un fluido con flusso di calore piú un campo scalare. Si discutono le proprietà termodinamiche di una soluzione specifica. Quando il flusso di calore si annulla, si ottengono le soluzioni omogenee nello spazio-tempo, trovate precedentemente dagli autori.

(*) *Traduzione a cura della Redazione.*

Класс неоднородных моделей Гёделевского типа.

Резюме (*). — Исследуется класс неоднородных решений Гёделевского типа полевых уравнений Эйнштейна. Источник геометрий представляет жидкость с потоком тепла плюс скалярное поле. Обсуждаются термодинамические свойства одного частного решения. Когда поток тепла обращается в нуль, получаются пространственно-временные однородые решения, ранее найденные автором.

(*) *Переведено редакцией.*