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Gravitational Bounce.

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Summary. — A solution of Einstein's field equations is derived which represents a thin spherical shell of charged dust falling in the spherically symmetric field of a charged massive body placed at its centre. Under suitable conditions the shell bounces reversibly at a nonzero minimal radius. A bounce is still possible even after the shell has collapsed inside the Schwarzschild sphere, so that the collapse as viewed externally is irreversible. The apparent paradox is explained in terms of the lattice-like structure of the analytically extended Reissner-Nordström manifold. The possible relevance of the results to the problem of realistic gravitational collapse is discussed.

1. – Introduction.

We shall be concerned in this paper with a conventional problem in general relativity: the gravitational collapse of a charged thin spherical shell falling in a spheri-symmetric external field. The study of such simple artificial problems, while of no direct relevance to astrophysics, can none the less serve a useful purpose, since it brings into relief basic issues of principle in the general relativistic theory of collapse which are still far from understood.

General relativity leads to the following picture for the evolution of a contracting spherical body [1]. Once the compression passes a certain critical limit, characterized roughly by the interior (Newtonian) potential becoming comparable with c^2, the subsequent history is one of continuing collapse which cannot be halted by pressure forces. The irreversibility of this

[1] See, *e.g.*, YA. B. ZEL'DOVICH and I. D. NOVIKOV: *Sov. Phys. Uspekhi*, **7**, 763 (1965); **8**, 522 (1966).

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picture is surprising, and differs radically from the corresponding Newtonian picture, where the motion is in general oscillatory. If one examines the relativistic derivation to see how the element of irreversibility enters, one finds that it stems from two largely unconnected causes.

A) External irreversibility: development of an event horizon at $r = 2m$. The surface of the contracting spherical body passes (in finite proper time) within the critical Schwarzschild sphere $r = 2m$. To an external observer, light emitted from this sphere suffers infinite gravitational red-shift, and $r = 2m$ therefore appears as an event horizon which the contracting body seems to be approaching asymptotically as $t \to \infty$. If ordinary ideas of causality are to be maintained, he can never see the body re-emerge from this sphere.

B) Intrinsic irreversibility: spacelike character of the curves $r = \text{const}$ *near* $r = 0$. The exterior (Schwarzschild) field of the body, analytically extended to $r = 0$, has the property that the curves $r = \text{const} < 2m$ are spacelike. The history of a particle on the surface of the body is a timelike curve of the exterior manifold. It is easy to see that the particle can reverse its inward motion at $r = r_0 < 2m$ only if *a*) its world-line is momentarily spacelike, and *b*) it subsequently travels into the past. Assuming that classical general relativity remains valid even under the extreme conditions prevailing near $r = 0$, one is forced to the conclusion that no rebound is possible and that the entire mass piles up irreversibly on the singular curve $r = 0$.

Of these two arguments, *A*) seems on surer ground, since it does not depend on extrapolation to extreme conditions. For masses of astrophysical interest, compression to $r = 2m$ does not produce immoderate densities or curvatures. Modifications due to quantized gravitation, possible inapplicability of Einstein's field equations at very high curvatures, or other new physical effects of an unanticipated kind might profoundly affect the situation near $r = 0$, but should not be important near $r = 2m$. The exact nature of such modifications is, of course, unknown. One could try to take their effects into account in a crude way by supposing that the standard Schwarzschild metric is modified (2) to

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r} + \frac{a}{r^2} + \ldots\right)^{-1} \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - \left(1 - \frac{2m}{r} + \frac{b}{r^2} + \ldots\right) \mathrm{d}t^2\,,$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\,.$$

(2) That the unmodified Schwarzschild metric cannot be trusted near the singular curve $r = 0$ is clear from the fact that this curve is spacelike: a body which has collapsed irreversibly to $r = 0$ would have to travel faster than light!

If the usual astronomical predictions of Einstein's theory are to be preserved, the constants a, b would have to be small compared with m^2. In that case, we still have an event horizon (the sphere on which $g_{00} = 0$), but the arguments of B) are clearly liable to break down. The possibility cannot be ruled out that a collapsing spherical body reverses its motion near $r = 0$ and re-expands. The intriguing question is how such a picture can be reconciled with the apparent irreversibility of the collapse as seen by an external observer.

We shall illustrate some of the possibilities by focussing attention on the special line element [3]

$$\mathrm{d}s^2 = f^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - f\,\mathrm{d}t^2\,, \qquad f = 1 - 2m/r + e^2/r^2\,, \tag{1}$$

which is formally the Reissner-Nordström metric for the gravitational field of a charged particle. In Sect. **2** and **3** we derive the equation of motion of a thin spherical shell in such a field. It is found (Sect. **4**, **7**) that bounce can occur under a great variety of conditions. In particular, the shell can bounce inside an event horizon. In that case the manifold represented by (1) is incomplete. If it is extended analytically (Sect. **5**) in the manner of GRAVES and BRILL [4], the extended space-time appears as a periodic lattice of geometrically similar asymptotically flat spaces, joined by « tunnels » in which space is closed. Re-emergence of the rebounding shell from the event horizon then takes place in a new, distinct space (Sect. **6**, **7**).

Our results are similar to those recently obtained by NOVIKOV [5], who has considered the homologous collapse of a uniformly charged ball of dust. The shell model has the advantage that the complete solution can be given in a simple explicit form. Novikov's results can be recovered as a special case (Sect. **7**).

The paper concludes with some general remarks (Sect. **8**).

2. – Dynamics of a thin shell.

The dynamics of a thin shell in vacuo has been considered in a previous paper by one of us [6]. The case where the shell falls in a continuous medium with nonvanishing energy tensor is a straightforward generalization, and we

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(3) As an illustration, it may be remarked that (1) represents the external field of an (uncharged) spherical body in the theory of F. HOYLE and J. V. NARLIKAR: *Proc. Roy. Soc.*, A **294**, 138 (1966).

(4) J. C. GRAVES and D. R. BRILL: *Phys. Rev.*, **120**, 1507 (1960).

(5) I. D. NOVIKOV: *JETP Lett.*, **3**, 142 (1966).

(6) W. ISRAEL: *Nuovo Cimento*, **44** B, 1 (1966); ibid, **48** B, 463 (1967).

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shall sketch it briefly, referring the reader to the previous paper for a fuller account of the basic ideas.

Let the timelike hypersurface Σ divide space-time into two parts V_-, V_+ which both contain Σ as part of their boundaries and are otherwise disjoint. Let $\boldsymbol{n}$ (directed from V_- to V_+) be the unit spacelike normal to Σ, ξ^a a set of intrinsic co-ordinates for Σ, $\boldsymbol{e}_{(a)}$ the triad of holonomic basis vectors tangent to Σ and associated with ξ^a (*i.e.* an infinitesimal displacement in Σ has the form $\boldsymbol{e}_{(a)}\,d\xi^a$) [7]. Further, let K^-_{ab}, K^+_{ab} denote the extrinsic curvatures of Σ associated with its imbeddings in V_-, V_+ respectively. Then Σ is the history of a thin shell if

$$\gamma_{ab} \equiv K^+_{ab} - K^-_{ab}$$

is nonvanishing. The surface energy tensor of the shell is given by the Lanczos equations [6]

$$\gamma_{ab} - g_{ab}\gamma = -8\pi S_{ab} \qquad (\gamma \equiv g^{ab}\gamma_{ab})\,, \tag{2}$$

the analogue of Einstein's field equations

$$G_{\alpha\beta} = -8\pi T_{\alpha\beta} \tag{3}$$

for the surrounding continuous medium.

In general, one has the following eight relations [6] between the extrinsic curvatures $K^\pm_{ab}$ and the normal components of the Einstein tensor on Σ:

$$^3R - K_{ab}K^{ab} + K^2|^\pm = -2G_{\alpha\beta}n^\alpha n^\beta|^\pm\,, \tag{4}$$

$$K^b_{a;b} - \partial_a K|^\pm = -G_{\alpha\beta}e_{(a)}{}^\alpha n^\beta|^\pm\,. \tag{5}$$

Here, $K \equiv K_p{}^p$, ^3R is the intrinsic curvature invariant of Σ, and the semicolon indicates covariant differentiation with respect to the intrinsic metric of Σ. The jumps of (4) and (5) across Σ can be written, with use of (2) and (3), in the form

$$S^{ab}(K^+_{ab} + K^-_{ab}) = 2[T_{\alpha\beta}n^\alpha n^\beta]\,, \tag{6}$$

$$S^b_{a;b} = -[T_{\alpha\beta}e_{(a)}{}^\alpha n^\beta]\,. \tag{7}$$

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[7] We adopt the following conventions: $G = c = 1$, signature of metric $+ + + -$, Greek indices refer to 4-dimensional, Latin indices to 3-dimensional quantities. Limits of the field quantity Ψ as the event P on Σ is approached from V_-, V_+ respectively are denoted by $\Psi(P)|^-$, $\Psi(P)|^+$. In Sect. **2** square brackets denote jump discontinuities: $[\Psi] \equiv \Psi|^+ - \Psi|^-$.

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We now specialize to the case of a *coherent shell of dust*, characterized by the surface energy tensor

$$S^{ab} = \sigma u^a u^b \,, \tag{8}$$

where the unit timelike vector $u^a = \mathrm{d}\xi^a/\mathrm{d}\tau$ tangent to Σ represents the 4-velocity of the dust particles and σ is the sum of their rest masses per unit area.

From (7) and (8) it follows that

$$(\sigma u^b)_{;b} = u^a [T_{\alpha\beta} e_{(a)}{}^\alpha n^\beta] = [T_{\alpha\beta} u^\alpha n^\beta] \tag{9}$$

$$\sigma u_{c;b} u^b = -(\delta_c{}^a + u_c u^a)[T_{\alpha\beta} e_{(a)}{}^\alpha n^\beta] \,. \tag{10}$$

The two 4-accelerations $\delta u^\alpha/\delta\tau|^\pm$ of a dust particle, as measured in V_-, V_+ can be resolved into components tangential and normal to Σ according to [6]

$$\delta u^\alpha/\delta\tau|^\pm = e_{(a)}{}^\alpha u^a{}_{;b} u^b - n^\alpha K_{ab} u^a u^b|^\pm \,. \tag{11}$$

From (11) and (16) we immediately obtain

$$\sigma n_\alpha \delta u^\alpha/\delta\tau|^+ + \sigma n_\alpha \delta u^\alpha/\delta\tau|^- = -2[T_{\alpha\beta} n^\alpha n^\beta] \,, \tag{12}$$

and also

$$n_\alpha \delta u^\alpha/\delta\tau|^+ - n_\alpha \delta u^\alpha/\delta\tau|^- = -\gamma_{ab} u^a u^b = 4\pi\sigma \,, \tag{13}$$

with the aid of (2) and (8).

3. – Charged spherical shell in a spheri-symmetric electrovac field.

We consider a charged spherical shell of dust falling in the electrovac field produced by a spherically symmetric concentration of mass and charge near its centre. For such a spherically symmetric (static or nonstatic) universe, an extension [8] of Birkhoff's theorem shows that the line element is reducible to the standard Reissner-Nordström metric (1)—with appropriate parameters e, m—in any region free of matter.

Let $r = R(\tau)$ be the equation of Σ, the history of the shell, and

$$(\mathrm{d}s^2)_\Sigma = \{R(\tau)\}^2 \mathrm{d}\Omega^2 - \mathrm{d}\tau^2 \tag{14}$$

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[8] B. HOFFMANN: *Quart. Journ. Math.*, **4**, 179 (1933), also in *Recent Developments in General Relativity* (Warsaw, 1962), p. 279; A. DAS: *Progr. Theor. Phys.*, **24**, 915 (1960).

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be its intrinsic metric, so that τ is proper time measured along the streamlines θ, $\varphi = \text{const}$. The interior and exterior line elements may be written

$$(\mathrm{d}s^2)_- = \{f_-(r)\}^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - f_-(r)\,\mathrm{d}t_-^2\,, \qquad (r < R(\tau))\,, \tag{15}$$

$$(\mathrm{d}s^2)_+ = \{f_+(r)\}^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 - f_+(r)\,\mathrm{d}t_+^2\,, \qquad (r > R(\tau))\,, \tag{16}$$

where

$$f_-(r) = 1 - 2m_1/r + e_1^2/r^2\,, \qquad f_+(r) = 1 - 2m_2/r + e_2^2/r^2\,. \tag{17}$$

Thus, the shell has charge $e_2 - e_1$ and gravitational mass $m_2 - m_1$.

Both (15) and (16) must induce the same intrinsic metric, namely (14), on Σ. Comparison of the coefficients of $\mathrm{d}\Omega^2$ confirms that the interior and exterior radial co-ordinates agree on Σ. Further,

$$\mathrm{d}\tau^2 = f_-(R)\,\mathrm{d}t_-^2 - \{f_-(R)\}^{-1}\,\mathrm{d}R^2 = f_+(R)\,\mathrm{d}t_+^2 - \{f_+(R)\}^{-1}\,\mathrm{d}R^2\,. \tag{18}$$

This fixes the relation between t_- and t_+ on Σ, and verifies, as expected, that the simultaneous imbedding of Σ in V_-, V_+ is possible.

We proceed to write out explicitly the dynamical equations (9), (10), (12), (13). Since $\boldsymbol{u}$, $\boldsymbol{n}$ are orthogonal unit vectors in the 2-space of $r(\equiv x_\pm^1)$, $t_\pm(\equiv x_\pm^4)$, we have

$$u_+^\alpha = \mathrm{d}x_+^\alpha/\mathrm{d}\tau = (\dot{R},\, 0,\, 0,\, X_+)\,, \tag{19}$$

$$n_\alpha^+ = (X_+,\, 0,\, 0,\, -\dot{R})\,, \tag{20}$$

where

$$f_+ X_+ = \{f_+(R) + \dot{R}^2\}^{\frac{1}{2}}\,, \qquad \dot{R} \equiv \mathrm{d}R/\mathrm{d}\tau\,, \tag{21}$$

with corresponding expressions for u_-^α, n_α^-. By intrinsic differentiation of $\boldsymbol{u}\cdot\boldsymbol{u} = -1$, we find

$$0 = u_\alpha\,\delta u^\alpha/\delta\tau|^+ = f_+^{-1}\,\dot{R}\,\delta^2 R/\delta\tau^2 - f_+ X_+\,\delta^2 t_+/\delta\tau^2\,,$$

and this may be used to eliminate $\delta^2 t_+/\delta\tau^2$ from

$$n_\alpha\,\delta u^\alpha/\delta\tau|^+ = X\,\delta^2 R/\delta\tau^2 - \dot{R}\,\delta^2 t/\delta\tau^2|^+$$

yielding

$$n_\alpha\,\delta u^\alpha/\delta\tau|^+ = (fX)^{-1}\,\delta^2 R/\delta\tau^2|^+ = (f_+X_+)^{-1}\{\mathrm{d}^2R/\mathrm{d}\tau^2 + \tfrac{1}{2}\,\mathrm{d}f_+(R)/\mathrm{d}R\}\,, \tag{22}$$

with an analogous expression for $n_\alpha\,\delta u^\alpha/\delta\tau|^-$.

The Reissner-Nordström metric (1) is associated with the energy tensor

$$-T^4_4 = -T^1_1 = T^2_2 = T^3_3 = e^2/8\pi r^4 \tag{23}$$

(other components zero), so that in our case

$$T^\beta_\alpha u^\alpha n_\beta|^\pm = 0\,, \tag{24}$$

$$[T^{\alpha\beta} n_\alpha n_\beta] = ({e_1}^2 - {e_2}^2)/8\pi r^4\,. \tag{25}$$

From (9) and (24),

$$(\sigma u^b)_{;b} = 0\,, \tag{26}$$

expressing conservation of the proper mass (or the number of particles) of the shell. For the co-moving co-ordinates employed in (14), $u^b = (0, 0, 1)$ and (26) simplifies to

$$4\pi R^2 \sigma = \text{const}\,. \tag{27}$$

Equations (12) and (13) yield, when (22) and (25) are substituted into them,

$$(f_+X_+)^{-1}\{\ddot{R} + m_2/R^2 - e_2^2/R^3\} + (f_-X_-)^{-1}\{\ddot{R} + m_1/R^2 - e_1^2/R^3\} = \\ = (e_2^2 - e_1^2)/4\pi R^4\sigma\,, \tag{28}$$

$$(f_+X_+)^{-1}\{\ddot{R} + m_2/R^2 - e_2^2/R^3\} - (f_-X_-)^{-1}\{\ddot{R} + m_1/R^2 - e_1^2/R^3\} = 4\pi\sigma\,. \tag{29}$$

We recall from (21) that

$$f_+X_+ = \{1 + \dot{R}^2 - 2m_2/R + e_2^2/R^2\}^{\frac{1}{2}}\,,$$

$$f_-X_- = \{1 + \dot{R}^2 - 2m_1/R + e_1^2/R^2\}^{\frac{1}{2}}\,.$$

Equations (28) and (29) are completely integrable. To obtain a first integral, multiply them together (thus eliminating σ), set $y \equiv \frac{1}{2}(1+\dot{R}^2)$ so that $\ddot{R} = \mathrm{d}y/\mathrm{d}R$, and employ $x \equiv 1/R$ as independent variable. The resulting first-order equation,

$$(y - m_1 x + \tfrac{1}{2}e_1^2x^2)(\mathrm{d}y/\mathrm{d}x - m_2 + e_2^2 x)^2 - (y - m_2 x + \tfrac{1}{2}e_2^2 x^2)(\mathrm{d}y/\mathrm{d}x - m_1 + e_1^2 x)^2 = \\ = 2(e_2^2 - e_1^2)(y - m_1 x + \tfrac{1}{2}e_1^2 x^2)(y - m_2 x + \tfrac{1}{2}e_2^2 x^2)\,,$$

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has a general solution of the form $2y = A + Bx + Cx^2$, where one of the three constants A, B, C is arbitrary. In this way we arrive at the first integral

$$1 + (\mathrm{d}R/\mathrm{d}\tau)^2 = A + B/R + C/R^2\,, \tag{30}$$

with A as the constant of integration, and

$$B = m_1 + m_2 - A(e_2^2 - e_1^2)/(m_2 - m_1)\,, \tag{31}$$

$$4C = A(e_2^2 - e_1^2)^2/(m_2 - m_1)^2 - 2(e_1^2 + e_2^2) + A^{-1}(m_2 - m_1)^2. \tag{32}$$

Substitution of (30) into (28) or (29) now gives

$$4\pi R^2 \sigma = A^{-\frac{1}{2}}(m_2 - m_1)\,, \tag{33}$$

in conformity with (27). For a physically meaningful solution it is necessary that A be nonnegative. Equation (33) enables us to interpret this constant in terms of the binding energy W, since

$$-W \equiv (m_2 - m_1)(1 - A^{-\frac{1}{2}})$$

represents the difference between the gravitational mass $m_2 - m_1$ of the shell (*i.e.* its total energy) and the sum of the rest masses of its constituent particles. It thus represents the contribution to the shell's gravitational mass due to its kinetic and potential energies.

The further integration of (30) would be elementary. However, the various physical possibilities emerge more clearly from a qualitative description of a few representative special cases. This will be our aim in the next few Sections.

4. – Charged shell in vacuo.

Suppose first that no mass or charge is present apart from the shell itself (mass m, charge e). Then $e_1 = m_1 = 0$, $e_2 = e$, $m_2 = m$ and the equation of motion (30) reduces to

$$\{1 + (\mathrm{d}R/\mathrm{d}\tau)^2\}^{\frac{1}{2}} = a - b/R\,, \tag{34}$$

where we have written $A^{\frac{1}{2}} = a$ and

$$b = (a^2e^2 - m^2)/2am\,. \tag{35}$$

If $b < 0$, we can imagine the shell as starting, either from infinity with initial velocity $\dot{R} = -(a^2 - 1)^{\frac{1}{2}}$ (for $a \geqslant 1$), or from rest at a finite maximal radius $R_{\max} = |b|/(1 - a)$ (for $0 < a < 1$). It accelerates as it falls inward and,

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upon reaching $R = 0$, produces a singularity. The subsequent history is therefore a matter of conjecture (the possibility of a rebound is, of course, not excluded).

More definite conclusions can be drawn when $b > 0$ (always obtainable for a shell with given mass and nonvanishing charge by taking a sufficiently large). In this case, it is necessary that $a > 1$. The shell is impelled inwards from infinity with initial speed $(a^2 - 1)^{\frac{1}{2}}$. It is decelerated, and comes to rest at a finite radius $R_{\min} = b/(a-1)$, then re-expands symmetrically to infinity.

We have been concerned with the intrinsic description of the motion, $r = R(\tau)$, as seen by a co-moving observer using the proper time τ. Since τ is related to the time t_- of stationary observers in the interior flat domain by $d\tau^2 = dt_-^2 - dR^2$, we obtain from (34)

$$(\mathrm{d}R/\mathrm{d}t_-)^2 = [(a-1)R - b][(a+1)R - b]/(aR - b)^2 . \tag{36}$$

The denominator does not vanish for $R \geqslant R_{\min}$. The motion as seen by an interior observer is thus qualitatively similar to the intrinsic description just given.

To an external observer, however, the sequence of events may appear quite different. From (34) and (16) we obtain

$$\begin{cases} \left(\dfrac{\mathrm{d}t_+}{\mathrm{d}R}\right)^2 = \dfrac{1}{f^2} + \dfrac{R^2}{f[(a-1)R - b][(a+1)R - b]} , \\ f(R) \equiv [(R - m)^2 + e^2 - m^2]/R^2 , \end{cases} \tag{37}$$

as the equation of motion in terms of the exterior time co-ordinate t_+ (essentially the proper time of a stationary observer with large radial co-ordinate). For a shell with $e^2 > m^2$, f never vanishes and the co-ordinates r, t_+ cover the complete exterior manifold: *qualitatively* the motion seen externally is as previously described. However, there is a quantitative divergence which increases without bound as $e^2 \to m^2$. Consider, for instance, the case where $b > 0$ and $\delta \equiv (e^2 - m^2)^{\frac{1}{2}}$ is small. Before reaching its minimal radius $R_{\min} \approx (a+1) \cdot m/2a < m$, the contracting shell passes through the « stagnant zone » $m - \delta < R < m + \delta$, where the term $1/f^2$ in (37) becomes large and dominant. The proper time required to traverse this zone is very small (of order δ), but the externally observed time is of order m^2/δ. Accordingly, the proper time required to implode from any given radius and re-expand to this radius is finite, but the externally observed time exceeds this by an amount of order $m^2/(e^2 - m^2)^{\frac{1}{2}}$ which is unbounded when $e^2 \to m^2$. We reach the curious conclusion that *an external observer never sees the re-expanding shell if $e^2 \leqslant m^2$; re-expansion to arbitrarily large radius nevertheless occurs in finite time according to an interior or a co-moving observer.*

5. – Analytic completion of Reissner-Nordström manifold for $e^2 \leqslant m^2$.

To resolve the apparent paradox of the previous Section, we require a picture of the exterior manifold when $e^2 \leqslant m^2$. The co-ordinates r, θ, φ, t then no longer furnish a complete map. The problem of analytically completing the Reissner-Nordström manifold has been dealt with by GRAVES and BRILL ([4]) (for $e^2 < m^2$) and by CARTER ([9]) (for $e^2 = m^2$). We shall present a somewhat simplified review.

i) *The case* $e^2 = m^2$. In this case the co-ordinate t remains timelike for all r. Introduce an angular timelike co-ordinate Θ, with range $-\infty < \Theta < \infty$, such that $t/2m = \operatorname{tg}\Theta$ for $-\pi/2 < \Theta < \pi/2$. The (formally) extended line element

$$ds^2 = (1 - m/r)^{-2}\, dr^2 + r^2\, d\Omega^2 - 4m^2(1 - m/r)^2(d\operatorname{tg}\Theta)^2\,,$$

represents a periodic space-time which has a geometrical singularity at $r = 0$ and is otherwise free of singularities. The r, Θ map is subject to local breakdown on the lines $r = m$, $\Theta = (n + \frac{1}{2})\pi$. That $r = m$ is actually a regular part of the manifold can be verified by expressing the line element in a form which is manifestly regular for $r > 0$:

$$ds^2 = 2\, dv'\, dr - (1 - m/r)^2\, dv'^2 + r^2\, d\Omega^2\,, \tag{38}$$

where the advanced time parameter v' is analytically related to r, t by

$$dv' = (1 - m/r)^{-2} dr + dt \qquad (r > m)\,. \tag{39}$$

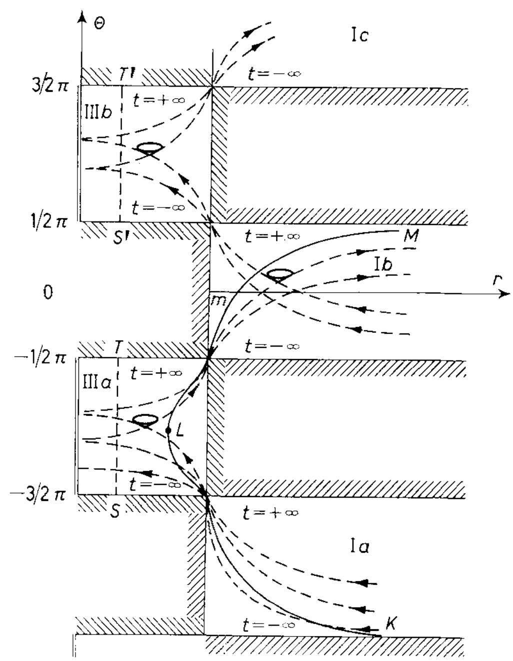

Fig. 1. – Schematic representation of the extended Reissner-Nordström manifold for $e = m$. Shaded sections of the map are not part of the manifold. Dashed lines represent radial null geodesics; the apparent constriction of these lines at $r = m$ is due to local defectiveness of the co-ordinates. The timelike curve KLM represents the history of a thin shell, which implodes in the space I a, reverses its motion at L after passing through the event horizon $r = m$, then re-expands in the space I b.

([9]) B. CARTER: *Phys. Lett.*, **21**, 423 (1966).

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In the v', r chart one can, for instance, follow any incoming radial null geodesic $v' = \text{const}$ originating in a region with $r > m$ (*e.g.* region I*a* of Fig. 1) down to $r = 0$ (in region III*a*). This chart thus provides a regular mapping of two adjoining regions such as I*a* and III*a*. By analogous use of a retarded time parameter we can construct a chart for III*a* and I*b*. An infinite chain of such overlapping co-ordinate patches enables us to follow any null or timelike geodesic down to the singularity $r = 0$ or to indefinitely large values of its affine parameter. Use of the r, Θ map means that allowance must be made for local breakdowns, but has the advantage of providing a clearer over-all picture.

ii) *The case* $e^2 < m^2$. In this case, the quadratic coefficient $f(r)$ in the Reissner-Nordström metric (1) has real unequal factors

$$f(r) = (r - r_1)(r - r_2)/r^2 \qquad (0 < r_2 < r_1)\,. \tag{40}$$

Incoming and outgoing radial null geodesics have equations $v = \text{const}$ and $u = \text{const}$ respectively, where

$$2ku^{-1}\,\mathrm{d}u = f^{-1}\,\mathrm{d}r - \mathrm{d}t\,, \tag{41}$$

$$2kv^{-1}\,\mathrm{d}v = f^{-1}\,\mathrm{d}r + \mathrm{d}t\,, \tag{42}$$

and k is an adjustable constant. In the u, v chart the line element (1) takes the form

$$\mathrm{d}s^2 = (4k^2 f/uv)\,\mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\Omega^2\,. \tag{43}$$

Integration of (41) and (42) yields

$$r + \frac{r_1^2}{r_1 - r_2}\ln\left|\frac{r}{r_1} - 1\right| - \frac{r_2^2}{r_1 - r_2}\ln\left|\frac{r}{r_2} - 1\right| = k\ln|uv|\,, \tag{44}$$

$$t = k\ln|v/u| \qquad (r > r_1 \text{ or } r < r_2)\,. \tag{45}$$

The constants of integration have been placed equal to zero for convenience.

Consider now the chart u_1, v_1 obtained by setting $k = k_1 \equiv r_1^2/(r_1 - r_2)$. We find from (44)

$$u_1 v_1 = \left(\frac{r}{r_1} - 1\right)\left(\frac{r}{r_2} - 1\right)^{-r_2^2/r_1^2}\exp\left[\frac{r_1 - r_2}{r_1^2}\,r\right] \qquad (r > r_2)\,, \tag{46}$$

and (43) exhibits no singularity at $r = r_1$. The chart u_1, v_1 in fact gives a regular mapping of any given subregion of the manifold which has $r > r_2$.

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A (co-ordinate) singularity does develop at $r = r_2$, however, and it is necessary to go over to another chart before that happens.

Define the chart u_2, v_2 by setting $k = k_2 \equiv -r_2^2/(r_1 - r_2)$ in (44) and (45). Then

$$u_2 v_2 = \left(\frac{r}{r_2} - 1\right)\left(1 - \frac{r}{r_1}\right)^{-r_1^2/r_2^2} \exp\left[-\frac{r_1 - r_2}{r_2^2}\, r\right] \qquad (r < r_1)\,, \tag{47}$$

and this provides a regular covering for any subregion with $r < r_1$.

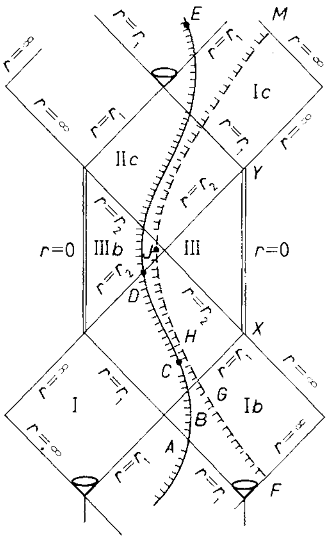

Fig. 2. – Schematic representation of the extended Reissner-Nordström manifold for $e < m$. Null lines are inclined at 45°. *FGHJM* is the history of a shell which collapses from infinity in the asymptotically flat space I *a*, passes through the event horizon $r = r_1$, comes to rest at *J* with a minimal radius smaller than r_2, then re-expands into the asymptotically flat space I *c*. *ABCDE* is the history of an oscillatory shell or uniformly charged sphere. Shading on the curves distinguishes the interior domain.

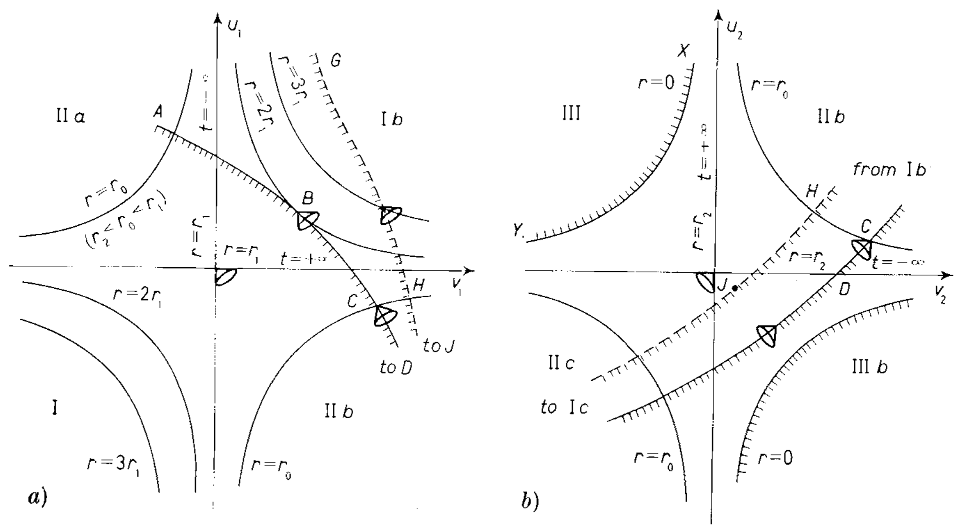

Fig. 3. – Kruskal-type diagrams for portions of the over-all map of Fig. 2, showing the same curves *ABCDE* and *FGHJM*. Figures 3*a*) and *b*) overlap in the region II *b*, and may be regarded as linked together along the curve $r = r_0$, where r_0 is any convenient value between r_1 and r_2.

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In the domain of overlap $r_2 < r < r_1$ the two charts are related by

$$|u_1|^{r_1^2} = |u_2|^{-r_2^2}, \qquad |v_1|^{r_1^2} = |v_2|^{-r_2^2} \qquad (r_2 < r < r_1). \tag{48}$$

The complete manifold for $e^2 < m^2$ is a periodic lattice of alternating regions of type I ($r > r_1$), type II ($r_2 < r < r_1$) and type III ($r < r_2$). Figure 2 (due to CARTER [9]) is a schematic over-all map with local singularities at some of the lattice points. Figures 3*a*) and *b*) are Kruskal-type diagrams which together give a faithful map of any subregion covered by a pair of overlapping charts u_1, v_1 and u_2, v_2.

Because of the cyclic character of the extended manifold, it is natural to raise the question of possible topological identifications. For instance, in Fig. 1 for $e^2 = m^2$, one might postulate that all points $(r, \Theta + 2n\pi)$, $n = 0, \pm 1, \ldots$, represent the same physical event. Such « space-saving » devices are tempting, but they lead to causal paradoxes. In addition, there would be dynamical difficulties connected with gravitational self-interaction, since a world-tube would then intersect a space $t = \text{const}$ more than once. These possibilities will not be considered further here.

6. – Charged shell with $e^2 \leqslant m^2$ in vacuo.

We now return to the discussion, begun in Sect. **4**, of the charged shell in empty space, and proceed to consider the exterior view of the motion for $b \geqslant 0$, $e^2 \leqslant m^2$, when an event horizon exists.

For a shell with $e^2 = m^2$, there is always a special solution ($a = 1$ in (34) (35)) which is static. The shell is then at rest (in neutral equilibrium) at any radius R. The world-line ST (Fig. 1) represents the history of such a shell with $R = \text{const} < m$. The extended manifold displays an infinite sequence of $r = 0$ physical singularities, *e.g.* for $\frac{1}{2}\pi < \Theta < \frac{3}{2}\pi$. If we wish, we can remove these singularities and maintain strict periodicity by introducing an endless number of « re-incarnations » of the shell, *e.g.* at $S'T'$. Space-time is then flat for $r < R$ and all Θ. The result is of some interest mathematically, since it represents a universe containing an event horizon ($r = m$) which is everywhere free of singularity [10].

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[10] This does not contradict a theorem on the inevitability of singularities due to R. PENROSE: *Phys. Rev. Lett.*, **14**, 57 (1965), since two of the hypotheses of that theorem are not satisfied here. In the first place, the manifold with $e = m$ contains no « trapped surface » (even though it contains an event horizon), since outgoing radial null geodesics have $dr/dt = (1 - m/r)^2 \geqslant 0$ and do not converge anywhere. Secondly, the manifold with $e \leqslant m$ does not admit a Cauchy hypersurface.

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The history of a shell with $e^2 = m^2$, $a > 1$ is represented by the timelike curve *KLM* in Fig. 1. To an external observer in the asymptotically flat space Ia the shell implodes, then appears to slow down as it approaches the observer's event horizon $r = m$, reaching it only asymptotically as $t \to \infty$. On the other hand, an observer moving with the shell finds that it passes rapidly and uneventfully through $r = m$, contracts to a nonzero minimal radius at L, then re-expands into a *new* space Ib, identical with Ia in its geometrical properties, but physically distinct from it. It appears that we are forced to accept this resolution of the paradox encountered in Sect. **4**.

The path *FGHJM* (Fig. 2 and 3) of a bouncing shell with $b > 0$, $e^2 < m^2$ has a similar general character: the bounce carries the shell into a different space. A new and peculiar feature is the appearance of a timelike singular curve $r = 0$ (the curve XY) in the vacuum region *outside* the shell. This singularity is connected with a temporary closure of the spaces $t =$ const. It has to be interpreted as the history of a particle with mass m and charge $-e$.

7. – Test shell; uniformly charged ball of dust.

We now turn briefly to the situation where the hollow interior of the shell contains nonvanishing charge e_1 and mass m_1. We shall confine our discussion to the case where the mass $\mu = m_2 - m_1$ and charge $\varepsilon = e_2 - e_1$ of the shell itself are small compared with m_1 and e_1, and for a qualitative description it will be sufficient to consider the limit of a «test shell» ($\mu \to 0$, $\varepsilon \to 0$ with ε/μ finite). In this limit we obtain from (30)

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = -\left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right) + A\left(1 - \frac{\varepsilon}{\mu}\frac{e}{R}\right)^2, \tag{49}$$

where we have written $e_1 \equiv e$, $m_1 \equiv m$.

It is to be expected that (49) will agree with the equation of motion of a radially moving charged test particle in the Reissner-Nordström field (1). The latter is obtainable from the Lagrangian

$$L\left(R, \frac{\mathrm{d}R}{\mathrm{d}t}\right) \equiv \mu A^{-\frac{1}{2}}\left(-g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}t}\frac{\mathrm{d}x^\beta}{\mathrm{d}t}\right)^{\frac{1}{2}} - \varepsilon\varphi_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \mu A^{-\frac{1}{2}}\{f - f^{-1}(\mathrm{d}R/\mathrm{d}t)^2\}^{\frac{1}{2}} - \varepsilon e/R$$

(where $\varphi_\mu = (0, 0, 0, e/r)$ is the electromagnetic vector potential) by forming the Hamiltonian integral $H = \mu$, and it does indeed reproduce (49).

If $|e/m|$, $|\varepsilon/\mu|$ and A are each less than unity, (49) shows that the shell's radius oscillates between a maximum larger than $r_1 = m + \sqrt{m^2 - e^2}$ and a minimum smaller than $r_2 = m - \sqrt{m^2 - e^2}$. The history of the shell is represented by the curve $ABCDE$ ... in Fig. 2 and 3. In each oscillation the shell enters a new space. As viewed by a co-moving or an interior observer the oscillation is strictly periodic; however the path $ABCDE$ ... in the exterior space-time is not cyclic, but subject to a systematic time-shift. If a given maximum occurs for $t = t_0$ in the space Ib (say), then succeeding maxima (in Ic, etc.) occur for $t = t_0 + C$, $t_0 + 2C$, etc. The constant C may be evaluated from (49) and (18) by an integration in the complex plane. For large maximal radius, C is nearly equal to the proper period of pulsation, and both agree closely with the corresponding period calculated from Newtonian theory ([11]).

The occurrence of a bounce is independent of the relative sign of ε and e, so it clearly has little to do with a contest between gravitational attraction and electrostatic repulsion. For a neutral shell ($\varepsilon = 0$) we obtain from (49) by differentiation,

$$\mathrm{d}^2R/\mathrm{d}\tau^2 = -M(R)/R^2\,, \qquad M(R) = m - e^2/R\,.$$

This brings out clearly the physical mechanism responsible for the bounce. Because the electrostatic field energy of the internal charge e is diffused throughout space, less and less of it contributes to the effective interior gravitational mass $M(R)$ as the shell contracts. Ultimately $M(R)$ becomes negative and there is a gravitational repulsion.

Finally, let us note another interesting special case. If we set $\varepsilon/\mu = e/m$, (49) may be regarded as the equation of motion of a particle on the outer surface $r = R(\tau)$ of a uniformly charged ball of dust with total charge e and mass m, which is collapsing homologously. For $e^2 < m^2$, $A < 1$, the motion is again oscillatory, and the history of the surface is given qualitatively by the curve $ABCDE$ ... in Fig. 2 and 3. This example has been discussed by NOVIKOV ([5]).

8. – Concluding remarks.

The collapse of a spherically symmetric body to an event horizon appears as an irreversible process to an external observer. As we have seen, the possibility cannot be ruled out that the body reverses its motion within the event horizon and re-expands symmetrically. It then appears necessary to believe in the existence of other asymptotically flat spaces geometrically similar to but distinct from ours, which will accommodate the re-expansion. This seems at least as fantastic as the alternative of irreversible collapse to virtually point-like dimensions.

([11]) In the Newtonian description the pulsating shell of course always remains in the same space.

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In assessing the possible relevance of these results to realistic gravitational collapse, it is, of course, necessary to keep in mind the various idealizations and hypotheses involved (asymptotic flatness, exact spherical symmetry, analytic continuability of the manifold etc.), each of which could be questioned.

As a null hypersurface (*i.e.* a characteristic hypersurface of the field equations), an event horizon is a possible locus of discontinuities of the field. It is not necessary, and perhaps not physically justified, to insist on analytic continuation of a manifold through an event horizon ([12]).

For the collapse of a stellar mass in our expanding universe, the idealization of asymptotic flatness is justified at the present epoch, but clearly not in the remote past. It will not always be justified in the future if the universe happens to be oscillatory. In fact, a lattice structure for space-time of the general type we have been describing would find a natural interpretation in terms of an oscillatory universe.

The following question is of more immediate concern. To what extent does the development of an event horizon in gravitational collapse (and hence the externally observed irreversibility) depend on the restrictive assumption of spherical symmetry? In astrophysical situations a considerable degree of asymmetry will nearly always be present. It has been claimed ([13]) that *small* departures from spherical symmetry will not affect the qualitative features of the collapse. Even this, however, does not yet seem to have been conclusively established.

* * *

It is a pleasure to express our thanks to Prof. J. L. Synge for the hospitality of the Dublin Institute for Advanced Studies, and to the staff and scholars of the Institute for their interest in this work. We gratefully acknowledge the financial support of the National Research Council of Canada.

([12]) Cf. A. Komar: *Phys. Rev.*, **137**, B 462 (1965).

([13]) A. G. Doroshkevich, Ya. B. Zel'dovich and I. D. Novikov: *Sov. Phys. JETP*, **22**, 122 (1966).

RIASSUNTO (*)

Si deduce una soluzione delle equazioni del campo di Einstein che rappresenta un sottile strato sferico di polvere carica che cade nel campo a simmetria sferica di un corpo carico dotato di massa posto al centro. In opportune condizioni lo strato rimbalza in modo

(*) *Traduzione a cura della Redazione.*

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reversibile a un raggio minimo non nullo. È possibile un rimbalzo anche dopo che lo strato è crollato entro la sfera di Schwarschild, cosicchè il collasso visto dall'esterno è irreversibile. Si spiega l'apparente paradosso per mezzo della struttura reticolare della molteplicità di Reissner-Nordström estesa analiticamente. Si discute la possibile influenza dei risultati sul problema del collasso gravitazionale realistico.

Гравитационная упругость.

Резюме (*). — Выводится решение полевых уравнений Эйнштейна, которое представляет тонкую сферическую оболочку заряженной пыли, падающей в центрально-симметричном поле заряженного массивного тела, помещенного в центре. При подходящих условиях оболочка отскакивает обратимо к ненулевому минимальному радиусу. Упругость еще оказывается возможной, даже после того, как оболочка коллапсировала внутрь сферы Шварцшильда, так что коллапс, когда рассматривается извне, является необратимым, Кажущийся парадокс объясняется в терминах рещетчато-подобной структуры аналитически продолженного многообразия Рейснера-Нордстрема. Обсуждается возможная уместность этих результатов в проблеме релятивистского гравитационного коллапса.

(*) *Переведено редакцией.*

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