IL NUOVO CIMENTO VOL. 81 B, N. 2 11 Giugno 1984

On the Shock-Wave–Generating Function in a Simple Mixture of Gases.

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(ricevuto il 24 Gennaio 1984)

Summary. — We present a study concerning the derivation of the so-called «shock generating function» in a flow of a simple mixture of ν ideal constituents. Due to the analytical complexity of this function, in general, numerical treatments have been discussed in some particular cases ($\nu = 2$ and $\nu = 3$). On the basis of these results, we discovered that, unlike the classical model of a single fluid (where only the supersonic shock is admitted), the mutual interaction of the constituents of the mixture allows the rising of a new type of k-shocks confined within intervals of low-shock Mach numbers, which satisfy the entropy principle. A procedure to symmetrize the system of the original balance equations in terms of the «main field» and the explicit computation of the jump of this field across the shock are also given in appendices.

PACS. 05.70. – Thermodynamics.

1. – Introduction.

Recent studies on quasi-linear hyperbolic systems of the first order have emphasized the important role played by those systems—in the context of continuum theories—which, written in conservative form, admit a supplementary conservation law and a convex density energy. A panorama on the status of these researches can be found in ([1-13]).

([1]) K. O. FRIEDRICHS and P. D. LAX: *Proc. Nat. Acad. Sci. USA*, **68**, 1686 (1971).
([2]) P. D. LAX: in *Contribution to Nonlinear Functional Analysis*, edited by E. H.

An important point worth noticing is that for such systems a « main field » exists, depending only on the field equations and the supplementary conservation law, in terms of which the original balance system assumes a symmetric hyperbolic conservative form, in the sense of Friedrichs [3], through a 4-vector generating function. This 4-vector behaves, therefore, like a sort of potential in that it generates the differential field equations [6,12]. Furthermore, for such systems, important properties for the shock structure hold: the existence of a generalized entropy—the so-called generating function of the shock—, which increases across a noncharacteristic shock [7], and the existence of limits for the speed of the shocks which cannot exceed the characteristic velocities. In a relativistic context this means that the shock velocity can never exceed the velocity of light in vacuum [10].

In the course of this work we shall derive the supplementary conservation law for a flow of a simple mixture of ν ideal constituents [14,15] and then shall construct the « shock generating function » (SGF) for this thermodynamic inviscid system. Numerical models will attempt to lighten some of the main properties of this function.

The plan of this paper is the following: in sect. **2** we present some outlines concerning the theory on which the work is based. In sect. **3** are summarized the governing balance equations of the mixture.

In sect. **4** we give a proof for the convexity of the assumed density entropy for the mixture and also derive the supplementary conservation law. Section **5** is devoted to the shock-generating function which shall be derived in terms of each of the shock Mach numbers of the components of the mixture. In sect. **6** we apply, under simplifying assumptions, the theoretical apparatus to cases of one-dimensional models to make easy numerical calculations. In this context, in *a*) we consider the characteristic polynomial for the eigenvalues of the system, in *b*) we discuss the global behaviour of the temperature jump across the shock and in *c*) we describe the numerical models and the profiles of both the temperature jump and the SGF. Finally, brief concluding remarks

ZARANTONELLO (Academic Press, New York, N. Y., 1971), p. 603.
(3) K. O. FRIEDRICHS: *Commun. Pure Appl. Math.*, **27**, 749 (1974).
(4) K. O. FRIEDRICHS: *Commun. Pure Appl. Math.*, **31**, 123 (1978).
(5) G. BOILLAT: *C. R. Acad. Sci. Paris, A* **274**, 1018 (1972).
(6) G. BOILLAT: *C. R. Acad. Sci. Paris A*, **278**, 909 (1974).
(7) G. BOILLAT: *C. R. Acad. Sci. Paris A*, **283**, 409 (1976*a*).
(8) G. BOILLAT: *C. R. Acad. Sci. Paris A*, **283**, 539 (1976*b*).
(9) G. BOILLAT: in *Wave Propagation*, Corso CIME (Bressanone, 1980).
(10) G. BOILLAT and T. RUGGERI: *C. R. Acad. Sci. Paris, A* **289**, 257 (1979).
(11) G. BOILLAT and T. RUGGERI: *Acta Mech.*, **35**, 271 (1980).
(12) T. RUGGERI and A. STRUMIA: *Ann. Inst. Henri Poincaré, XXXIV A*, **65** (1981).
(13) T. RUGGERI: *Acta Mech.*, **47**, 167 (1983).
(14) I. MÜLLER: *Arch. Ration. Mech. Anal.*, **28**, 1 (1968).
(15) I. MÜLLER: *Thermodynamics and Constitutive Equations*, Corso CIME (Noto, 1982).

are provided in sect. 7. In appendices the procedure to build up the Hessian coefficient matrices for the symmetrized system and the jump of the main field are explicitly derived. These calculations may be of some utility to search numerical solutions.

2. – Recalls and outline of the general theory.

Some general recalls of the theory concerned with are required for a better understanding of the paper. To this aim, let

$$\partial_\omega \boldsymbol{F}^\omega(\boldsymbol{U}) = \boldsymbol{f}(\boldsymbol{U}) \quad \text{or} \quad A^\omega(\boldsymbol{U})\partial_\omega \boldsymbol{U} = \boldsymbol{f}(\boldsymbol{U})\,, \tag{2.1}$$

with $A^\omega \equiv \boldsymbol{\nabla}_U \boldsymbol{F}^\omega(\boldsymbol{U})$ a $N\times N$ matrix ($\omega = 0, i$; $i = 1, 2, 3$; $\boldsymbol{\nabla}_U \equiv \partial/\partial \boldsymbol{U}$ is the gradient with respect to the components of the field $\boldsymbol{U}$), be the first-order quasi-linear hyperbolic system in conservative form describing the flow, where $\boldsymbol{U} = \boldsymbol{U}(x_\omega) \in \mathscr{D} \subseteq \boldsymbol{R}^N$ is the unknown N-vector, $\mathscr{D}$ a convex open domain, $\partial_\omega \equiv \partial/\partial x_\omega$, $\boldsymbol{F}^\omega$ and $\boldsymbol{f}$ column vectors of $\boldsymbol{R}^N$, with the convection that $\boldsymbol{F}_0 = \boldsymbol{U}$ and $x_0 = t$ (time). Supposing that a supplementary conservation law, consequence of the field equations (2.1), exists and is given by

$$\partial_\omega h^\omega(\boldsymbol{U}) = g(\boldsymbol{U})\,, \tag{2.2}$$

then, if $h^0(\boldsymbol{U})$ is a convex density function of $\boldsymbol{U}$, defined in $\mathscr{D}$, the « main field » is given by [6,12]

$$\boldsymbol{U}' = \boldsymbol{U}'(\boldsymbol{U}) = \check{\boldsymbol{\nabla}}_U h^0(\boldsymbol{U}) \quad (\check{}\ \text{denotes transposition}). \tag{2.3}$$

The field $\boldsymbol{U}' \in \boldsymbol{R}^N$ allows us then to build up the « 4-vector–generating function »

$$h'^0 = \boldsymbol{U}'\cdot\boldsymbol{U} - h^0\,, \qquad h'^i = \boldsymbol{U}'\cdot\boldsymbol{F}^i - h^i\,. \tag{2.4}$$

Since from (2.3) it turns out that the Jacobian matrix

$$\frac{\partial \boldsymbol{U}'}{\partial \boldsymbol{U}} = \boldsymbol{\nabla}_U(\check{\boldsymbol{\nabla}}_U h^0) = \frac{\partial^2 h_0}{\partial \boldsymbol{U}\cdot\partial \boldsymbol{U}}$$

is symmetric and, thanks to the convexity of h^0, positive definite, a well-known theorem [16] ensures the global invertibility of the mapping $\boldsymbol{U}'(\boldsymbol{U})$. In view of this, one has also that $\boldsymbol{U} = \boldsymbol{\nabla}_{U'} h'^0$.

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[16] M. BERGER and M. BERGER: *Perspectives in Nonlinearity* (W. Benjamin Inc., New York, N. Y., 1968).

By taking, therefore, U' as new field, it is a simple matter to prove that

$$F^{\omega} = \check{\nabla}_{U'} h'^{\omega} . \tag{2.5}$$

This is an important result in that it expresses that, for all those systems of type (2.1) endowed with a supplementary conservation law (2.2) with h^0 a convex function of U, the vectors F^{ω} are nothing but the transposed of the gradients of h'^{ω} with respect to the new field U'. In other words, the 4-vector h'^{ω} plays the role of a potential and for this reason h'^{ω} are called the « generating functions » of system (2.1) ([12]).

From $(2.4)_1$ it turns out that U and U' are conjugate of the other through the simplest contact transformation (the so-called Le Gendre transformation), so that h'^0 too is a convex function of U'.

The substitution of (2.5) into (2.1) yields the following symmetrized conservative system:

$$\frac{\partial^2 h'^{\omega}}{\partial U' \cdot \partial U'} \partial_{\omega} U' = f(U') , \tag{2.6}$$

which, through the generating functions h'^{ω}, describes the flow in terms of the main field U' (*). System (2.6) is further hyperbolic (in the sense of Friedrichs ([3])) in that the Hessian coefficient matrix of the time derivative of U', $\partial^2 h'^0/\partial U' \cdot \partial U'$, turns out to be positive definite. The convexity of h^0 provides, therefore, a sufficient condition for the hyperbolicity of the original balance system.

As remarked in ([12]), the field U' possesses special privileges both from the mathematical standpoint and for its physical relevance. In fact, U' is not affected by the transformation of U, *i.e.* it is independent of the choice of U. The components of U' are generally expressed in terms of thermomechanics quantities, velocity, absolute temperature and free enthalpy, namely in terms of the « observable » properties of the physical system. They play, furthermore, as pointed out by the mentioned authors, the same role as the so-called « Lagrange multipliers » ([18]) used in connection with the entropy principle proposed and elaborated in ([15,19,20]) in the context of a new thermodynamic theory of the mixtures. The supplementary conservation law (2.2) may be obtained,

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(*) The search for idoneous field vectors, able to reduce in symmetric form some special physical systems, is found first in ([17]). The general approach is given in ([6]) with necessary and sufficient conditions to construct a conservation supplementary law for a given conservative hyperbolic system.

([17]) S. K. GODUNOV: *Sov. Math.*, **2**, 947 (1961).

([18]) LIU I-SHIH: *Arch. Ration. Mech. Anal.*, **41**, 131 (1972).

([19]) I. MÜLLER: *Arch. Ration. Mech. Anal.*, **40**, 1 (1971).

([20]) I. MÜLLER: *Arch. Ration. Mech. Anal.*, **41**, 319 (1972).

in fact, as a linear combination of the field equations by using the components of U' as multiplier factors.

It is worthwhile recalling finally that, for a symmetric conservative system, a well-known existence and uniqueness theorem holds [21]. This theorem ensures that, if the initial data of the problem belong to a Sobolev space $H^S(\mathbf{R}^N)$ with index $S \geqslant 4$, then in the neighbourhood of $t = 0$ the system has a unique solution belonging itself to H^S.

In view of the foregoing statements a number of important properties concerning either the evolution of weak discontinuities, or the shock waves has been straightforwardly demonstrated.

Limiting our speech to the shock wave topic, for what concerns the present paper, the following main properties are worth being recalled [7,10,11]:

i) the existence of a function which generalizes the jump of the thermodynamic entropy, the already mentioned SGF, which is an increasing function of the shock velocity;

ii) the finiteness of the shock speeds, confined within the range of the characteristic velocities, which are real and bounded.

Property i) descends from the fact that the Rankine-Hugoniot jump conditions lose their validity when applied to the supplementary law (2.2). In fact, the quantity (*)

$$\tilde{\eta} \equiv [\![h^\omega]\!] \partial_\omega \varphi \tag{2.7}$$

is, in general, nonvanishing (the brackets $[\![\ ]\!]$ shall denote hereafter the jump of any quantity across the shock front $\varphi(x_\omega) = 0$).

By indicating by s the velocity of the shock front, it has been shown that $\partial\tilde{\eta}(s)/\partial s > 0$ (**) and $\tilde{\eta} \geqslant 0$ for $s \geqslant \lambda$ with λ denoting any one of the eigenvalues of system (2.1). This behaviour of $\tilde{\eta}$ implies the well-known growth of the thermodynamic entropy across a shock. Furthermore, in the case of a noncharacteristic shock (namely, when the shock speed $s \neq \lambda$), indicating by $\boldsymbol{U}_0$ the unperturbed field, the jump of U' can be also expressed by

$$[\![\boldsymbol{U}']\!] = \frac{\partial \tilde{\eta}}{\partial \boldsymbol{U}_0} (A_0^\omega \partial_\omega \varphi)^{-1} . \tag{2.8}$$

([21]) A. FISCHER and D. P. MARSDEN: *Commun. Math. Phys.*, **28**, 1 (1972).

(*) Unlike as usually done, we have used, for convenience, the symbol $\tilde{\eta}$ to indicate the SGF preserving the symbol η for the density entropy.

(**) This result, first derived in ([2]) by introducing in the field equations an artificial viscosity, has been further discussed in ([7]) under the only condition of convexity for the density energy. A generalization of this result in a covariant formalism has been made in ([12]).

In other words, with only the knowledge of the function $\tilde{\eta}(U_0, s)$, one knows the jump of U'; $\tilde{\eta}$ behaves, therefore, as a «generator» of the shock.

Applications of this theory have been made in several physical fields: hyperelastic media subject to finite strains (11), relativistic fluid dynamics (12), extended thermodynamics (13), nonlinear electrodynamics (22), classical fluid dynamics (17,23,24).

3. – Balance equations for the flow of a mixture of ν constituents.

According to the usual formalism (14), the leading system of partial differential equations describing the flow of a simple mixture of ν ideal constituents, determining the fields of densities ϱ_α, velocities v_i^α and absolute temperature T or total energy E, writes

$$(3.1)_1 \qquad \frac{\partial \varrho_\alpha}{\partial t} + \frac{\partial}{\partial x_j}(\varrho_\alpha v_j^\alpha) = \tau_\alpha \qquad \text{(balance of masses)},$$

$$(3.1)_2 \qquad \frac{\partial(\varrho_\alpha v_i^\alpha)}{\partial t} + \frac{\partial}{\partial x_j}(\varrho_\alpha v_i^\alpha v_j^\alpha - t_{ij}^\alpha) = m_i^\alpha \quad \text{(balance of momenta)},$$

$$(3.1)_3 \qquad \frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j}(E v_j - t_{ij} v_i + q_j) = 0 \quad \text{(balance of energy for the mixture)}$$

$$(\alpha = 1, 2, ..., \nu;\ i, j = 1, 2, 3),$$

where we have neglected in eqs. $(3.1)_2$ the specific external-body forces and in eqs. $(3.1)_3$ both the specific energy supply due to radiation and the exchange of energy among the constituents. Since in our context the constituents are not reacting, the source terms τ_α on the r.h.s. of eqs. $(3.1)_1$ might be also cancelled. All these simplifications do not reduce the general results of the present work in that the neglected terms do not belong to the differential part of the system.

To the above balance equations one must associate the constitutive relations compatible with the linear representation (14,15) for a nondissipative simple mixture of ideal constituents.

(22) G. Boillat: *C. R. Acad. Sci. Paris A*, **290**, 259 (1980).
(23) D. Fusco: *Atti Semin. Mat. Fis. Univ. Modena*, *XXVIII*, 223 (1979).
(24) A. I. Volpert and S. I. Hudiaev: *Math. USSR Sbornik.*, **10**, 571 (1972).

Even though all the variables in (3.1) can be easily identified with the index α denoting the constituent, we list the following symbols:

$\varrho = \sum_\alpha \varrho_\alpha$	(density of the mixture as a whole),
$E = \sum_\alpha \varrho_\alpha \left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2}\right] = \varrho\left(\varepsilon + \frac{v^2}{2}\right)$	(total energy/cm³ of the mixture),
$\varepsilon = \varepsilon_\mathrm{I} + \frac{1}{\varrho}\sum_\alpha \varrho_\alpha \frac{(u^\alpha)^2}{2}$	(internal energy/g of the mixture),
$v_i = \frac{1}{\varrho}\sum_\alpha \varrho_\alpha v_i^\alpha$	(i-th velocity component of the mixture),
$v = \frac{1}{\varrho}\sum_\alpha \varrho_\alpha v^\alpha$	(velocity of the mixture),
t_{ij}^α	(symmetric stress tensor on the constituent α),
$t_{ij} = \sum_\alpha (t_{ij}^\alpha - \varrho_\alpha u_i^\alpha u_j^\alpha)$	(symmetric stress tensor on the mixture),
$q_i = q_i^\mathrm{I} + \sum_\alpha \varrho_\alpha \frac{(u^\alpha)^2}{2} u_i^\alpha$	(internal energy flux of the mixture),
$u_i^\alpha = v_i^\alpha - v_i$	(i-th diffusion velocity component of the constituent α),
$u^\alpha = v^\alpha - v$	(diffusion velocity of the constituent α),
τ_α	(mass production of the constituent α due to chemical reactions),
m_i^α	(momentum production due to exchange).

The balance laws of the mixture as a whole impose the constraints $\sum_\alpha \tau_\alpha = 0$ and $\sum_\alpha m_i^\alpha = 0$, so that total mass and momentum of the mixture are conserved.

The constitutive relations read

$$(3.2)\qquad \begin{cases} t_{ij}^\alpha = -p_\alpha \delta_j^i & \text{with } p_\alpha = \dfrac{\mathscr{R}}{\mathscr{M}_\alpha}\varrho_\alpha T\,, \\ \varepsilon_\mathrm{I} = \dfrac{1}{\varrho}\sum_\alpha \varrho_\alpha \varepsilon_\alpha & \text{with } \varepsilon_\alpha = z_\alpha \dfrac{\mathscr{R}}{\mathscr{M}_\alpha} T = c_V^\alpha T\,, \\ q_i^\mathrm{I} = \sum_\alpha q_\nu^\alpha (v_i^\alpha - v_i^\nu) & \text{with } q_\nu^\alpha = -\varrho_\alpha \sum_\beta \dfrac{\varrho_\beta}{\varrho}\left(\varepsilon_\beta + \dfrac{p_\beta}{\varrho_\beta}\right) + \varrho_\alpha\varepsilon_\alpha + p_\alpha\,, \end{cases}$$

ε_I and q_i^I denote, respectively, the intrinsic values of internal energy and the flux of internal energy as defined in ([15]). In the above relations $\mathscr{M}_\alpha$ is the molecular weight of the constituent α, $\mathscr{R}$ is the ideal-gas constant, $z_\alpha = 1/(\gamma_\alpha - 1) = \frac{3}{2}$, $\frac{5}{2}$ and 3 (according to whether the gas is monoatomic, biatomic, or poliatomic), $\gamma_\alpha = c_p^\alpha / c_V^\alpha$ the ratio of specific heats.

The constitutive relations for m_i^α and τ_α need not to be explicated in the context of the present work.

By setting

$$F^0 \equiv U \equiv \begin{pmatrix} \varrho_\alpha \\ \varrho_\alpha v_i^\alpha \\ E \end{pmatrix}, \qquad F^j \equiv \begin{pmatrix} \varrho_\alpha v_j^\alpha \\ \varrho_\alpha v_i^\alpha v_j^\alpha - t_{ij}^\alpha \\ Ev_j - t_{ij} v_i + q_j \end{pmatrix}, \qquad f \equiv \begin{pmatrix} \tau_\alpha \\ m_i^\alpha \\ 0 \end{pmatrix}, \tag{3.3}$$

system (3.1) of $4\nu + 1$ equations in the $4\nu + 1$ unknown functions ϱ_α, v_i^α and E writes

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}^j}{\partial x_j} = \boldsymbol{f}.$$

4. – Supplementary conservation law and convex density entropy for the balance system.

a) The supplementary conservation law. In view of the Gibbs equation for a mixture [15]

$$d(\varrho\eta) = \frac{1}{T}\left[d(\varrho\varepsilon_{\mathrm{I}}) - \sum_\alpha \mu_\alpha^{\mathrm{I}}\, \mathrm{d}\varrho_\alpha\right], \tag{4.1}$$

we get

$$\frac{\partial(\varrho\eta)}{\partial t} = \frac{1}{T}\left[\frac{\partial(\varrho\varepsilon_{\mathrm{I}})}{\partial t} - \sum_\alpha \mu_\alpha^{\mathrm{I}} \frac{\partial\varrho_\alpha}{\partial t}\right], \tag{4.2}$$

where $\eta = (1/\varrho)\sum_\alpha \varrho_\alpha\eta_\alpha$ is the specific entropy of the mixture with $\eta_\alpha = c_V^\alpha \log(p_\alpha \varrho_\alpha^{-\gamma_\alpha})$ the entropy of the constituent α; $\mu_\alpha^{\mathrm{I}} = \varepsilon_\alpha - T\eta_\alpha + p_\alpha/\varrho_\alpha$ is the intrinsic chemical potential of the constituent α (*i.e.* the specific free entalpy).

To write down explicitly the supplementary conservation law (2.2) for our balance system (3.1), we need few steps. In fact, eq. $(3.1)_3$ may be rewritten as

$$\frac{\partial}{\partial t}\left[\varrho\varepsilon_{\mathrm{I}} + \sum_\alpha \varrho_\alpha \frac{(v^\alpha)^2}{2}\right] = -\frac{\partial}{\partial x_j}(Ev_j - t_{ij}v_i + q_j), \tag{4.3}$$

so that, since

$$\varrho_\alpha (v^\alpha)^2 = \sum_i \varrho_\alpha (v_i^\alpha)^2 \quad \text{and} \quad \frac{\partial[\varrho_\alpha (v_i^\alpha)^2]}{\partial t} = \varrho_\alpha v_i^\alpha \frac{\partial v_i^\alpha}{\partial t} + v_i^\alpha \frac{\partial(\varrho_\alpha v_i^\alpha)}{\partial t},$$

in view of $(3.1)_{1,2}$ one finds

$$(4.4)\qquad \frac{\partial[\varrho_\alpha(v^\alpha)^2]}{\partial t}=\frac{\partial}{\partial t}\,[\sum_i \varrho_\alpha(v_i^\alpha)^2]=2\sum_i\left(m_i^\alpha v_i^\alpha-v_i^\alpha\,\frac{\partial p_\alpha}{\partial x_i}\right)-\tau_\alpha(v^\alpha)^2-\\ -\sum_i\frac{\partial}{\partial x_i}\,[\varrho_\alpha v_i^\alpha(v^\alpha)^2]\,.$$

Besides, although tedious, it is not difficult to show the identity

$$(4.5)\qquad Ev_j-t_{ij}v_i+q_j=\sum_\alpha v_j^\alpha\left\{\varrho_\alpha\left[\varepsilon_\alpha+\frac{(v^\alpha)^2}{2}\right]+p_\alpha\right\}.$$

Substitution of (4.4) and (4.5) into (4.3), after some simplifications, yields

$$(4.6)\qquad \frac{\partial(\varrho\varepsilon_{\mathrm{I}})}{\partial t}=\sum_\alpha\left[\tau_\alpha\,\frac{(v^\alpha)^2}{2}-\sum_i m_i^\alpha v_i^\alpha-\sum_i\frac{\partial}{\partial x_i}\,(\varrho_\alpha\varepsilon_\alpha v_i^\alpha)-p_\alpha\sum_i\frac{\partial v_i^\alpha}{\partial x_i}\right].$$

Finally, combining (4.2) and (4.6), eliminating $\partial\varrho_\alpha/\partial t$ and manipulating, one finds the requested supplementary conservation law

$$(4.7)\qquad \sum_\alpha\left[\frac{\partial}{\partial t}\,(\varrho_\alpha\eta_\alpha)+\sum_i\frac{\partial}{\partial x_i}\,(\varrho_\alpha\eta_\alpha v_i^\alpha)\right]=\frac{1}{T}\sum_\alpha\left\{\left[\frac{(v^\alpha)^2}{2}-\mu_\alpha^{\mathrm{I}}\right]\tau_\alpha-\sum_i m_i^\alpha v_i^\alpha\right\}.$$

In the case of a single constituent (adiabatic motion of an ideal fluid in continuum mechanics), since the production density and the exchange of momentum are identically zero, eq. (4.7) reduces to the well-known « equation of continuity for the entropy » and reads

$$\frac{\partial(\varrho\eta)}{\partial t}+\boldsymbol{\nabla}\cdot(\varrho\eta\boldsymbol{v})=0$$

with $\varrho\eta\boldsymbol{v}$ the entropy flux density.

Returning to the general case, let us set

$$(4.8)\qquad \left|\begin{aligned} &h^0(\boldsymbol{U})=-\sum_\alpha\varrho_\alpha\eta_\alpha=-\varrho\eta\,,\\ &h^i(\boldsymbol{U})=-\sum_\alpha\varrho_\alpha\eta_\alpha v_i^\alpha\,,\\ &g(\boldsymbol{U})=\frac{1}{T}\sum_\alpha\left\{\sum_i v_i^\alpha m_i^\alpha-\left[\frac{(v^\alpha)^2}{2}-\mu_\alpha^{\mathrm{I}}\right]\tau_\alpha\right\},\end{aligned}\right.$$

then eq. (4.7) takes the compact form (2.2).

b) Main field and convex density entropy. As sketched in sect. **2**, the main field $\boldsymbol{U}'$ to be associated to the field $\boldsymbol{U}=\boldsymbol{F}^0$, as given by $(3.3)_1$, can be found once a convex density function $h^0(\boldsymbol{U})$ is known.

From the definition of $U' = \check{\nabla}_U h^0$, the main field may be directly found by expressing (4.1) in terms of the components of the differential field $\mathrm{d}U$, namely in terms of $\mathrm{d}\varrho_\alpha$, $\mathrm{d}(\varrho_\alpha v_i^\alpha)$ and $\mathrm{d}E$. This may be obtained through a simple manipulation which allows us to write

$$\varepsilon_{\mathrm{I}} = \frac{1}{\varrho}\left\{E - \sum_\alpha\left[\frac{\sum_i \varrho_\alpha^2 (v_i^\alpha)^2}{2\varrho_\alpha}\right]\right\}$$

and then

$$\mathrm{d}\varepsilon_{\mathrm{I}} = \frac{1}{\varrho}\left\{\sum_\alpha\left[\frac{(v^\alpha)^2}{2} - \varepsilon_{\mathrm{I}}\right]\mathrm{d}\varrho_\alpha - \sum_\alpha\sum_i v_i^\alpha\,\mathrm{d}(\varrho_\alpha v_i^\alpha) + \mathrm{d}E\right\}.$$

The combination of this expression with (4.1) gives

$$\mathrm{d}(\varrho\eta) = -\,\mathrm{d}h^0(U) = \frac{1}{T}\left\{\sum_\alpha\left[\frac{(v^\alpha)^2}{2} - \mu_\alpha^{\mathrm{I}}\right]\mathrm{d}\varrho_\alpha - \sum_\alpha\sum_i v_i^\alpha\,\mathrm{d}(\varrho_\alpha v_i^\alpha) + \mathrm{d}E\right\}. \tag{4.9}$$

From this last one derives at once

$$\check{U}' \equiv -\frac{1}{T}\left[\frac{(v^\alpha)^2}{2} - \mu_\alpha^{\mathrm{I}},\ -v_i^\alpha,\ 1\right].$$

Let us now evaluate the thermodynamical restrictions which are requested in order that h^0 is a convex function of U. For this it suffices to find the conditions under which the Hessian matrix $\partial^2 h^0/\partial U \partial U$ or, equivalently, the quadratic form

$$\frac{\partial^2 h^0}{\partial U\cdot\partial U}\,\mathrm{d}U\cdot\mathrm{d}U = \mathrm{d}(\check{\nabla}_U h^0)\,\mathrm{d}U = \mathrm{d}U'\cdot\mathrm{d}U$$

is positive definite.

Now, since

$$\mathrm{d}\check{U} \equiv \left\{\mathrm{d}\varrho_\alpha,\ \varrho_\alpha\,\mathrm{d}v_i^\alpha + v_i^\alpha\,\mathrm{d}\varrho_\alpha,\ \left(\varepsilon + \frac{v^2}{2}\right)\mathrm{d}\varrho + \varrho(\mathrm{d}\varepsilon + v\,\mathrm{d}v)\right\}$$

and

$$\mathrm{d}\check{U}' \equiv -\frac{1}{T}\left\{v^\alpha\,\mathrm{d}v^\alpha + \eta_\alpha\,\mathrm{d}T - \frac{1}{\varrho_\alpha}\,\mathrm{d}p_\alpha + \left[\mu_\alpha^{\mathrm{I}} - \frac{(v^\alpha)^2}{2}\right]\frac{\mathrm{d}T}{T},\ -\mathrm{d}v_i^\alpha + v_i^\alpha\frac{\mathrm{d}T}{T},\ -\frac{\mathrm{d}T}{T}\right\},$$

it follows that

$$\begin{aligned} T\,\mathrm{d}U\cdot\mathrm{d}U' = -\sum_\alpha\Bigg(&-\varrho_\alpha\sum_i(\mathrm{d}v_i^\alpha)^2 - \frac{1}{\varrho_\alpha}\,\mathrm{d}p_\alpha\,\mathrm{d}\varrho_\alpha + \\ &+\left\{\left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2} + \frac{p_\alpha}{\varrho_\alpha}\right]\mathrm{d}\varrho_\alpha + \varrho_\alpha v^\alpha\,\mathrm{d}v^\alpha\right\}\frac{\mathrm{d}T}{T}\Bigg) + \left[\left(\varepsilon + \frac{v^2}{2}\right)\mathrm{d}\varrho + \varrho v\,\mathrm{d}v + \varrho\,\mathrm{d}\varepsilon\right]\frac{\mathrm{d}T}{T}. \end{aligned} \tag{4.10}$$

By using the Gibbs equation, one finds that

$$\mathrm{d}\varepsilon = T\,\mathrm{d}\eta - v\,\mathrm{d}v + \frac{1}{\varrho}\sum_\alpha\left\{\left[\frac{(v^\alpha)^2}{2} + \varepsilon_\alpha - \left(\varepsilon + \frac{v^2}{2}\right) + T(\eta - \eta_\alpha) + \frac{p_\alpha}{\varrho_\alpha}\right]\mathrm{d}\varrho_\alpha + \varrho_\alpha v^\alpha\,\mathrm{d}v_\alpha\right\},$$

which, introduced into (4.10), allows the quadratic form to be written as

$$\mathrm{d}\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{U}' = \sum_\alpha \frac{\varrho_\alpha}{T}\left[\sum_i (\mathrm{d}v_i^\alpha)^2 - (\mathrm{d}p_\alpha\,\mathrm{d}\mathscr{V}_\alpha - \mathrm{d}T\,\mathrm{d}\eta_\alpha)\right], \tag{4.11}$$

where $\mathscr{V}_\alpha = 1/\varrho_\alpha$ is the specific volume of the constituent α. It is evident that (4.11) is positive definite if the quantity

$$\sum_\alpha (\mathrm{d}p_\alpha\,\mathrm{d}\mathscr{V}_\alpha - \mathrm{d}T\,\mathrm{d}\eta_\alpha) \tag{4.12}$$

is negative definite.

To show this, we note that, differentiating μ_α^{I} and then eliminating $\mathrm{d}\eta_\alpha = (\mathrm{d}\varepsilon_\alpha + p_\alpha\,\mathrm{d}\mathscr{V}_\alpha)/T$, one finds that $\mathrm{d}\mu_\alpha^{\mathrm{I}} = -\eta_\alpha\,\mathrm{d}T + \mathscr{V}_\alpha\,\mathrm{d}p_\alpha$. Setting for brevity $G \equiv G(p_\alpha, T) = \mu_\alpha^{\mathrm{I}}(p_\alpha, T)$, it follows that

$$G_T = \left(\frac{\partial G}{\partial T}\right)_{p_\alpha} = -\eta_\alpha \quad\text{and}\quad G_{p_\alpha} = \left(\frac{\partial G}{\partial p_\alpha}\right)_T = \mathscr{V}_\alpha .$$

In view of these expressions, each term of quantity (4.12) writes

$$\mathrm{d}p_\alpha\,\mathrm{d}\mathscr{V}_\alpha - \mathrm{d}T\,\mathrm{d}\eta_\alpha = G_{p_\alpha p_\alpha}(\mathrm{d}p_\alpha)^2 + 2G_{p_\alpha T}\,\mathrm{d}p_\alpha\,\mathrm{d}T + G_{TT}(\mathrm{d}T)^2 . \tag{4.13}$$

It turns out that (4.13) is negative definite if and only if the following inequalities hold:

$$\begin{cases} G_{p_\alpha p_\alpha} = \left(\dfrac{\partial\mathscr{V}_\alpha}{\partial p_\alpha}\right)_T < 0\,, \qquad G_{p_\alpha T} = \left(\dfrac{\partial\mathscr{V}_\alpha}{\partial T}\right)_{p_\alpha} = -\dfrac{p_\alpha}{T}\left(\dfrac{\partial\mathscr{V}_\alpha}{\partial p_\alpha}\right)_T = G_{Tp_\alpha} > 0\,, \\ G_{TT} = -\left(\dfrac{\partial\eta_\alpha}{\partial T}\right)_{p_\alpha} = -\dfrac{c_V^\alpha}{T} - \dfrac{p_\alpha}{T}\left(\dfrac{\partial\mathscr{V}_\alpha}{\partial T}\right)_{p_\alpha} < 0\,, \end{cases} \tag{4.14}$$

so that the discriminant of (4.13) turns out to be

$$-\frac{c_V^\alpha}{T}\left(\frac{\partial\mathscr{V}_\alpha}{\partial p_\alpha}\right)_T > 0\,.$$

In other words, under conditions (4.14) which for a single fluid are but the usual thermodynamical-equilibrium functions, we have established that the function h^0 is, in the large, a convex function of $\boldsymbol{U}$ in any convex domain $\mathscr{D} \subseteq R^{4\nu+1}$ and, as a consequence, that the balance system is hyperbolic.

We possess now all the elements to construct the 4-vector $h'^{\omega}(U')$. From definition (2.4) it turns out, in fact, that

$$(4.15)\quad \begin{cases} h'^0 = U'\cdot U - h^0 = \dfrac{1}{T}\sum_\alpha \left\{\varrho_\alpha\left[\mu^{\mathrm{I}}_\alpha - \dfrac{(v^\alpha)^2}{2}\right] + \varrho_\alpha (v^\alpha)^2\right\} - \dfrac{E}{T} + \varrho\eta = \dfrac{p}{T}\,, \\ h'^i = U'\cdot F^i - h^i = \dfrac{1}{T}\sum_\alpha \varrho_\alpha v^\alpha_i(\mu^{\mathrm{I}}_\alpha - \varepsilon_\alpha + T\eta_\alpha) = \dfrac{1}{T}\sum_\alpha p_\alpha v^\alpha_i\,, \end{cases}$$

which, in the case of a single fluid, reduce, respectively, to $h'^0 = p/T$ and $h'^i = pv_i/T$ [25,26].

Expressing h'^{ω} in terms of the components of the main field U', the Hessian coefficient matrices of the relative symmetrized system (2.6) are explicitly given in appendix A.

5. – The shock-generating function.

a) The Rankine-Hugoniot equations. As known, the compatibility conditions for a shock to be a weak solution to system (3.1) are the so-called Rankine-Hugoniot jump conditions. These can be simply derived from eqs. (3.1) through the formal substitutions [27]

$$(5.1)\qquad \frac{\partial}{\partial t} \to -s[\![\;]\!] \quad\text{and}\quad \frac{\partial}{\partial x_i} \to n_i[\![\;]\!]\,,$$

where s denotes the normal velocity of the shock front represented by a C^2-surface $\Gamma\colon \varphi(x_\omega) = 0$ and $\boldsymbol{n} \equiv (n_i)$ is the positive unit normal vector to Γ. $[\![X]\!] = \overset{1}{X} - \overset{0}{X}$ denotes the difference of the limits taken from the two sides of Γ.

In view of (5.1), from (3.1) we get

$$(5.2)\quad \begin{cases} -s[\![\varrho_\alpha]\!] + [\![\varrho_\alpha v^\alpha_n]\!] = 0\,, \\ -s[\![\varrho_\alpha \boldsymbol{v}^\alpha]\!] + [\![\varrho_\alpha \boldsymbol{v}^\alpha v^\alpha_n + p_\alpha \boldsymbol{n}]\!] = 0\,, \\ -s[\![E]\!] + \left[\!\!\left[\sum_\alpha \varrho_\alpha v^\alpha_n\left[\varepsilon_\alpha + \dfrac{(\boldsymbol{v}^\alpha)^2}{2} + \dfrac{p_\alpha}{\varrho_\alpha}\right]\right]\!\!\right] = 0\,, \end{cases}$$

where $v_n = \boldsymbol{v}\cdot\boldsymbol{n}$. In writing $(5.2)_3$ we made use of identity (4.5). The ex-

([25]) T. Ruggeri: in *Wave Propagation*, Corso CIME (Bressanone, 1980).
([26]) T. Ruggeri: in *Lectures at VI Scuola Estiva di Fisica Matematica* (Ravello, 1981).
([27]) A. Jeffrey and T. Taniuti: *Non Linear Wave Propagation* (Academic Press, New York, N. Y., 1964).

plicit form of (5.2), which shall be useful in the following, writes

$$\text{(5.3)}\quad \begin{cases} \overset{1}{\varrho}_\alpha(\overset{1}{v}{}_n^\alpha - s) = \overset{0}{\varrho}_\alpha(\overset{0}{v}{}_n^\alpha - s)\,, \\ \overset{1}{\varrho}_\alpha\,\overset{1}{\boldsymbol{v}}{}^\alpha(\overset{1}{v}{}_n^\alpha - s) + \overset{1}{p}_\alpha \boldsymbol{n} = \overset{0}{\varrho}_\alpha\,\overset{0}{\boldsymbol{v}}{}^\alpha(\overset{0}{v}{}_n^\alpha - s) + \overset{0}{p}_\alpha \boldsymbol{n}\,, \\ \sum_\alpha \left\{\overset{1}{\varrho}_\alpha\left[\overset{1}{\varepsilon}_\alpha + \dfrac{(\overset{1}{\boldsymbol{v}}{}^\alpha)^2}{2}\right](\overset{1}{v}{}_n^\alpha - s) + \overset{1}{p}_\alpha \overset{1}{v}{}_n^\alpha\right\} = \sum_\alpha \left\{\overset{0}{\varrho}_\alpha\left[\overset{0}{\varepsilon}_\alpha + \dfrac{(\overset{0}{\boldsymbol{v}}{}^\alpha)^2}{2}\right](\overset{0}{v}{}_n^\alpha - s) + \overset{0}{p}_\alpha \overset{0}{v}{}_n^\alpha\right\}. \end{cases}$$

Defining the shock Mach number (SMN) in the unperturbed flow as

$$\text{(5.4)}\qquad \overset{0}{M}_\alpha = \frac{\overset{0}{v}{}_n^\alpha - s}{\overset{0}{c}_\alpha} \quad \text{with } \overset{0}{c}_\alpha = (\gamma_\alpha \overset{0}{p}_\alpha / \overset{0}{\varrho}_\alpha)^{\frac{1}{2}}$$

and setting $Y_\alpha = [\![v_n^\alpha]\!]$, simple manipulations upon $(5.3)_{1,2}$ lead to

$$\text{(5.5)}\qquad [\![\varrho_\alpha]\!] = -\frac{\overset{0}{\varrho}_\alpha Y_\alpha}{Y_\alpha + \overset{0}{M}_\alpha \overset{0}{c}_\alpha} \qquad \text{and} \qquad [\![p_\alpha]\!] = -\overset{0}{\varrho}_\alpha \overset{0}{M}_\alpha \overset{0}{c}_\alpha Y_\alpha\,.$$

On the other hand, we may also write

$$\text{(5.6)}\quad [\![p_\alpha]\!] = \frac{\mathscr{R}}{\mathscr{M}_\alpha}[\![\varrho_\alpha T]\!] = \frac{\mathscr{R}}{\mathscr{M}_\alpha}\{[\![\varrho_\alpha]\!][\![T]\!] + \overset{0}{\varrho}_\alpha[\![T]\!] + \overset{0}{T}[\![\varrho_\alpha]\!]\} = \frac{K\overset{0}{M}_\alpha \overset{0}{c}_\alpha - Y_\alpha}{\overset{0}{M}_\alpha \overset{0}{c}_\alpha + Y_\alpha}\,\overset{0}{p}_\alpha\,,$$

where we have set $[\![T]\!] = K\overset{0}{T}$ with $K > -1$ a real constant to be determined. From $(5.5)_2$ and (5.3) it turns out that

$$\text{(5.7)}\qquad Y_\alpha = \frac{\overset{0}{c}_\alpha}{2\gamma_\alpha \overset{0}{M}_\alpha}(1 - \gamma_\alpha \overset{0}{M}{}_\alpha^2 + \sigma_\alpha W_\alpha)\,,$$

where σ_α can take the value $+1$ or -1 and we have put for brevity $W_\alpha \equiv \equiv [(1-\gamma_\alpha \overset{0}{M}{}_\alpha^2)^2 - 4K\gamma_\alpha \overset{0}{M}{}_\alpha^2]^{\frac{1}{2}}$. From (5.7) it follows also that, in order that Y_α be real, K must satisfy the constraint

$$\text{(5.8)}\qquad -1 < K \leqslant \min_\alpha \frac{(1-\gamma_\alpha \overset{0}{M}{}_\alpha^2)^2}{4\gamma_\alpha \overset{0}{M}{}_\alpha^2}\,.$$

In view of (5.7), expressions (5.5) become

$$\text{(5.9)}\qquad \begin{cases} [\![\varrho_\alpha]\!] = -\dfrac{\overset{0}{\varrho}_\alpha}{\overset{0}{c}_\alpha(1+K)}(\gamma_\alpha \overset{0}{M}_\alpha Y_\alpha + K\overset{0}{c}_\alpha)\,, \\[2ex] [\![p_\alpha]\!] = -\dfrac{\gamma_\alpha \overset{0}{M}_\alpha}{\overset{0}{c}_\alpha}\overset{0}{p}_\alpha Y_\alpha\,, \end{cases}$$

and, as a consequence,

$$
\left\{
\begin{aligned}
&[\![\mathscr{V}_\alpha]\!] = \frac{\overset{0}{c}_\alpha K + \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha - \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}\,\overset{0}{\mathscr{V}}_\alpha\,,\\
&\frac{[\![p_\alpha]\!]}{[\![\mathscr{V}_\alpha]\!]} = \left(\frac{\gamma_\alpha \overset{0}{M}_\alpha Y_\alpha - \overset{0}{c}_\alpha}{\gamma_\alpha \overset{0}{M}_\alpha Y_\alpha + \overset{0}{c}_\alpha K}\right)\frac{\gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha}\,\overset{0}{p}_\alpha\,\overset{0}{\varrho}_\alpha\,.
\end{aligned}
\right.
\tag{5.10}
$$

When the mixture is considered as a whole, one has

$$
\left\{
\begin{aligned}
&[\![\varrho]\!] = \sum_\alpha [\![\varrho_\alpha]\!] = -\frac{1}{1+K}\left(\sum_\alpha \frac{\gamma_\alpha \overset{0}{M}_\alpha Y_\alpha \overset{0}{\varrho}_\alpha}{\overset{0}{c}_\alpha} + K\overset{0}{\varrho}\right),\\
&[\![\mathscr{V}]\!] = \frac{\overset{0}{\mathscr{V}} \sum\limits_\alpha \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha \overset{0}{\varrho}_\alpha/\overset{0}{c}_\alpha + K}{\overset{0}{\varrho} - \sum\limits_\alpha \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha \overset{0}{\varrho}_\alpha/\overset{0}{c}_\alpha}\,,\\
&[\![p]\!] = -\sum_\alpha \frac{\gamma_\alpha \overset{0}{M}_\alpha \overset{0}{p}_\alpha Y_\alpha}{\overset{0}{c}_\alpha}\,,\\
&\frac{[\![p]\!]}{[\![\mathscr{V}]\!]} = \left(\sum_\alpha \frac{\gamma_\alpha \overset{0}{M}_\alpha \overset{0}{p}_\alpha Y_\alpha}{\overset{0}{c}_\alpha}\right)\left(\frac{\sum\limits_\alpha \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha \overset{0}{\varrho}_\alpha/\overset{0}{c}_\alpha - \varrho_0}{\overset{0}{\mathscr{V}} \sum\limits_\alpha \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha \overset{0}{\varrho}_\alpha/\overset{0}{c}_\alpha + K}\right).
\end{aligned}
\right.
\tag{5.11}
$$

Other jumps which are useful in the following calculations are explicitly given by

$$
\left\{
\begin{aligned}
&[\![\boldsymbol{v}^\alpha]\!] = Y_\alpha \boldsymbol{n}\,,\\
&[\![(\boldsymbol{v}^\alpha)^2]\!] = \overset{1}{\boldsymbol{v}}{}^\alpha[\![\boldsymbol{v}^\alpha]\!] + \overset{0}{\boldsymbol{v}}{}^\alpha[\![\boldsymbol{v}^\alpha]\!] = \{[\![\boldsymbol{v}^\alpha]\!] + 2\overset{0}{\boldsymbol{v}}{}^\alpha\}\,[\![\boldsymbol{v}^\alpha]\!] = Y_\alpha^2 + 2\overset{0}{v}{}^\alpha_n Y_\alpha =\\
&\qquad = \frac{\overset{0}{c}_\alpha}{\gamma_\alpha \overset{0}{M}_\alpha}\left[\left(1 - \gamma_\alpha \overset{0}{M}{}^2_\alpha + 2\frac{\gamma_\alpha \overset{0}{M}_\alpha \overset{0}{v}{}^\alpha_n}{\overset{0}{c}_\alpha}\right)Y_\alpha - K\,\overset{0}{c}_\alpha\,\overset{0}{M}_\alpha\right],\\
&[\![\varepsilon_\alpha]\!] = \frac{\overset{0}{\varepsilon}_\alpha[\![T]\!]}{\overset{0}{T}} = \frac{K\overset{0}{c}{}^2_\alpha}{\gamma_\alpha(\gamma_\alpha - 1)}\,,\\
&[\![p_\alpha v^\alpha_n]\!] = \overset{1}{p}_\alpha[\![v^\alpha_n]\!] + \overset{0}{v}{}^\alpha_n\,[\![p_\alpha]\!] = \{[\![p_\alpha]\!] + \overset{0}{p}_\alpha\}[\![v^\alpha_n]\!] + \overset{0}{v}{}^\alpha_n[\![p_\alpha]\!] =\\
&\qquad = \overset{0}{M}_\alpha \overset{0}{p}_\alpha\left[\gamma_\alpha\left(\overset{0}{M}_\alpha - \frac{\overset{0}{v}{}^\alpha_n}{\overset{0}{c}_\alpha}\right)Y_\alpha + K\overset{0}{c}_\alpha\right].
\end{aligned}
\right.
\tag{5.12}
$$

Equation $(5.3)_3$, in view of $(5.3)_1$ and (5.4), may be conveniently written as

$$\sum_\alpha \left[\overset{0}{c}_\alpha \overset{0}{M}_\alpha \overset{0}{\varrho}_\alpha \left[\!\left[\varepsilon_\alpha + \frac{(\boldsymbol{v}^\alpha)^2}{2}\right]\!\right] + [\![p_\alpha v_n^\alpha]\!]\right] = 0\,,$$

which, expressed in terms of the foregoing jumps (5.12), gives the following equation for K:

$$\sum_\alpha \overset{0}{c}_\alpha \overset{0}{p}_\alpha \left[\Gamma_\alpha \overset{0}{M}_\alpha K + (1 + \gamma_\alpha \overset{0}{M}{}_\alpha^2)\frac{Y_\alpha}{\overset{0}{c}_\alpha}\right] = 0 \text{ with } \Gamma_\alpha = \frac{\gamma_\alpha + 1}{\gamma_\alpha - 1}\,. \tag{5.13}$$

It is worthwhile remarking that the values of K, implicit solutions of (5.13), must also satisfy the foregoing constraints (5.8).

In the case of a single fluid, eq. (5.13) reduces to

$$K = \frac{2(\gamma - 1)}{(\gamma + 1)^2}\,\frac{\overset{0}{M}{}^2 - 1}{\overset{0}{M}{}^2}\,(\gamma \overset{0}{M}{}^2 + 1)\,. \tag{5.14}$$

Making use of this value, algebraic manipulations give

$$Y = [\![v_n]\!] = \frac{2\overset{0}{c}(1 - \overset{0}{M}{}^2)}{(\gamma + 1)\overset{0}{M}}\,, \qquad [\![\varrho]\!] = \frac{2(\overset{0}{M}{}^2 - 1)\overset{0}{\varrho}}{(\gamma - 1)\overset{0}{M}{}^2 + 2}\,,$$

$$[\![\mathscr{V}]\!] = -\frac{2\overset{0}{\mathscr{V}}(\overset{0}{M}{}^2 - 1)}{(\gamma + 1)\overset{0}{M}{}^2}\,, \qquad [\![p]\!] = \frac{2\gamma(\overset{0}{M}{}^2 - 1)\overset{0}{p}}{\gamma + 1}\,,$$

which are but the well-known Rankine-Hugoniot relations in ordinary fluid dynamics ([28]).

b) The function $\tilde{\eta}$. As sketched in sect. **2**, the Rankine-Hugoniot conditions, when applied to the supplementary conservation law of a physical system written in conservative form, do not generally lead to an identically zero quantity as they do for the field equations. On the contrary, from (4.7), in view of (5.1), we get

$$\begin{aligned} -\sum_\alpha \left[-s[\![\varrho_\alpha \eta_\alpha]\!] + [\![\varrho_\alpha \eta_\alpha v_n^\alpha]\!]\right] &= \sum_\alpha \left[\overset{1}{\varrho}_\alpha(s - \overset{1\alpha}{v}_n)\overset{1}{\eta}_\alpha - \overset{0}{\varrho}_\alpha(s - \overset{0\alpha}{v}_n)\overset{0}{\eta}_\alpha\right] = \\ &= \sum_\alpha \overset{0}{\varrho}_\alpha(s - \overset{0\alpha}{v}_n)[\![\eta_\alpha]\!] \equiv \tilde{\eta} \end{aligned} \tag{5.15}$$

([28]) L. D. LANDAU and E. M. LIFSHITZ: *Fluid Mechanics* (Pergamon Press, London, 1959).

and, in general, this quantity is not zero. Besides, by using expressions (5.9), the entropy jump of the constituent α takes the explicit form

$$[\![\eta_\alpha]\!] = c_V^\alpha \log \left[\left(\frac{\overset{0}{c}_\alpha - \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha} \right)^{1-\gamma_\alpha} (1+K)^{\gamma_\alpha} \right].$$

For the mixture as a whole we get

$$(5.16) \qquad [\![\eta]\!] = \frac{1}{\overset{1}{\varrho}} \sum_\alpha \overset{1}{\varrho}_\alpha \overset{1}{\eta}_\alpha - \frac{1}{\overset{0}{\varrho}} \sum_\alpha \overset{0}{\varrho}_\alpha \overset{0}{\eta}_\alpha = \frac{1}{\overset{1}{\varrho}} \Big[\sum_\alpha \big(\overset{1}{\varrho}_\alpha [\![\eta_\alpha]\!] + \overset{0}{\eta}_\alpha [\![\varrho_\alpha]\!] \big) - \overset{0}{\eta} [\![\varrho]\!] \Big]$$

with

$$\overset{0}{\eta} = \frac{1}{\overset{0}{\varrho}} \sum_\alpha \overset{0}{\varrho}_\alpha \overset{0}{\eta}_\alpha = \sum_\alpha \nu_\alpha c_V^\alpha \log \left(\frac{\overset{0}{p}_\alpha}{\overset{0}{\varrho}_\alpha^{\gamma_\alpha}} \right),$$

$$\overset{1}{\varrho}_\alpha = \frac{\overset{0}{c}_\alpha - \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha (1+K)} \overset{0}{\varrho}_\alpha , \qquad \overset{1}{\varrho} = \sum_\alpha \overset{1}{\varrho}_\alpha$$

and ν_α the concentration of the constituent α.

Finally, expressing (5.15) in terms of the SMN, the general expression of the SGF for a mixture of ν constituents takes the form

$$(5.17) \qquad \tilde{\eta} = \sum_\alpha c_V^\alpha \overset{0}{c}_\alpha \overset{0}{M}_\alpha \overset{0}{\varrho}_\alpha \log \left[\left(\frac{\overset{0}{c}_\alpha - \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha} \right)^{\gamma_\alpha - 1} (1+K)^{-\gamma_\alpha} \right].$$

In the case of a single constituent, it turns out that $\tilde{\eta} \propto [\![\eta]\!]$ and simple calculations lead to

$$\tilde{\eta} = c_V \overset{0}{c} \overset{0}{M} \overset{0}{\varrho} \log \left\{ \left[\frac{\gamma+1}{(1-\gamma) + 2\gamma \overset{0}{M}{}^2} \right] \left[\frac{(\gamma+1) \overset{0}{M}{}^2}{(\gamma-1) \overset{0}{M}{}^2 + 2} \right]^\gamma \right\}.$$

By setting $\mu^2 = (\gamma - 1)/(\gamma + 1)$, the foregoing expression exactly coincides with the same law as found in ([23]) (*).

As was to be expected, except for the case of a single fluid, the geometrical representation of $\tilde{\eta}$ is very complicated. Its strong dependence on the param-

(*) Let us remark that in the single-fluid model the value $K = -1$ (devoid of any physical meaning!) leads, through (5.14), to the limiting SMN values $\overset{0}{M} = \pm$ $\pm [(\gamma - 1)/2\gamma]^{\frac{1}{2}}$. In a $(\overset{0}{M} - \tilde{\eta})$ framework these values correspond to two vertical asymptotes for $\tilde{\eta}$.

eter K, *i.e.* on the temperature jump, as given implicitly by eq. (5.13), makes the problem one to be resolved only numerically.

6. – Study of the function $\tilde{\eta}$.

As mentioned, the analytical study of $\tilde{\eta}$, as given by (5.17), turns out to be extremely complicated. We shall confine, therefore, to numerical solutions relative to one-dimensional models. The results of these cases shall lighten the problem and make a better understanding of the general behaviours of this function possible.

Let us note, first of all, that the graphycal image of $\tilde{\eta}$ turns out to be very useful in that it helps to distinguish among its branches those which are compatible with physical shocks, namely those along which $[\![\eta]\!] > 0$. The relative, acceptable ranges of the SMN may be also visualized. Furthermore, since for $\tilde{\eta} \neq 0$ also $[\![\eta]\!] \neq 0$ across the shock, the irreversible thermodynamical character of a nonlinear shock is well enhanced. Finally, in view of the general inequality [7,26]

$$\frac{\partial \tilde{\eta}}{\partial s} = -h^0(\boldsymbol{U}) + h^0(\boldsymbol{U}_0) + \frac{\partial h^0}{\partial \boldsymbol{U}}(\boldsymbol{U} - \boldsymbol{U}_0) > 0 ,$$

the slope of $\tilde{\eta}$ gives a measure of the shock strength amplitude.

To start with, we recall that the eigenvalues of the system describing the flow—which are but the propagation velocity of weak perturbations along the characteristic lines—are also roots of the function $\tilde{\eta}$. This important fact allows us to check separately the numerical values of both the eigenvalues as the roots of the characteristic polynomial related to system (3.1) and the zeros of $\tilde{\eta}$ as found from (5.17) among which there are also these roots.

a) *Eigenvalues.* In the one-dimensional case, system (3.1) may be conveniently set into the form

$$\text{(6.1)} \qquad \begin{cases} \dfrac{\partial \varrho_\alpha}{\partial t} + \dfrac{\partial}{\partial x}(\varrho_\alpha v^\alpha) = \tau_\alpha , \\[2ex] \dfrac{\partial v^\alpha}{\partial t} + v^\alpha \dfrac{\partial v^\alpha}{\partial x} + \dfrac{1}{\varrho_\alpha}\dfrac{\partial p_\alpha}{\partial x} = \dfrac{1}{\varrho_\alpha}(m_\alpha - \tau_\alpha v^\alpha) , \\[2ex] \dfrac{\partial E}{\partial t} + \dfrac{\partial}{\partial x}[(E - t_{11})v + q_1] = 0 \end{cases}$$

The eigenvalues of this system may be found—as usually is formally made in treating weak-discontinuity propagation— as those values of λ to which correspond nontrivial solutions of the following algebraic system (*):

$$
(6.2)\qquad \begin{cases} (-\lambda + v^\alpha)\,\delta\varrho_\alpha + \varrho_\alpha \delta v^\alpha = 0\,, \\[2mm] (-\lambda + v^\alpha)\,\delta v^\alpha + \dfrac{1}{\varrho_\alpha}\,\delta p_\alpha = 0\,, \\[2mm] (-\lambda + v)\,\delta E + (E - t_{11})\,\delta v - v\,\delta t_{11} + \delta q_1 = 0\,, \end{cases}
$$

obtained from (6.1) through the formal substitution (see, for instance, ([27]))

$$
\frac{\partial}{\partial t} \to -\lambda\delta\,, \qquad \frac{\partial}{\partial x} \to \delta\,.
$$

Now, with the help of (3.2), one finds

$$
t_{11} = p - \sum \varrho_\alpha (u^\alpha)^2 = \sum_\alpha \{\varrho_\alpha [v^2 - (v^\alpha)^2] - p_\alpha\}\,,
$$

$$
q_1 = \sum \left[\frac{\varrho_\alpha}{2}(v^\alpha - v)^3 + p_\alpha v^\alpha (z_\alpha + 1)\right] - v(\varrho\varepsilon_{\mathrm{I}} + p)\,.
$$

Besides, since $E = \varrho\varepsilon_{\mathrm{I}} + \sum_\alpha \varrho_\alpha (v^\alpha)^2/2$, we obtain

$$
E - t_{11} = \varrho\left(3\varepsilon - 2\varepsilon_{\mathrm{I}} + \frac{v^2}{2}\right) + p\,.
$$

By differentiating these relations, the characteristic polynomial of system (6.1) may be obtained through subsequent eliminations. After some rearrangements, one gets

$$
(6.3)\qquad \begin{cases} \delta v^\alpha = \dfrac{\lambda - v^\alpha}{A^\alpha}\,\dfrac{p_\alpha}{\varrho_\alpha}\,\dfrac{\delta T}{T} \quad \text{with } A_\alpha = (\lambda - v^\alpha)^2 - \dfrac{p_\alpha}{\varrho_\alpha}\,, \\[3mm] \delta\varrho_\alpha = \dfrac{p_\alpha}{A_\alpha}\,\dfrac{\delta T}{T}\,, \\[3mm] \delta v \;= \dfrac{\lambda - v}{\varrho}\,\dfrac{\delta T}{T}\sum_\alpha \dfrac{p_\alpha}{A_\alpha}\,, \\[3mm] \delta E = \left\{\varrho\varepsilon_{\mathrm{I}} + \sum_\alpha \left[\varepsilon_\alpha - \dfrac{(v^\alpha)^2}{2} + \lambda v^\alpha\right]\dfrac{p_\alpha}{A_\alpha}\right\}\dfrac{\delta T}{T}\,, \\[3mm] \delta t_{11} = -(\lambda - v)^2\,\dfrac{\delta T}{T}\sum_\alpha \dfrac{p_\alpha}{A_\alpha}\,, \end{cases}
$$

(*) In the following, all the involved quantities should be evaluated at the unperturbed state; for simplicity of notation we remove the superposed circles 0.

$$
(6.3) \quad \left\{
\begin{aligned}
\delta q_1 = \sum_\alpha \Big\{ & (v^\alpha - v)^2 \left(\frac{3}{2}\lambda - \frac{1}{2} v - v^\alpha\right) + \\
& + (z_\alpha + 1)(\lambda - v^\alpha)\left[(\lambda - v^\alpha)(v^\alpha - v) + \frac{p_\alpha}{\varrho_\alpha}\right] + \\
& \qquad + \left(2\varepsilon_{\mathrm{I}} - 3\varepsilon - \frac{p}{\varrho}\right)(\lambda - v)\Big\} \frac{p_\alpha}{\varrho_\alpha}\frac{\delta T}{T} .
\end{aligned}
\right.
$$

The combination of all these quantities with the last of (6.2) leads to the following algebraic equation of order $2\nu + 1$:

$$
\varrho\varepsilon_{\mathrm{I}}(v - \lambda) + \sum_\alpha \frac{R_\alpha \lambda^2 + S_\alpha \lambda + T_\alpha}{(\lambda - v^\alpha)^2 - p_\alpha/\varrho_\alpha} = 0 , \tag{6.4}
$$

where we have set for brevity

$$
\begin{aligned}
R_\alpha &= z_\alpha (v^\alpha - v) p_\alpha , \\
S_\alpha &= \left[\frac{p_\alpha}{\varrho_\alpha} - 2 z_\alpha v^\alpha (v^\alpha - v)\right] p_\alpha , \\
T_\alpha &= \left\{[\varepsilon_\alpha - z_\alpha (v^\alpha)^2](v - v^\alpha) - \frac{p_\alpha}{\varrho_\alpha} v^\alpha\right\} p_\alpha
\end{aligned}
$$

and have used the relation $\varepsilon_\alpha \varrho_\alpha = z_\alpha p_\alpha = p_\alpha/(\gamma_\alpha - 1)$.

To handle analytically eq. (6.4) is not a simple task, except in some particular case. Let us take, therefore, as unperturbed field, a state in which the velocity of each constituent coincides with the velocity of the mixture as a whole, namely, whatever α may be, we set $v^\alpha = v = 0$. Then, after simple algebra, eq. (6.4) becomes

$$
\lambda \sum_\alpha p_\alpha \left(\frac{1}{\gamma_\alpha - 1} - \frac{p_\alpha}{\varrho_\alpha}\frac{1}{\lambda^2 - p_\alpha/\varrho_\alpha}\right) = 0 , \tag{6.5}
$$

so that $\lambda = 0$ is a standard root of this equation.

Consider

I) $\nu = 1$ (single fluid), then $\gamma_\alpha = \gamma$, $p_\alpha = p$, $\varrho_\alpha = \varrho$ and from (6.5) $\lambda^2 = \gamma p/\varrho = c^2$. We obtain in this case the well-known eigenvalues $\lambda = 0$ and $\lambda^\pm = \pm c$.

II) $\nu = 2$ (case of a binary mixture). Equation (6.5) becomes

$$
A\lambda^4 - (Bc_1^2 + Cc_2^2)\lambda^2 + Dc_1^2 c_2^2 = 0 , \tag{6.6}
$$

where

$$A=\frac{p_1}{\gamma_1-1}+\frac{p_2}{\gamma_2-1}\,,\qquad B=\frac{p_1}{\gamma_1-1}+\frac{p_2}{\gamma_1(\gamma_2-1)}\,,\qquad C=\frac{p_2}{\gamma_2-1}+\frac{p_1}{\gamma_2(\gamma_1-1)}\,,$$

$$D=\frac{p_1}{\gamma_2(\gamma_1-1)}+\frac{p_2}{\gamma_1(\gamma_2-1)}\,,$$

c_1 and c_2 being the sound velocities related, respectively, to each single fluid. As expected, due to the hyperbolicity of the leading system, the four roots of (6.6), which we know are real and distinct, are given by

$$\lambda=\pm\left(\frac{Bc_1^2+Cc_2^2\pm\sqrt{(Bc_1^2+Cc_2^2)^2-4ADc_1^2c_2^2}}{2A}\right)^{\frac{1}{2}}.$$

Notice that the reality of these roots might be also checked at once since it is very easy to prove that $BC>AD$.

III) $\nu=3$. Simple manipulations allow us to write eq. (6.5) in the form

$$A\lambda^6-B\lambda^4+C\lambda^2-D=0\,, \tag{6.7}$$

where this time

$$A=\frac{p_1}{\gamma\;-1}+\frac{p_2}{\gamma_2-1}+\frac{p_3}{\gamma_3-1}\,,$$

$$B=\frac{p_1}{\gamma_1-1}\left(c_1^2+\frac{c_2^2}{\gamma_2}+\frac{c_3^2}{\gamma_3}\right)+\frac{p_2}{\gamma_2-1}\left(\frac{c_1^2}{\gamma_1}+c_2^2+\frac{c_3^2}{\gamma_3}\right)+\frac{p_3}{\gamma_3-1}\left(\frac{c_1^2}{\gamma_1}+\frac{c_2^2}{\gamma_2}+c_3^2\right),$$

$$\begin{aligned}C=\frac{p_1}{\gamma_1-1}\left[c_1^2\left(\frac{c_2^2}{\gamma_2}+\frac{c_3^2}{\gamma_3}\right)+\frac{c_2^2c_3^2}{\gamma_2\gamma_3}\right]&+\frac{p_2}{\gamma_2-1}\left[c_2^2\left(\frac{c_1^2}{\gamma_1}+\frac{c_3^2}{\gamma_3}\right)+\frac{c_1^2c_3^2}{\gamma_1\gamma_3}\right]+\\&+\frac{p_3}{\gamma_3-1}\left[c_3^2\left(\frac{c_1^2}{\gamma_1}+\frac{c_2^2}{\gamma_2}\right)+\frac{c_1^2c_2^2}{\gamma_1\gamma_2}\right],\end{aligned}$$

$$D=\frac{c_1^2c_2^2c_3^2}{\gamma_1\gamma_2\gamma_3}\left(\frac{\gamma_1p_1}{\gamma_1-1}+\frac{\gamma_2p_2}{\gamma_2-1}+\frac{\gamma_3p_3}{\gamma_3-1}\right).$$

Again, due to the hyperbolicity of the problem, the six real and distinct roots of eq. (6.7) have been computed numerically.

b) Global behaviour of the temperature jump. We report here the global study of the temperature jump K whose behaviour is strictly related to $\bar{\eta}$. To this aim observe that, having assumed $v^\alpha=v=0$, $\forall\alpha$, we may write, for instance, $M_\alpha=f_\alpha M_1$ with $f_\alpha=c_1/c_\alpha=(\gamma_1\mathcal{M}_\alpha/\gamma_\alpha\mathcal{M}_1)^{\frac{1}{2}}$. In such a way all the SMN differ from each other for a constant which depends only on the constituent α.

Equation (5.13) then becomes

$$\sum_\alpha c_\alpha p_\alpha \left[\Gamma_\alpha f_\alpha M_1^2 K + (1+\gamma_\alpha f_\alpha^2 M_1^2)\cdot \right.$$
$$\left. \cdot\left(\frac{1-\gamma_\alpha f_\alpha^2 M_1^2 + \sigma_\alpha\sqrt{(1-\gamma_\alpha f_\alpha^2 M_1^2)^2 - 4\gamma_\alpha f_\alpha^2 M_1^2 K}}{2\gamma_\alpha f_\alpha}\right)\right] = 0 . \tag{6.8}$$

It is a simple matter to realize that, for any fixed M_1, the root $K = 0$ can be always obtained by choosing a suitable set of signs σ_α. (One gets in this case that all the jumps of the field functions are identically zero, in agreement with the fact that the continuum solution too satisfies the Rankine-Hugoniot jump conditions.) Equation (6.8), which is of order 2^ν in K, distributes, in fact, all its roots in the set of the 2^ν dispositions of the signatures σ_α, say, for instance, $(++, +-, -+, --)$ or $(+++, ---, ++-, --+, +-+, -+-, +--, -++)$ according to whether $\nu = 2$ or 3, respectively.

The behaviour of K for very large or very low values of M_1^2, as reported below, can help for a better understanding of the profile of $\bar{\eta}$.

i) For very high values of M_1^2, eq. (6.8) may be substituted by

$$\sum_\alpha c_\alpha f_\alpha p_\alpha (2\Gamma_\alpha K - \gamma_\alpha f_\alpha^2 M_1^2 + \sigma_\alpha\sqrt{\gamma_\alpha^2 f_\alpha^4 M_1^4 - 4\gamma_\alpha f_\alpha^2 M_1^2 K}) = 0 \tag{6.9}$$

and it is not difficult to prove that K and M_1^2 must have the same order of infinite. In view of this, the following law

$$K = LM_1^2 \tag{6.10}$$

must hold, with L a necessarily positive constant. Substituting (6.10) into (6.9), we obtain the following algebraic equation of order 2^ν in L:

$$\sum_\alpha B_\alpha (2\Gamma_\alpha N_\alpha L - 1 + \sigma_\alpha \sqrt{1 - 4N_\alpha L}) = 0 , \tag{6.11}$$

where $B_\alpha = \gamma_\alpha f_\alpha^3 c_\alpha p_\alpha$ and $N_\alpha = 1/\gamma_\alpha f_\alpha^2$.

From (6.11) one sees that L must satisfy the following constraint:

$$0 < L \leqslant \frac{1}{4 \max\limits_\alpha N_\alpha} . \tag{6.12}$$

In view of the parabolic law (6.10), the branches of the curve $K = K(M_1^2)$ for high SMN are, therefore, as many as the roots of eq. (6.11) which satisfy (6.12). Discarding, in view of what stated above, the solution $L = 0$, we have numerically found only one branch of K which extends to infinite, namely only one real root $\neq 0$.

ii) For very small values of M_1^2 eq. (6.8) reduces to

$$\sum_\alpha c_\alpha p_\alpha (2\gamma_\alpha \Gamma_\alpha f_\alpha^2 M_1^2 K + 1 + \sigma_\alpha \sqrt{1 - 4\gamma_\alpha f_\alpha^2 M_1^2 K}) = 0 \,. \tag{6.13}$$

One sees at once that, when $M_1^2 \to 0$, none of the following situations $KM_1^2 \to \infty$ or $KM_1^2 \to 0$, or $KM_1^2 \to h \neq (0, \infty)$ can be verified (*). This allows

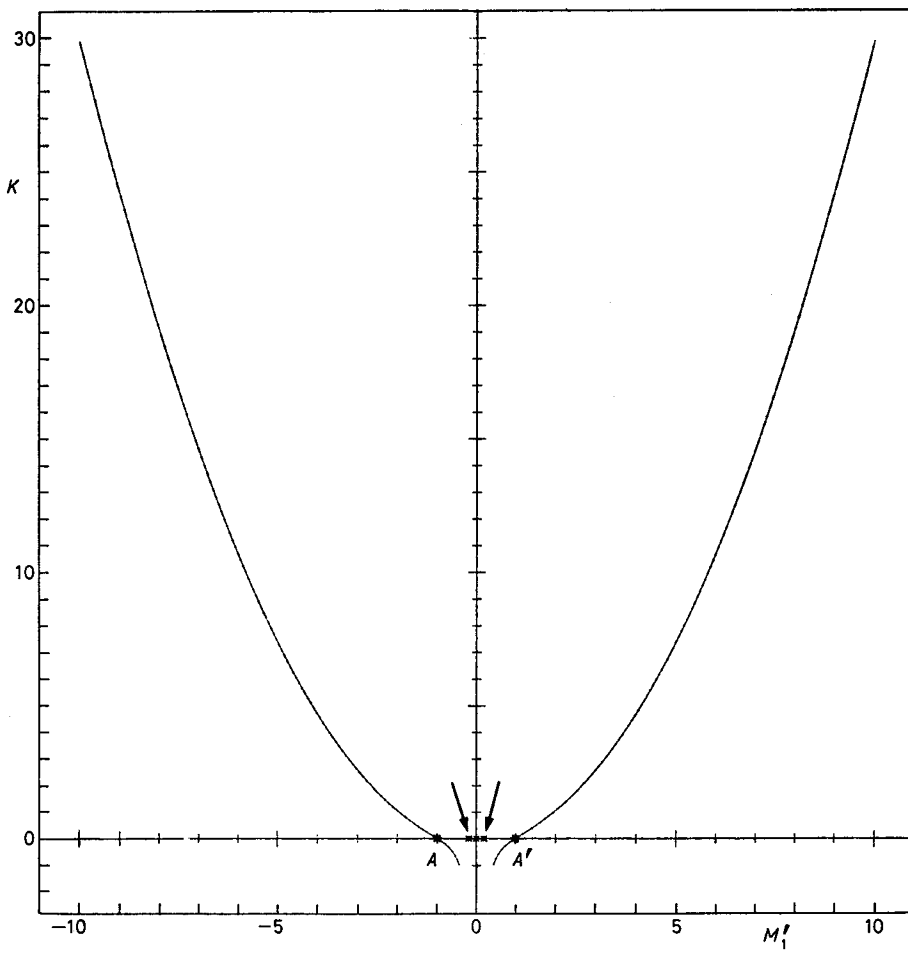

Fig. 1. – Model A). The parabolic profile of the temperature jump K as given by eq. (6.10) for high SMN values, as a function of $M_1' = s/c_1$ (see the text).

(*) In fact, $KM_1^2 \to \pm \infty$ implies K to be an infinite of order larger than M_1^{-2}, but this is not possible in view of the constraint $\inf K > -1$ (see (5.8)). For K positive, the reality of (6.13) imposes that $KM_1^2 < 1/4\gamma_\alpha f_\alpha^2$.

If, on the other hand, $KM_1^2 \to 0$, then, for small values of KM_1^2, eq. (6.13) would reduce to

$$\sum_\alpha c_\alpha p_\alpha [2\gamma_\alpha \Gamma_\alpha f_\alpha^2 M_1^2 K + 1 + \sigma_\alpha (1 - 2\gamma_\alpha f_\alpha^2 M_1^2 K)] = 0 \,. \tag{6.13$'$}$$

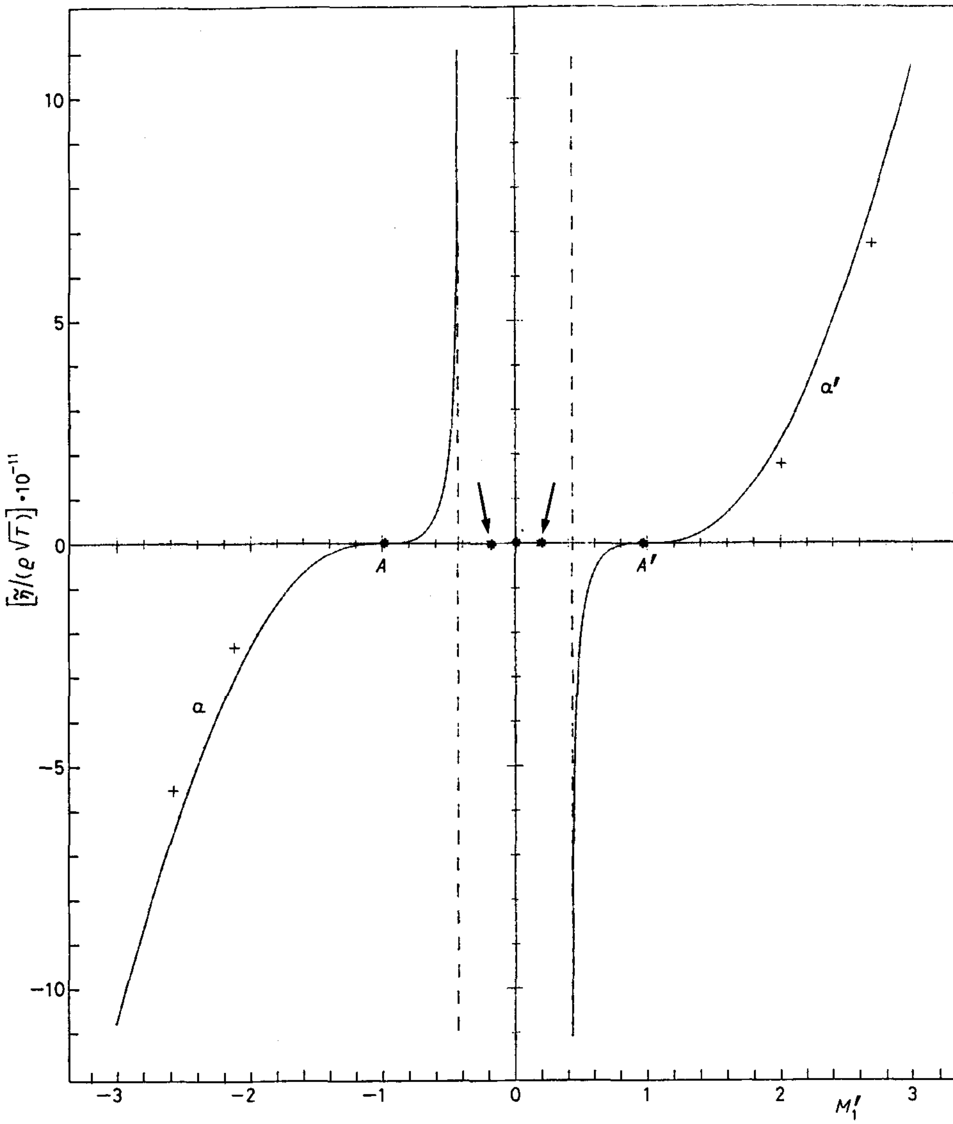

Fig. 2. – Model A). The profile of the shock-generating function $\tilde{\eta}$ for high SMN (see the text).

This equation can never be satisfied for $K \neq 0$ in that, if $\sigma_\alpha = -1$, $\forall \alpha$, then $\sum_\alpha c_\alpha p_\alpha \gamma_\alpha f_\alpha^2(\Gamma_\alpha + 1) \neq 0$; on the contrary, if some of the σ_α is $\neq -1$, then one at least of the terms in (6.13)′ is equal to 2 overcoming eventually the remaining infinitesimal terms of the summation.

Analogously one can prove immediately that the situation $KM_1^2 \to h(> 0)$ is not allowed, *i.e.* K and M_1^{-2} cannot be infinite of the same order.

one to conclude that a neighbourhood of zero for M_1 must exist at which the curve $K = K(M_1^2)$ is not allowed. Properties i) and ii) are clearly exhibited by the plots of K as we shall see later on in discussing numerical models.

c) *Discussion of the numerical results.* The theory exposed has been numerically experimented for the following three models of gaseous mixtures:

A) 3 constituents ($X_H = 0.4$, $X_O = 0.3$, $X_{H_2O} = 0.3$),

B) 2 constituents ($X_H = 0.5$, $X_O = 0.5$),

C) 2 constituents ($X_H = 0.1$, $X_O = 0.9$).

The symbols indicate the concentrations and are self-explicative. In the following we shall use indices 1, 2 and 3 for, respectively, H, O and H_2O. In discussing the graphs related to each model, the reader should take into account that suitable, nonlinear numerical scales have been often used to allow the global plotting of the functions K and $\tilde{\eta}$ at the smallest SMN values. The scale effect remarkably distorts the natural shape of the profiles.

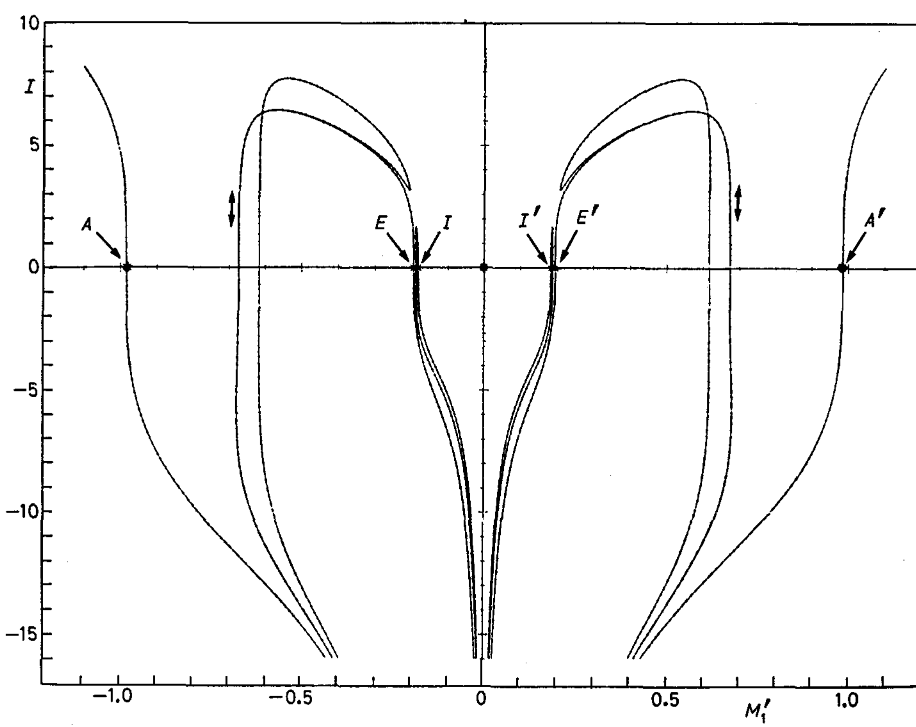

Fig. 3. – Model A). The profile of the temperature jump K, as implicitly given by eq. (6.8), in a narrow interval of SMN. For plotting reasons we found suitable to use in ordinate the mapping $|K| = 10E(-4 + \log_2|I|) = 10^{-4}|I|^a$ ($a = \log_2 10$) with I denoting a length and with the convention of taking $K < 0$ for $I < 0$ (see the text).

Model A). In fig. 1 is shown, on linear scale for both co-ordinates, the parabolic behaviour of the function K for high SMN, $M'_\alpha = s/c_\alpha$ (with $M'_\alpha = -M_\alpha$) (*). This figure does not exhibit the profile of K in its wholeness because, owing to the scales adopted, that part of the graph for the smallest SMN would get crushed upon the abscissa axis. The entire profile is given in fig. 3, where the inner part is clearly shown. Asterisks in this figure indicate those values of M'_1 at which the eigenvalues drop. Except the origin, each of the innermost asterisks, indicated by an arrow, is representative of two indistinguishable eigenvalues. In fig. 3, where only in abscissa the scale is linear, the defor-

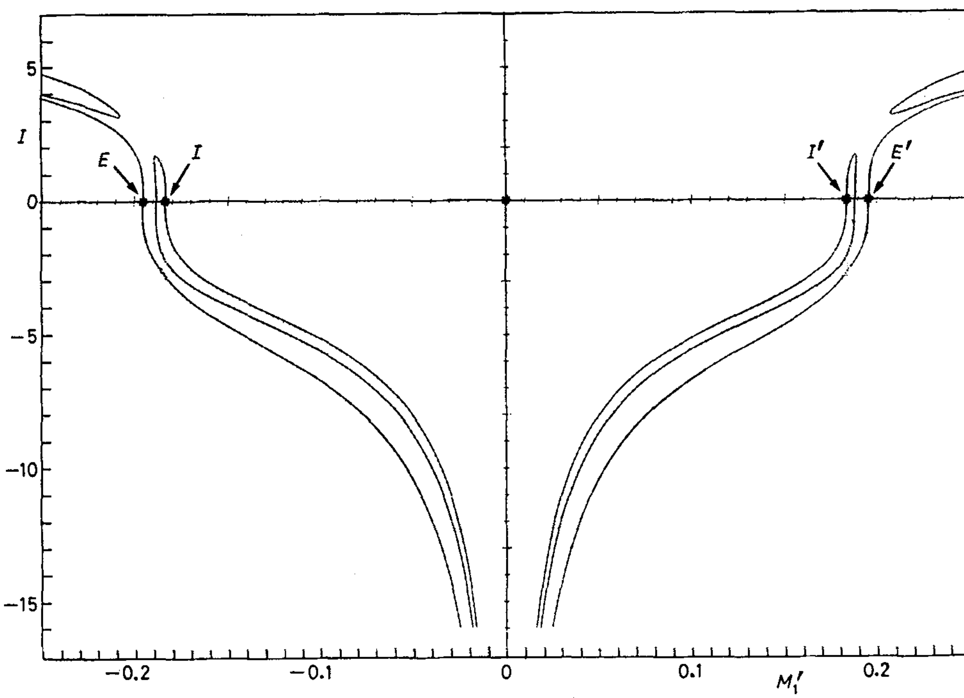

Fig. 3*a*. – Model *A*). An enlargement of fig. 3 in the neighbourhood of $M'_1 = 0$.

(*) Hereafter, as a rule, we shall use, as abscissa, $M'_1 = s/c_1$. To read the abscissa in terms of a generic M'_α we recall that $M'_\alpha = (c_1/c_\alpha)\, M'_1$ with $(c_1, c_2, c_3)/\sqrt{T} = 1.174\cdot 10^4$, $2.692\cdot 10^3$ and $5.676\cdot 10^3$, respectively. For convenience of the reader we report also the mean sound speed in each of the mixtures here considered. Since the mean specific-heat ratio in a mixture writes $\gamma = \left(\sum_\alpha \gamma_\alpha \nu_\alpha/[\mathscr{M}_\alpha(\gamma_\alpha - 1)]\right)\Big/\left(\sum_\alpha \nu_\alpha/[\mathscr{M}_\alpha(\gamma_\alpha - 1)]\right)$ and $p/\varrho = \sum_\alpha \mathscr{R}\nu_\alpha\, T/\mathscr{M}_\alpha$, one finds, respectively, $(c_A, c_B, c_C)/\sqrt{T} = (7.763, 8.501, 4.462)\cdot 10^3$, where the symbols are self-explicative. The mean SMN, expressed in terms of M'_1, is then, respectively, given by $M = M'_1 \times (c_1/c_A,\, c_1/c_B,\, c_1/c_C) = M'_1 \times (1.514, 1.382, 2.633)$. In other words, the values of M'_1 such that $M = 1$, namely $M'_1 = 0.661$, 0.723, 0.380, discriminate between supersonic and subsonic shocks (physically acceptable if $[\![\eta]\!] > 0$) for the models *A*, *B* and *C*, respectively.

mation of the outermost symmetric branches, when compared with those exhibited in fig. 1, is remarkable (see the explanation of fig. 3). Two branches, those indicated by a double-pointed arrow, are yet indistinguishable. An enlargement of the inner part of fig. 3 is shown in fig. 3*a*; here the distincted couples of points, E, I and E', I', indicate the points, indistinguishable in fig. 1 (indicated by arrows), representative of the inner eigenvalues.

In fig. 2 is shown, on linear scales, the partial profile of $\tilde{\eta}$ for high SMN, together with two asymptotes in correspondence with those values of M_1' at which $K = -1$ (see eq. (5.17)). Each vertical line is representative of three indistinguishable asymptotes. The profile of this figure looks, in shape, like the entire profile one obtains in the case of a single fluid with the nonconvexity zone delimited by the two asymptotes and with the two horizontal flex points in correspondence with the eigenvalues ([26]). In our case, on the contrary, the scales we have used do not allow us to see the inner branches for small SMN. As in fig. 1 each of the asterisks, indicated by an arrow, is representative of two eigenvalues.

The profile of $\tilde{\eta}$ in its wholeness is shown in fig. 4. Here the branches b

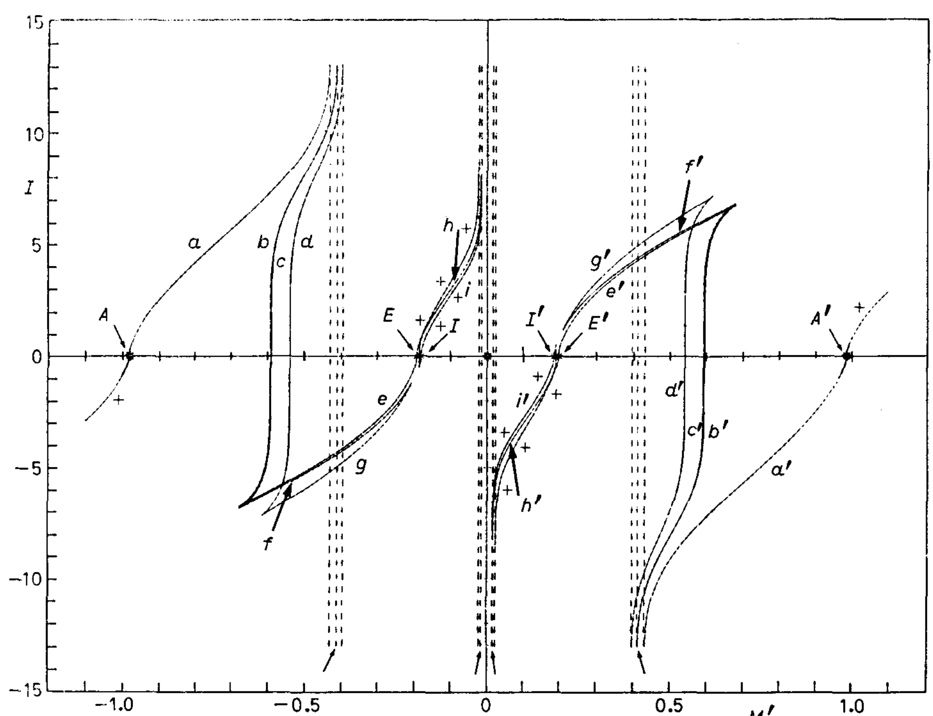

Fig. 4. – Model A). The plot of $\tilde{\eta}$ in a narrow interval of SMN. We have used in ordinate the mapping $|\tilde{\eta}|/(\varrho\sqrt{T}) = 10^{6.5}|I|^{1.5a}(a = \log_2 10)$ with the convention of taking $\tilde{\eta} < 0$ for $I < 0$. Each one of the dashed vertical lines, indicated by arrows, represents two indistinguishable asymptotes (see the text).

and c and the antisymmetric ones b' and c' are yet indistinguishable; besides, due to scale effect, the branches a and a' undergo now to an evident distorsion (compare with fig. 2) and the two flex points are no longer horizontal as they appear in fig. 2 (*).

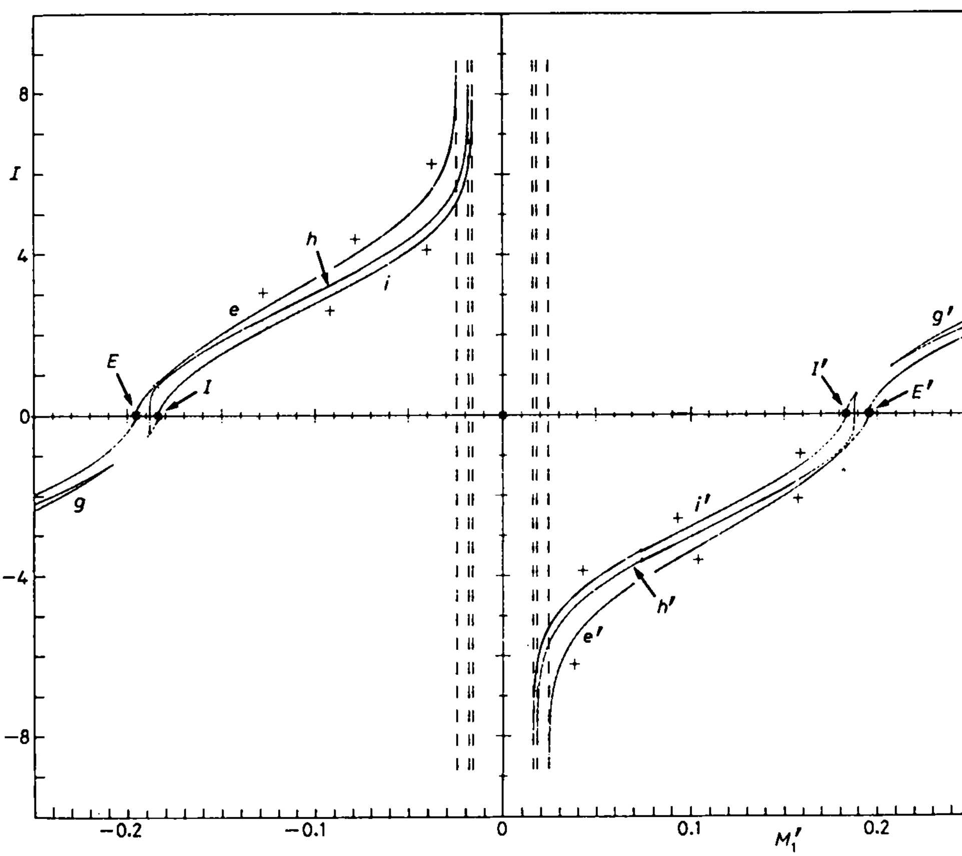

Fig. 4a. – Model A). An enlargement of fig. 4 in the neighborhood of $M_1' = 0$. Notice the two inner asymptotes which are now separated.

(*) In order to get the plot of $\tilde{\eta}$ in its wholeness, we have used in fig. 4 the mapping $|\tilde{\eta}|/\varrho\sqrt{T} = 10E(6.5 + 1.5\log_2|I|) = 10^{6.5}|I|^{1.5a}$ with $a = \log_2 10$. This mapping largely distorts the natural profile of $\tilde{\eta}$, but this is the price one has to pay to get an idea on all the branches which form the curve $\tilde{\eta}$. It follows that

$$1/(\varrho\sqrt{T})\,\mathrm{d}|\tilde{\eta}|/\mathrm{d}M_\alpha = 10^{6.5}\times 1.5a|I|^{(1.5a-1)}\mathrm{d}|I|/\mathrm{d}M_\alpha\,.$$

This relation justifies the transformation of a horizontal flex point in the $(M_\alpha - \tilde{\eta})$-framework, say, for instance, the point A of fig. 2, into the vertical flex point A of fig. 4 in the $(M_\alpha - |I|)$-framework. The mapping is, in fact, such that $\tilde{\eta} = 0 \Leftrightarrow I = 0$, whereas $\mathrm{d}|\tilde{\eta}|/\mathrm{d}M_\alpha = 0$ may come also, through the above differential relation, from an infinite value of $\mathrm{d}|I|/\mathrm{d}M_\alpha$.

TABLE I. – *Summary of some relevant numerical results related to model A* (*).

Branch	Interval of existence	crossing point at M_1'	Asymptote at M_1'	$[\![p_\alpha]\!]$	$[\![V_\alpha]\!]$	$[\![p]\!]$	$[\![V]\!]$	$[\![\eta]\!]$	Remarks
a	$-[\infty, 0.4309)$	-0.9850	-0.4309	+	—	+	—	+	(1)
				+	—				
				+	—				
e	$-[0.6796, 0.0245)$	-0.1955	-0.0245	+	—	—	+	+	(2) (4)
				—	+				
				—	+				
i	$-[0.1893, 0.0161)$	-0.1833	-0.0161	+	—	—	+	+	(3) (4)
				+	—				
				—	+				

(*) The numerical values reported in this table refer to the branches a, e and i of fig. 4. The same deductions hold for the antisymmetric branches a', e' and i', respectively. For the quantities affected by index α (fifth and sixth column) signs in the first, second and third row refer, respectively, to the constituent of the mixture according to the order as written in subsect. **6***c*).

(1) Signs are given for $|M_1'| > 0.9850$ and undergo inversion for $|M_1'| < 0.9850$. At the crossing point $s = \lambda_A = -0.9850c_1$.

(2) Signs are given for $|M_1'| < 0.1955$ and undergo inversion for $|M_1'| > 0.1955$. At the crossing point $s = \lambda_B = -0.1955c_1$.

(3) Signs are given for $|M_1'| > 0.1833$ and undergo inversion for $|M_1'| > 0.1833$. At the crossing point $s = \lambda_I = -0.1833c_1$.

(4) It is worthwhile remarking that, whereas for each single constituent $[\![p_\alpha]\!] > 0$ and $[\![\mathscr{V}_\alpha]\!] < 0$ so that $[\![p_\alpha]\!]/[\![\mathscr{V}_\alpha]\!] < 0$ (in agreement with the entropy principle), for the mixture as a whole one finds numerically that, in correspondence with each of the asymptotes related to the branches e and i, respectively, a left neighbourhood of M_1' exists at which $[\![\mathscr{V}]\!] < 0$. For these branches it results that $[\![p]\!] < 0$, therefore, $[\![p]\!]/[\![\mathscr{V}]\!] > 0$; it also happens that the sign of $[\![\eta]\!]$ changes from positive to negative so that the shock loses its physical meaning. This behaviour of $[\![\eta]\!]$ is at once explained. In fact, for $K \to -1$ (this condition lets $\tilde{\eta}$ become infinite, see (5.17)), $\overset{1}{T} \to 0$ so that $\overset{1}{\eta}_\alpha \to -\infty$. From (5.16) it turns out, therefore, that $[\![\eta]\!]$ becomes definitively negative.

TABLE II. – *Summary of the relevant numerical results related to model B* (*).

Branch	Interval of existence	Crossing point at M_1'	Asymptote at M_1'	$[\![p_\alpha]\!]$	$[\![\mathscr{V}_\alpha]\!]$	$[\![p]\!]$	$[\![\mathscr{V}]\!]$	$[\![\eta]\!]$	Remarks
a	$-[\infty, 0.426)$	-0.981	-0.426	+	—	+	—	+	(1)
				+	—				
c	$-[0.654, 0.021)$	-0.196	-0.021	+	—	—	+	+	(2) (3)
				—	+				

(*) The numerical values here reported are related to the branches a and c of fig. 6. The same deductions as given in the explanation of table I hold.

(1) Signs are given for $|M_1'| > 0.981$ and undergo inversion for $|M_1'| < 0.981$. At the crossing point $s = \lambda_A = -0.981c_1$.

(2) Signs are given for $|M_1'| < 0.196$ and undergo inversion for $|M_1'| > 0.196$. At the crossing point $s = \lambda_C = -0.196c_1$.

(3) For branch c hold the same remarks as made for the jumps $[\![\mathscr{V}]\!]$ and $[\![\eta]\!]$ in Table I, for the inner branches of model A.

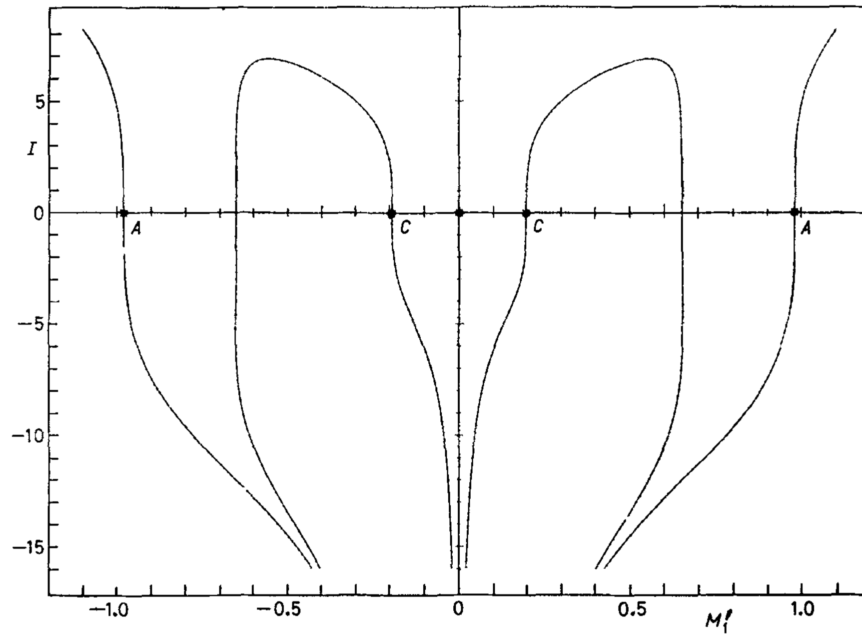

Fig. 5. – Model *B*). The profile of *K*. The same mapping of fig. 3 have been used.

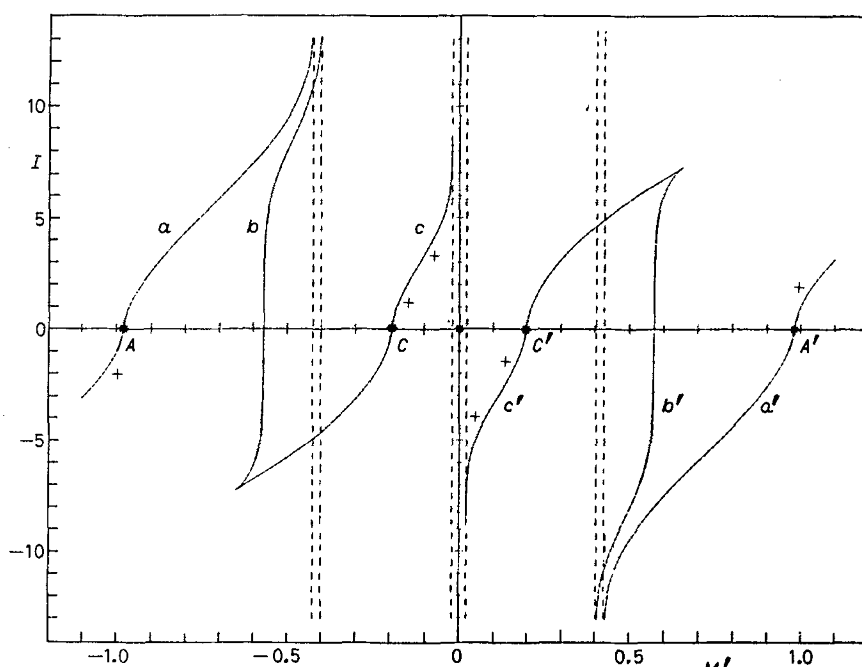

Fig. 6. – Model *B*). The profile of function $\tilde{\eta}$. The same mapping of fig. 4 have been adopted.

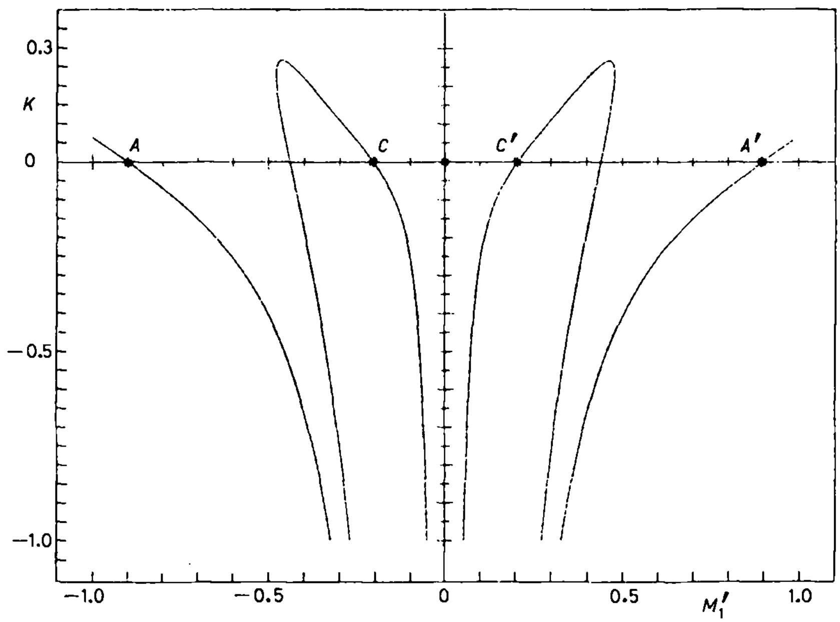

Fig. 7. – Model *C*). The profile of the temperature jump *K*. Here linear scales have been used.

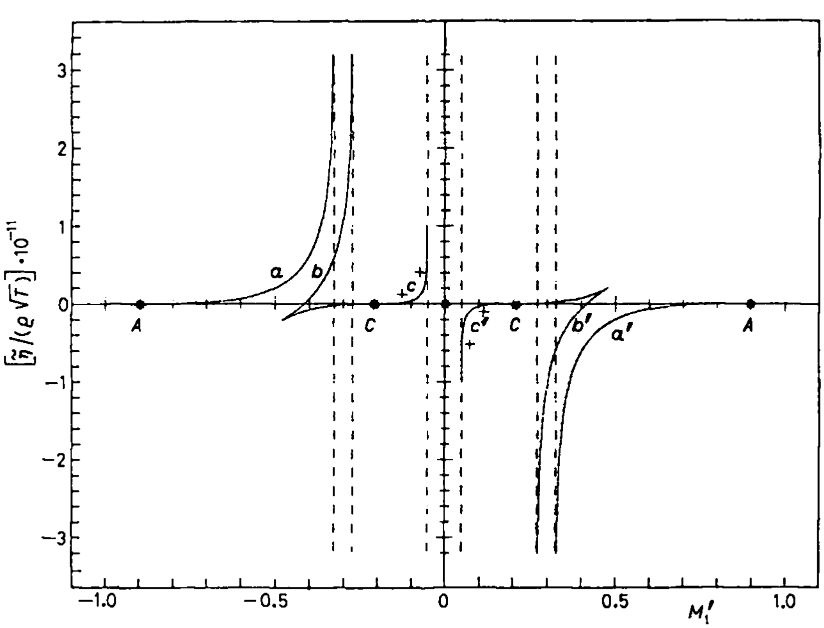

Fig. 8. – Model *C*). The profile of $\tilde{\eta}$ on linear scales. Notice the well-distincted horizontal flex points.

Notice in this figure the zero points of the branches b, c, d and h and their antisymmetric b', c', d' and h'. These zeros are «spurious» roots for $\tilde{\eta}$. In fact, whereas at A, E, I and A', E', I' both $\tilde{\eta}$ and K are zero, so that the shock is identically zero (in all these cases the k-shock condition, namely $\lim_{s\to\lambda}[\![U]\!]=0$, as given in ([12,26]) is well verified), on the contrary, at the crossing points of the foregoing mentioned branches, solely $\tilde{\eta}$ is zero, whereas $K\neq 0$ ensures the existence of a shock.

In table I, which summarizes some of the numerical results, we shall confine our discussion only to the k-shocks, *viz.* to the shocks related to the branches a, e and i and their antisymmetric ones. Signs $+$ along these branches denote the points and so delimit the SMN intervals, where $[\![\eta]\!]>0$, namely where shocks are physically acceptable.

Figure 4a) is a partial enlargement of fig. 4 with the innermost asymptotes clearly separated. The zone in between these asymptotes is the region of non-convexity for the density function $h^0(\boldsymbol{U})$: $\tilde{\eta}$ cannot enter this zone!

Model B). This model is illustrated by fig. 5 and 6 where the graphs of the temperature jump K and of $\tilde{\eta}$ are, respectively, exhibited.

For plotting reasons, as those adduced for model A), we have used the same mapping $K\leftrightarrow I$ and $\tilde{\eta}\leftrightarrow I$ as explicitly given in the explanations of fig. 3 and 4, respectively. Except for a smaller number of branches, the behaviour of each of these graphs is similar to the corresponding of model A).

Table II summarizes some of the numerical results.

Model C). This model has been computed to see how the percentage change of the constituents influences the shape of the plots. The results are similar to those of model B) and are exhibited in fig. 7 and 8. Numerical data are summarized in table III. This time, however, it has been possible to use linear

TABLE III. – *Summary of the relevant numerical results related to model C* (*).

Branch	Interval of existence	Crossing point at M_1'	Asymptote at M_1'	$[\![p_\alpha]\!]$	$[\![\mathscr{V}_\alpha]\!]$	$[\![p]\!]$	$[\![\mathscr{V}]\!]$	$[\![\eta]\!]$	Remarks
a	$-[\infty, 0.327)$	-0.899	-0.327	$+$ $+$	$-$ $-$	$+$	$-$	$+$	(1)
c	$-[0.478, 0.052)$	-0.207	-0.052	$+$ $-$	$-$ $+$	$-$	$+$	$+$	(2) (3)

(*) The numerical values here reported are related to the branches a and c of fig. 8 (or 9). The same deductions as given in the explanation of table I, hold.

(1) Signs are given for $|M_1'|>0.899$. They undergo inversion for $|M_1'|<0.899$. At the crossing point $s=\lambda_A=-0.899c_1$.

(2) Signs are given for $|M_1'|<0.207$. They undergo inversion for $|M_1'|>0.207$. At the crossing point $s=\lambda_C=0.207c_1$.

(3) For branch c hold the same remarks as made for the jumps $[\![\mathscr{V}]\!]$ and $[\![\eta]\!]$ in table I, for the inner branches of model A).

scales for both graphs K and $\tilde{\eta}$. Just to make a direct comparison, we have repeated in fig. 9 the plot of $\tilde{\eta}$ by using a nonlinear scale in ordinate. This last figure shows the desappearance of the horizontal flex point which, on the contrary, is clearly exhibited in fig. 8.

It is worthwhile noticing finally that the cuspidal points of all the graphs concerning $\tilde{\eta}$ discend from the property that, since $\partial\tilde{\eta}/\partial s > 0$, $\tilde{\eta}$ is an increasing function of the shock velocity.

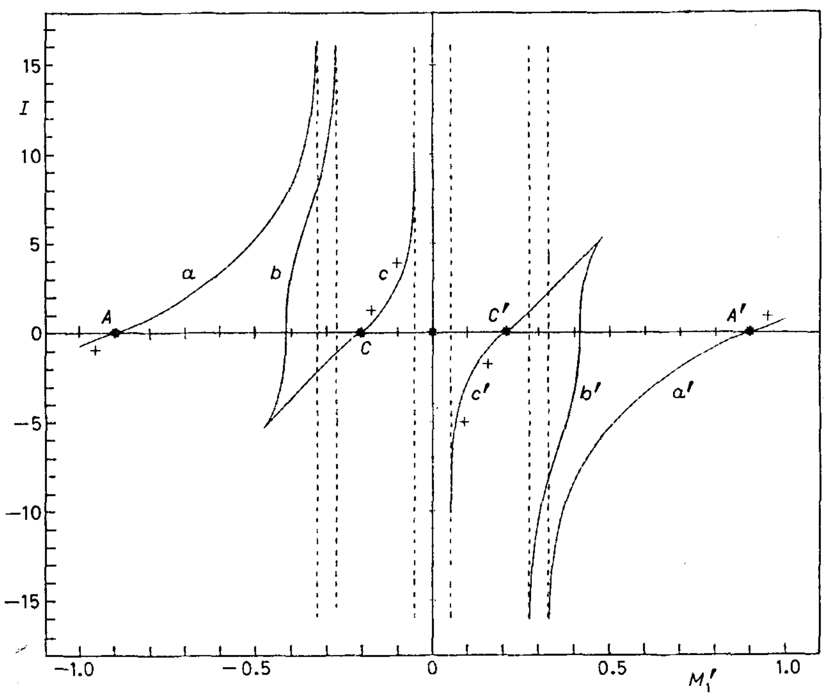

Fig. 9. – Model C). The same as in fig. 8. In ordinate we have used the mapping $|\tilde{\eta}|/(\varrho\sqrt{T}) = 10^{8.5}|I|^{0.751a}$ with the same convention as in fig. 4.

7. – Concluding remarks.

As well known, a shock is physically acceptable when the jump of the specific entropy across the shock itself is positive. In fig. 4, 6 and 8 this property is indicated by signs + along the branches of $\tilde{\eta}$ in agreement with the numerical data as given in the tables I, II and III, respectively.

Our computations have shown that in a mixture of fluids, in addition to the supersonic shock, which is unique in the case of a single constituent, a new type of shock arises (those related to the branches e and i in fig. 4 and to branch c in both figures 6 and 8). These new shocks, confined in narrow ranges of low SMN ($0.0245 < |M_1| < 0.1955$ for branch e and $0.0161 < |M_1| < 0.1833$ for branch i

of model A); $0.021 < |M_1| < 0.196$ and $0.052 < |M_1| < 0.207$ for branch c of model B) and C), respectively) (*) satisfy the thermodynamical principle $[\![\eta]\!] > 0$ and may be explained as due to the mutual interaction of the single components of the mixture.

If we look at the plots of the temperature jump K, we see that, in correspondence to these SMN intervals, $K < 0$, *i.e.* the temperature decreases across these shocks. However, since $[\![\eta]\!] > 0$, this means that some of the densities of the constituents should also decrease sufficiently to keep the entropy jump positive. This mechanism is clearly exhibited by the numerical tables in which one finds that some of the jumps are positive.

As one sees, the problem turns out to be much complicated and a correct physical interpretation might perhaps be given both numerically and experimentally through an accurate evolutive analysis of shock decay and measures of shock amplitudes.

* * *

We would like to thank G. Boillat, P. J. Chen, I. Müller and T. Ruggeri, for useful discussions and appreciations to this work.

This work was largely inspired by the lectures given by I. Müller at the Corso CIME on *Thermodynamics and Constitutive Equations* (Noto, Italy, 1982) and by the lectures on *Propagazione Ondosa* given by G. Boillat and T. Ruggeri at the VI Scuola Estiva di Fisica Matematica (Ravello, Italy, 1981).

This research was supported by the C.N.R. (National Research Council) through the G.N.F.M. (National Group for Mathematical Physics).

Appendix A

Although the understanding of this paper does not require the explicit symmetrization of system (3.1), we give here, for the aim of completeness and in view of applications, the procedure—somewhat heavy—to construct the related Hessian matrices in terms of the main field $\boldsymbol{U}'$. To start with, let us sketch the procedure to write the matrix $H'^0 = \partial^2 h'^0/\partial \boldsymbol{U}' \cdot \partial \boldsymbol{U}'$. We need first to express the differentials $\mathrm{d}h'^0$ in terms of the components of $\mathrm{d}\boldsymbol{U}'$. From $(4.15)_1$ we have

$$\mathrm{d}h'^0 = \mathrm{d}\frac{p}{T} = \frac{1}{T}\sum_\alpha \frac{p_\alpha}{\varrho_\alpha}\,\mathrm{d}\varrho_\alpha . \tag{A.1}$$

(*) We remark that for a mixture of fluids the eigenvalues are no longer simply expressed in terms of the sound velocity (as in the case of a single fluid), but are complicated expressions like those given as roots of eq. (6.4). Besides, the mean sound speed in the mixture does not coincide in general with anyone of the eigenvalues.

In view of what said in the footnote (*) on p. 221, one sees at once that all these new-type shocks are subsonic.

By using thermodynamic relations, simple manipulations allow us to express $d\varrho_\alpha$ in terms of the components of $d\boldsymbol{U}'$. We have

$$\text{(A.2)}\qquad d\varrho_\alpha = \frac{\varrho_\alpha^2 T}{p_\alpha}\left\{d\left(\frac{1}{T}\left[\mu_\alpha^{\mathrm{I}} - \frac{(v^\alpha)^2}{2}\right]\right) + \left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2}\right]d\left(-\frac{1}{T}\right) + \sum_i v_i^\alpha\, d\left(\frac{v_i^\alpha}{T}\right)\right\}.$$

Combining (A.1) and (A.2), we may then write

$$dh'^0 = \sum_\alpha \varrho_\alpha\left\{d\left(\frac{1}{T}\left[\mu_\alpha^{\mathrm{I}} - \frac{(v^\alpha)^2}{2}\right]\right) + \sum_i v_i^\alpha\, d\left(\frac{v_i^\alpha}{T}\right) + \left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2}\right]d\left(-\frac{1}{T}\right)\right\}.$$

It follows, therefore, that

$$\text{(A.3)}\qquad \frac{\partial h'_0}{\partial\{(1/T)[\mu_\alpha^{\mathrm{I}} - (v^\alpha)^2/2]\}} = \varrho_\alpha\,,\qquad \frac{\partial h'^0}{\partial(v_i^\alpha/T)} = \varrho_\alpha v_i^\alpha\,,\qquad \frac{\partial h'^0}{\partial(-1/T)} = E\,.$$

To get the second partial derivatives of h'^0, the best way is then to express $d(\varrho_\alpha v_i^\alpha)$ and dE in terms of the components of $d\boldsymbol{U}'$ as made for $d\varrho_\alpha$. Cumbersome calculations lead to

$$\text{(A.4)}\qquad \left\{\begin{aligned} d(\varrho_\alpha v_j^\alpha) &= a_\alpha v_j^\alpha\, d\hat\mu_\alpha + a_\alpha v_j^\alpha \hat\theta_\alpha\, d\sigma + \sum_i a_\alpha\left((v_i^\alpha v_j^\alpha + \frac{p_\alpha}{\varrho_\alpha}\delta_j^i\right)d\nu_i^\alpha\,,\\ dE &= \sum_\alpha\left(a_\alpha\theta_\alpha\, d\hat\mu_\alpha + \sum_i a_\alpha v_i^\alpha\hat\theta_\alpha\, d\nu_i^\alpha\right) + \left[(2E - \varrho\varepsilon_{\mathrm{I}})T + \sum_\alpha a_\alpha\theta_\alpha^2\right]d\sigma\,,\end{aligned}\right.$$

where we have set for brevity

$$\text{(A.5)}\qquad \left\{\begin{aligned} &\hat\mu_\alpha = \frac{1}{T}\left[\mu_\alpha^{\mathrm{I}} - \frac{(v^\alpha)^2}{2}\right], \qquad \nu_i^\alpha = \frac{v_i^\alpha}{T}\,, \qquad \sigma = -\frac{1}{T}\,,\\ &a_\alpha = \frac{\varrho_\alpha^2 T}{p_\alpha}\,, \qquad \theta_\alpha = \varepsilon_\alpha + \frac{(v^\alpha)^2}{2}\,, \qquad \hat\theta_\alpha = \theta_\alpha + \frac{p_\alpha}{\varrho_\alpha}\,,\end{aligned}\right.$$

so that the elements of the matrix H'^0 may be arranged as

$$\frac{\partial^2 h'^0}{\partial\hat\mu_\alpha\partial\hat\mu_\beta} = a_\alpha\delta_\beta^\alpha\,,\qquad \frac{\partial^2 h'^0}{\partial\hat\mu_\alpha\partial\nu_i^\beta} = a_\alpha v_i^\alpha\delta_\beta^\alpha\,,\qquad \frac{\partial^2 h'^0}{\partial\hat\mu_\alpha\partial\sigma} = a_\alpha\theta_\alpha\,,$$

$$\frac{\partial^2 h'^0}{\partial\nu_j^\alpha\partial\hat\mu_\beta} = a_\alpha v_j^\alpha\delta_\beta^\alpha\,,\qquad \frac{\partial^2 h'^0}{\partial\nu_j^\alpha\partial\nu_i^\beta} = a_\alpha\left(v_i^\alpha v_j^\beta + \frac{p_\alpha}{\varrho_\alpha}\delta_j^i\right)\delta_\beta^\alpha\,,\qquad \frac{\partial^2 h'^0}{\partial\nu_j^\alpha\partial\sigma} = a_\alpha v_j^\alpha\hat\theta_\alpha\,,$$

$$\frac{\partial^2 h'^0}{\partial\sigma\partial\hat\mu_\beta} = a_\beta\theta_\beta\,,\qquad \frac{\partial^2 h'_0}{\partial\sigma\partial\nu_i^\beta} = a_\beta v_i^\beta\hat\theta_\beta\,,\qquad \frac{\partial^2 h'^0}{\partial\sigma\partial\sigma} = (2E - \varrho\varepsilon_{\mathrm{I}})T + \sum_{k=1}^{\nu} a_k\theta_k^2\,.$$

The $(4\nu + 1)\times(4\nu + 1)$ matrix H'^0 may be explicitly written by taking the indices α and β in the Kröneker symbol as, respectively, row index and column index. In the following scheme, the matrix H'^0 may be explicitly written by fixing, for instance, the index $\alpha(=1, 2, ..., \nu)$ and taking, each time, for each term of the fixed row, the index β variable from 1 to ν:

$$\begin{bmatrix} a_\alpha \delta^\alpha_\beta & a_\alpha v^\alpha_1 \delta^\alpha_\beta & a_\alpha v^\alpha_2 \delta^\alpha_\beta & a_\alpha v^\alpha_3 \delta^\alpha_\beta & a_\alpha \theta_\alpha \\ a_\alpha v^\alpha_1 \delta^\alpha_\beta & a_\alpha[(v^\alpha_1)^2 + p_\alpha/\varrho_\alpha]\delta^\alpha_\beta & a_\alpha v^\alpha_1 v^\alpha_2 \delta^\alpha_\beta & a_\alpha v^\alpha_1 v^\alpha_3 \delta^\alpha_\beta & a_\alpha v^\alpha_1 \hat{\theta}_\alpha \\ a_\alpha v^\alpha_2 \delta^\alpha_\beta & a_\alpha v^\alpha_1 v^\alpha_2 \delta^\alpha_\beta & a_\alpha[(v^\alpha_2)^2 + p_\alpha/\varrho_\alpha]\delta^\alpha_\beta & a_\alpha v^\alpha_2 v^\alpha_3 \delta^\alpha_\beta & a_\alpha v^\alpha_2 \hat{\theta}_\alpha \\ a_\alpha v^\alpha_3 \delta^\alpha_\beta & a_\alpha v^\alpha_1 v^\alpha_3 \delta^\alpha_\beta & a_\alpha v^\alpha_2 v^\alpha_3 \delta^\alpha_\beta & a_\alpha[(v^\alpha_3)^2 + p_\alpha/\varrho_\alpha]\delta^\alpha_\beta & a_\alpha v^\alpha_3 \hat{\theta}_\alpha \\ a_\beta \theta_\beta & a_\beta v^\beta_1 \hat{\theta}_\beta & a_\beta v^\beta_2 \hat{\theta}_\beta & a_\beta v^\beta_3 \hat{\theta}_\beta & (2E - \varrho\varepsilon_I)T + \sum_{k=1}^{\nu} a_k \theta_k^2 \end{bmatrix}.$$

The calculations to construct the matrices $H'^i = \partial^2 h'^i/\partial \boldsymbol{U}' \cdot \partial \boldsymbol{U}'$ $(i = 1, 2, 3)$ are a little more complicated. From $(4.15)_2$ one has

$$\text{(A.6)} \qquad \mathrm{d}h'^i = -\sum_\alpha p_\alpha v^\alpha_i \,\mathrm{d}\left(-\frac{1}{T}\right) + \frac{1}{T}\sum_\alpha \left[v^\alpha_i\left(\frac{p_\alpha}{\varrho_\alpha}\,\mathrm{d}\varrho_\alpha + \frac{p_\alpha}{T}\,\mathrm{d}T\right) + p_\alpha \,\mathrm{d}v^\alpha_i\right]$$

besides

$$\text{(A.7)} \qquad \mathrm{d}v^\alpha_i = \mathrm{d}\left(T\,\frac{v^\alpha_i}{T}\right) = v^\alpha_i T\,\mathrm{d}\left(-\frac{1}{T}\right) + T\,\mathrm{d}\left(\frac{v^\alpha_i}{T}\right).$$

Combining (A.2), (A.7) and relations (A.5) with (A.6), we obtain

$$\text{(A.8)} \qquad \mathrm{d}h'^i = \sum_\alpha \left[\varrho_\alpha v^\alpha_i \,\mathrm{d}\hat{\mu}_\alpha + \sum_j (\varrho_\alpha v^\alpha_i v^\alpha_j + p_\alpha \delta^i_j)\,\mathrm{d}\boldsymbol{v}^\alpha_j + \varrho_\alpha v^\alpha_i \hat{\theta}_\alpha \,\mathrm{d}\sigma\right].$$

Therefore, we readily have

$$\text{(A.9)} \qquad \begin{cases} \dfrac{\partial h'^i}{\partial \hat{\mu}_\alpha} = \varrho_\alpha v^\alpha_i\,, \\[2ex] \dfrac{\partial h'^i}{\partial \boldsymbol{v}^\alpha_j} = \varrho_\alpha v^\alpha_j v^\alpha_i + p_\alpha \delta^i_j\,, \\[2ex] \dfrac{\partial h'^i}{\partial \sigma} = \sum_\alpha \varrho_\alpha v^\alpha_i \hat{\theta}_\alpha = [\text{in view of } ((4.5)]Ev_i + q_i - t_{ji}v_j\,. \end{cases}$$

Let us remark, at this point, that expressions (A.3) and (A.9) are, as was to be expected, in perfect agreement with the vectorial equation (2.5). This gives a good check for all the calculations.

To compute the second derivatives of h'^i, the best way is to first differentiate the quantities on the r.h.s. of $(A.9)_{2,3}$ and then express these differentials in terms of the components of $\mathrm{d}\boldsymbol{U}'$. Making use of previous expressions, simple manipulations lead to

$$\text{(A.10)}\qquad \mathrm{d}(\varrho_\alpha v_i^\alpha v_j^\alpha + p_\alpha \delta_j^i) = \varrho_\alpha v_i^\alpha \mathrm{d}v_j^\alpha + v_j^\alpha \mathrm{d}(\varrho_\alpha v_i^\alpha) + \left(\frac{p_\alpha}{\varrho_\alpha}\mathrm{d}\varrho_\alpha + \frac{p_\alpha}{T}\mathrm{d}T\right)\delta_j^i =$$

$$= a_\alpha\left(v_i^\alpha v_j^\alpha + \frac{p_\alpha}{\varrho_\alpha}\delta_j^i\right)\mathrm{d}\hat{\mu}_\alpha + \sum_k a_\alpha\left[v_i^\alpha v_j^\alpha v_k^\alpha + \frac{p_\alpha}{\varrho_\alpha}(v_i^\alpha\delta_k^j + v_j^\alpha\delta_k^i + v_k^\alpha\delta_j^i)\right]\mathrm{d}v_k^\alpha +$$

$$+ a_\alpha\left[v_i^\alpha v_j^\alpha\left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha}\right) + \frac{p_\alpha}{\varrho_\alpha}\hat{\theta}_\alpha\delta_j^i\right]\mathrm{d}\sigma\,.$$

In view of (4.5)

$$\text{(A.11)}\qquad \mathrm{d}(Ev_i + q_i - t_{ji}v_j) = \sum_\alpha\left\{\left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2} + \frac{p_\alpha}{\varrho_\alpha}\right]\mathrm{d}(\varrho_\alpha v_i^\alpha) + \right.$$

$$\left. + \varrho_\alpha v_i^\alpha\,\mathrm{d}\left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2} + \frac{p_\alpha}{\varrho_\alpha}\right]\right\}.$$

On the other hand, we have

$$\mathrm{d}\varepsilon_\alpha = T\varepsilon_\alpha\,\mathrm{d}\left(-\frac{1}{T}\right),$$

$$\mathrm{d}\,\frac{(v^\alpha)^2}{2} = \sum_j v_j^\alpha\,\mathrm{d}v_j^\alpha = [\text{in view of (A.7)}]\; T(v^\alpha)^2\,\mathrm{d}\left(-\frac{1}{T}\right) + \sum_j Tv_j^\alpha\,\mathrm{d}\left(\frac{v_j^\alpha}{T}\right),$$

$$\mathrm{d}\left(\frac{p_\alpha}{\varrho_\alpha}\right) = \frac{p_\alpha T}{\varrho_\alpha}\,\mathrm{d}\left(-\frac{1}{T}\right),$$

so that

$$\text{(A.12)}\qquad \mathrm{d}\left[\varepsilon_\alpha + \frac{(v^\alpha)^2}{2} + \frac{p_\alpha}{\varrho_\alpha}\right] = T\left\{\left[\varepsilon_\alpha + (v^\alpha)^2 + \frac{p_\alpha}{\varrho_\alpha}\right]\mathrm{d}\left(-\frac{1}{T}\right) + \sum_j v_j^\alpha\,\mathrm{d}\left(\frac{v_j^\alpha}{T}\right)\right\}.$$

Substituting $(\text{A.4})_1$ and (A.12) into (A.11), we can finally write

$$\text{(A.13)}\qquad \mathrm{d}(Ev_i + q_i - t_{ji}v_j) =$$

$$= \sum_\alpha\left\{a_\alpha v_i^\alpha\hat{\theta}_\alpha\,\mathrm{d}\hat{\mu}_\alpha + \sum_j a_\alpha\left[v_i^\alpha v_j^\alpha\left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha}\right) + \frac{p_\alpha}{\varrho_\alpha}\hat{\theta}_\alpha\delta_j^i\right]\mathrm{d}v_j^\alpha + a_\alpha v_i^\alpha\left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha}\right)\hat{\theta}_\alpha\,\mathrm{d}\sigma\right\}.$$

Starting from (A.9) and using relations (A.4), (A.10) and (A.13), the second derivatives h'^i write

$$\frac{\partial(\varrho_\alpha v_i^\alpha)}{\partial\hat{\mu}_\beta} = a_\alpha v_i^\alpha\delta_\beta^\alpha\,,\qquad \frac{\partial(\varrho_\alpha v_i^\alpha)}{\partial v_j^\beta} = a_\alpha\left(v_i^\alpha v_j^\alpha + \frac{p_\alpha}{\varrho_\alpha}\delta_j^i\right)\delta_\beta^\alpha\,,\qquad \frac{\partial(\varrho_\alpha v_i^\alpha)}{\partial\sigma} = a_\alpha v_i^\alpha\hat{\theta}_\alpha\,,$$

$$\frac{\partial(\varrho_\alpha v_i^\alpha v_j^\alpha + p_\alpha\delta_j^i)}{\partial\hat{\mu}_\beta} = a_\alpha\left(v_i^\alpha v_j^\alpha + \frac{p_\alpha}{\varrho_\alpha}\delta_j^i\right)\delta_\beta^\alpha\,,$$

$$\frac{\partial(\varrho_\alpha v_i^\alpha v_j^\alpha + p_\alpha\delta_j^i)}{\partial v_k^\beta} = a_\alpha\left[v_i^\alpha v_j^\alpha v_k^\alpha + \frac{p_\alpha}{\varrho_\alpha}(v_i^\alpha\delta_k^j + v_j^\alpha\delta_k^i + v_k^\alpha\delta_j^i)\right]\delta_\beta^\alpha\,,$$

$$\frac{\partial(\varrho_\alpha v_i^\alpha v_j^\alpha + p_\alpha \delta_j^i)}{\partial \sigma} = a_\alpha \left[v_i^\alpha v_j^\alpha \left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha} \right) + \frac{p_\alpha}{\varrho_\alpha} \hat{\theta}_\alpha \delta_j^i \right],$$

$$\frac{\partial(Ev_i + q_i - t_{ji}v_j)}{\partial \hat{\mu}_\alpha} = a_\alpha v_i^\alpha {}_\alpha\hat{\theta},$$

$$\frac{\partial(Ev_i + q_i - t_{ji}v_j)}{\partial v_k^\alpha} = a_\alpha \left[v_i^\alpha v_k^\alpha \left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha} \right) + \frac{p_\alpha}{\varrho_\alpha} \hat{\theta}_\alpha \delta_k^i \right],$$

$$\frac{\partial(Ev_i + q_i - t_{ji}v_j)}{\partial \sigma} = \sum_\alpha a_\alpha v_i^\alpha \left(\hat{\theta}_\alpha + \frac{p_\alpha}{\varrho_\alpha} \right) \hat{\theta}_\alpha .$$

According to the previous convection concerning the indices of the Krönecker symbol, matrices H'^i can be promptly constructed.

Appendix B

The jump of the main field U'.

To search the jump of $\boldsymbol{U}'$, we do not need to proceed by using system (2.8), as suggested by the theory; more simply we can find it straightforwardly from the components of $\boldsymbol{U}'$ themselves. We have, in fact,

$$[\![\check{\boldsymbol{U}}']\!] \equiv \left\{ \left[\!\!\left[\frac{1}{T}\left[\mu_\alpha^{\mathrm{I}} - \frac{(v^\alpha)^2}{2} \right] \right]\!\!\right], \quad \left[\!\!\left[\frac{\boldsymbol{v}^\alpha}{T} \right]\!\!\right], \quad \left[\!\!\left[-\frac{1}{T} \right]\!\!\right] \right\}.$$

With the help of previous results as given in *b*) of sect. **5**, we find

$$\left[\!\!\left[\frac{1}{T}\right]\!\!\right] = -\frac{K}{(K+1)\overset{0}{T}}, \qquad \left[\!\!\left[\frac{\mu_\alpha^{\mathrm{I}}}{T}\right]\!\!\right] = \left[\!\!\left[\frac{\varepsilon_\alpha}{T} - \eta_\alpha + \frac{p_\alpha}{\varrho_\alpha T}\right]\!\!\right] = -[\![\eta_\alpha]\!],$$

$$\left[\!\!\left[\frac{(v^\alpha)^2}{T}\right]\!\!\right] = \frac{1}{(K+1)\overset{0}{T}} \left\{ \frac{\overset{0}{c}_\alpha}{\gamma_\alpha \overset{0}{M}_\alpha} \left(1 - \gamma_\alpha \overset{0}{M}{}_\alpha^2 + 2\gamma_\alpha \frac{\overset{0}{M}_\alpha \overset{0}{v}{}_n^\alpha}{\overset{0}{c}_\alpha} \right) Y_\alpha - K\left[(\overset{0}{v}_\alpha)^2 + \frac{\overset{0}{c}{}_\alpha^2}{\gamma_\alpha} \right]\right.$$

and

$$\left[\!\!\left[\frac{\boldsymbol{v}_\alpha}{T}\right]\!\!\right] = \frac{Y_\alpha \boldsymbol{n} - K\overset{0}{\boldsymbol{v}}{}^\alpha}{(K+1)\overset{0}{T}},$$

so that

$$[\![\check{\boldsymbol{U}}']\!] \equiv \frac{1}{(K+1)\overset{0}{T}} \left\{ -\overset{0}{\varepsilon}_\alpha (K+1) \log\left[\left(\frac{\overset{0}{c}_\alpha - \gamma_\alpha \overset{0}{M}_\alpha Y_\alpha}{\overset{0}{c}_\alpha} \right)^{1-\gamma_\alpha} (1+K)^{\gamma_\alpha} \right] - \right.$$

$$\left. - \frac{\overset{0}{c}_\alpha}{2\gamma_\alpha \overset{0}{M}_\alpha} \left(1 - \gamma_\alpha \overset{0}{M}{}_\alpha^2 + 2\gamma_\alpha \overset{0}{M}_\alpha \frac{\overset{0}{v}{}_n^\alpha}{\overset{0}{c}_\alpha} \right) Y_\alpha + \frac{K}{2}\left[(\overset{0}{v}{}^\alpha)^2 + \frac{\overset{0}{c}{}_\alpha^2}{\gamma_\alpha} \right], \quad Y_\alpha \boldsymbol{n} - K\overset{0}{\boldsymbol{v}}{}^\alpha, K \right\}.$$

In the case of a single constituent, after a great deal of algebra that is omitted here, this jump writes

$$[\![\check{U}']\!] = \frac{1}{Q\overset{0}{T}}\left\{-\overset{0}{\varepsilon}Q\log\left(\frac{1-\gamma+2\gamma\overset{0}{M}{}^2}{1+\gamma}\left[\frac{(\gamma-1)\overset{0}{M}{}^2+2}{(\gamma+1)\overset{0}{M}{}^2}\right]^\gamma\right)-2\overset{0}{c}{}^2(\overset{0}{M}{}^2-1)+\right.$$

$$+\overset{0}{v}(\overset{0}{M}{}^2-1)[(2(\gamma+1)\overset{0}{c}\overset{0}{M}+(\gamma-1)\overset{0}{v}(\gamma\overset{0}{M}{}^2+1)],$$

$$2(\gamma+1)\overset{0}{c}\overset{0}{M}(1-\overset{0}{M}{}^2)\boldsymbol{n}-2(\gamma-1)(\gamma\overset{0}{M}{}^2+1)(\overset{0}{M}{}^2-1)\overset{0}{\boldsymbol{v}},$$

$$\left.2(\gamma-1)(\gamma\overset{0}{M}{}^2+1)(\overset{0}{M}{}^2-1)\right\},$$

where we have set for brevity

$$Q = [(\gamma-1)\overset{0}{M}{}^2+2](1-\gamma+2\gamma\overset{0}{M}{}^2).$$

● RIASSUNTO

Si calcola e si studia la cosiddetta «funzione generatrice dell'urto» in una miscela semplice di ν costituenti ideali. Essendo in generale tale funzione abbastanza complicata, si discutono alcuni modelli numerici nel caso particolare di fluidi composti da 2 o 3 costituenti. Sulla base di tali risultati si scopre che, a differenza di quanto accade nel caso classico di un singolo fluido (dove è ammesso il solo urto supersonico), la mutua interazione dei costituenti della miscela fa nascere un nuovo tipo di k-urti che soddisfano il principio di entropia in corrispondenza a limitati intervalli di piccoli numeri di Mach. Si riportano infine i procedimenti per simmetrizzare le equazioni originali di bilancio in funzione del «campo principale» e per costruire il salto di quest'ultimo attraverso l'urto.

О призводящей функции ударных волн в простой смеси газов.

Резюме (*). — Мы исследуем вывод так называемой производящей функции ударных волн в потоке простой смеси ν идеальных компонент. Из-за аналитической сложности этой функции в общем случае, обсуждаются численные результаты для некоторых частных случаев ($\nu = 2$ и $\nu = 3$). На основе полученных результатов мы обнаружили, что в отличие от классической модели для одной среды (где может существовать только суперзвуковая ударная волна) взаимодействие компонент смеси допускает образование нового типа k ударных волн, в ограниченном интервале малых чисел Маха и которые удовлетворяют принципу энтропии. В приложении приводится процедура симметризации системы исходных уравнений баланса в терминах функций « главного поля » и явное вычисление скачка этого поля через ударную волну.

(*) *Переведено редакцией.*