The Classical Motion of an Extended Charged Particle Revisited.

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Summary. — The motion of a nonrelativistic extended self-interacting particle is analysed. The equation of motion is integro-differential and generates, at variance with the pointlike case, a strictly causal behaviour, thus overcoming all the fundamental shortcomings of the Abraham-Lorentz theory. The motion is endowed with memory, which generates effects totally absent in the structureless case, such as the existence of characteristic damped oscillations, whose frequency and number are determined by the specific structure.

1. – Introduction.

The classical movement of an electrically charged particle is commonly described by an equation due to ABRAHAM and LORENTZ (1),

$$m\ddot{\boldsymbol{r}} = \boldsymbol{F}_{\text{ext}} + m\tau\dddot{\boldsymbol{r}}\,, \tag{1}$$

where self-interaction is taken into account by considering the mass m of the particle to be the sum of an inertial mass and of an electromagnetic

(1) M. ABRAHAM: *Phys. Z.*, **5**, 576 (1904); H. A. LORENTZ: *The Theory of Electrons* (New York, N. Y., 1952).

contribution and by including the radiation reaction $m\tau\dddot{\boldsymbol{r}}$ which contains the caracteristic time

$$\tau = \frac{2e^2}{3mc^3}\,. \tag{2}$$

This is, except for a geometric factor $\frac{2}{3}$, the time needed for light to traverse the particle classical radius e^2/mc^2. Equation (1) has, for a time-dependent external force $\boldsymbol{F}_{ext}(t)$ applied at $t = 0$, the general solution

$$\ddot{\boldsymbol{r}}(t) = \exp[t/\tau]\left[\ddot{\boldsymbol{r}}(0) - \frac{1}{m\tau}\int_0^t \exp[-t'/\tau]\boldsymbol{F}_{ext}(t')\,\mathrm{d}t'\right].$$

But eq. (1) and its solution have well-known drawbacks, which we need not do more than list here [2-6]:

i) The Abraham-Lorentz equation is only approximate and does not apply at all to the free particle [4].

ii) For a point charge the electromagnetic mass diverges.

iii) The electromagnetic contribution to the mass appears with an odd factor $\frac{4}{3}$. For later convenience, we present here a brief account of the origin of this difficulty. The Abraham-Lorentz equation is derived from the assumption that the electromagnetic moment contained within the (extended) particle is given by

$$\boldsymbol{p} = \int \frac{\boldsymbol{S}}{c^2}\,\mathrm{d}^3r\,, \tag{3}$$

where $\boldsymbol{S}$ is the Poynting vector. However, the relativistic version of this equation, namely

$$p_\mu = \frac{i}{c}\int T_{\mu 4}\,\mathrm{d}^3r\,, \tag{4}$$

where $T_{\mu\nu}$ is the stress-energy tensor of the self-field, is not Lorentz covariant;

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(2) E. N. PLASS: *Rev. Mod. Phys.*, **33**, 37 (1961).
(3) J. L. JIMÉNEZ and O. L. FUCHS: *The integrodifferential version of the Abraham-Lorentz equation*, preprint OFIN 19-80, Facultad de Ciencias, UNAM (to be published). This is a recent review of the subject.
(4) L. LANDAU and E. LIFSHITZ: *The Classical Theory of Fields* (Cambridge, Mass., 1951).
(5) P. A. M. DIRAC: *Proc. R. Soc. London Ser. A*, **167**, 148 (1938).
(6) J. D. JACKSON: *Classical Electrodynamics* (New York, N. Y., 1962).

to recover covariance, eq. (4) must be modified to become

$$p_\mu = \frac{\gamma}{c}\int T_{\mu\nu} n_\nu \, \mathrm{d}^3 r \,, \tag{5}$$

where $n_\nu = (\gamma(\boldsymbol{v}/c), i\gamma)$; this eliminates the undesired factor $\frac{4}{3}$ (6).

iv) A particle moving with constant acceleration should radiate, according to the Larmor formula, but for this case the Abraham-Lorentz equation admits solutions with vanishing radiation reaction.

v) An extra boundary condition is needed for a third-order equation such as (1), though the exact original problem does not. (It is sometimes stated, erroneously, that such an additional boundary condition conflicts with Newtonian mechanics.)

vi) The solutions to eq. (1) can exhibit «run-away» behaviour, with an acceleration increasing exponentially even in the absence of external forces.

vii) The solutions to eq. (1) can show the so-called preacceleration which occurs in time before the corresponding force appears.

One of the last two problems, both of which imply a noncausal behaviour, can be eliminated by a suitable choice of the initial value for the acceleration, but not both. Thus $\ddot{\boldsymbol{r}}(0) = 0$ avoids preacceleration, while

$$\ddot{\boldsymbol{r}}(0) = \frac{1}{m\tau}\int_0^\infty \exp[-t/\tau]\boldsymbol{F}_{\text{ext}}(t)\,\mathrm{d}t\,,$$

apparently proposed first by IVANENKO and SOKOLOV (7) and HAAG (8) (see also PLASS (2) and ROHRLICH (9)), removes the run-away solutions. The fully covariant version of the theory due to DIRAC solves automatically difficulty iii) and sidesteps difficulty ii) by using both advanced and retarded potentials to cancel out the divergent terms; it offers the possibility of a final boundary condition to eliminate the run-away solutions, but preacceleration remains a feature of it (9). It is, in fact, generally agreed that there is no satisfactory classical equation of motion for a radiating point charge moving in an external field.

(7) D. IVANENKO and A. A. SOKOLOV: *Klassische Feldtheorie* (Berlin, 1953) (translated from the Russian edition (1949)).
(8) R. HAAG: *Z. Naturforsch. Teil A*, **10**, 752 (1955).
(9) F. ROHRLICH: *Classical Charged Particles* (Reading, Mass., 1965).

It is, therefore, reasonable to consider the alternative extended model and attempt the construction of a consistent classical theory without all these drawbacks. The idea goes back to LORENTZ and has usually been explored in a relativistic framework ([9-13]). In the present paper we shall develop a coherent though simple description of the motion of an extended charge with a completely causal behaviour. Our model is not new, having been to some extent anticipated by KAUP ([12]), but our approach is essentially nonrelativistic, though formulated in such a way that the entire effect of the retarded self-interaction of the extended charge is taken into account; the requirements that come from Lorentz covariance discussed in point iii) above are also satisfied.

It must be stressed that, in allowing for the structure of the particle, even in the nonrelativistic approximation, we completely recover causality, contrary to a common belief that in this way only a finite electromagnetic mass is achieved. Hence a theory of this kind satisfies a physical *desideratum*: to offer an elementary solution to an elementary problem.

The equation of motion that we derive in sect. **2** predicts a number of new properties for the motion; these will be discussed in sect. **3** to **5**. Among other significant points, the theory contains three different mass parameters, rather than the usual two, of which only one corresponds to the Newtonian mass. Furthermore, the extended structure of the charged particle gives rise to at least one characteristic oscillation frequency, together with a corresponding decay time. Moreover, it is interesting to mention that, to guarantee causality, the effective radius of the charge distribution cannot be less than a certain minimum of the order of the classical radius of the electron.

Section **4** explores the behaviour of the particle for a particular model, namely when the charge distribution is of Yukawa type. This appears to be physically more convincing than other models such as the rigid shell of charge that have been considered ([12]) and lends itself to a simple mathematical treatment. In sect. **5** we discuss the possibility of relating the characteristic frequency to that of the zitterbewegung or pair creation frequency $2mc^2/h$; with this criterion the size of the electron comes out to be just that predicted by quantum electrodynamics.

The theory as developed here has several limitations. Firstly, we have considered the charge distribution to be rigid and neglected nonlinear corrections. Secondly, we ignore the possibility of the charge distribution rotating; taking rotations into account may prove to be far from simple, but, since the theory in its nonrelativistic form will apply above all to fairly slowly changing

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([10]) J. S. NODVIK: *Ann. Phys.* (*N. Y.*), **28**, 225 (1964).
([11]) J. D. KAUP: *Phys. Rev.*, **152**, 1130 (1966).
([12]) H. LEVINE, E. J. MONIZ and O. H. SHARP: *Am. J. Phys.*, **45**, 75 (1977); E. J. MONIZ and O. H. SHARP: *Phys. Rev. D*, **15**, 2850 (1977).
([13]) H. M. FRANÇA, G. C. MARQUES and A. J. DA SILVA: *Nuovo Cimento A*, **48**, 65 (1978).

forces, the torques developed over the diameter of an electron should be small, and neglecting them a realistic approximation.

The theory should in principle apply to any charged lepton; because of their fundamental importance, we shall, however, mostly consider electrons. The electrical structure of leptons is essentially unknown, hence special care is taken to show that the fundamental properties of the motion of the extended particle are largely independent of the details of this structure. The Yukawa-type model described in sect. **4** is, therefore, intended chiefly as an illustrative case; but, because of its physically pleasing nature, we propose to use it in our future work ([14]).

2. – The equation of motion for extended particles.

Our starting point is the expression that gives the self-force or radiation reaction force for an extended particle, according to the Lorentz model. If we neglect nonlinear terms in time derivatives of $\boldsymbol{v}$—which are all of order $(v/c)^2$ times the linear terms or smaller—, this expression is ([7])

$$\boldsymbol{F}_{\text{self}} = -\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}\left(\frac{\partial^n\dot{\boldsymbol{v}}}{\partial t^n}\right)\int\mathrm{d}^3r\int\mathrm{d}^3r'|\boldsymbol{r}-\boldsymbol{r}'|^{n-1}\varrho(\boldsymbol{r})\varrho(\boldsymbol{r}')\,, \tag{6}$$

where the charge density $\varrho(\boldsymbol{r})$ is normalized to unity. This force is found by direct calculation of the rate of change of the momentum of the particle, this last being given by eq. (3), *i.e.* the nonrelativistic approximation to eq. (4). Now, as stated above, eq. (4) lacks the appropriate Lorentz transformation properties and must be replaced by eq. (5). The nonrelativistic approximation to the spatial components of eq. (5) is ([6])

$$\boldsymbol{p} = \frac{1}{c^2}\int(\boldsymbol{S}+\overleftrightarrow{T}\cdot\boldsymbol{v})\,\mathrm{d}^3r\,. \tag{7}$$

Thus we must add to eq. (6) the contribution of the term $(1/c^2)\int\overleftrightarrow{T}\cdot\boldsymbol{v}\,\mathrm{d}^3r$. A direct calculation of this contribution, according to, *e.g.*, JACKSON ([6]), shows that this term contributes $-\frac{1}{4}$ times the $n=0$ term in eq. (6) for spherically symmetric charge distributions. In fact, we have that for a spherically symmetric field, the first term in

$$\frac{1}{c^2}\int\overleftrightarrow{T}\cdot\boldsymbol{v}\,\mathrm{d}^3r = \frac{1}{8\pi c^2}\int(2\boldsymbol{E}(\boldsymbol{E}\cdot\boldsymbol{v})-E^2\boldsymbol{v})\,\mathrm{d}^3r$$

([14]) L. DE LA PEÑA: *Stochastic electrodynamics for the free particle*, preprint IFUNAM 80-21 (to be published).

gives two-thirds of the second; thus one gets

$$\frac{1}{c^2}\int \overleftrightarrow{T}\cdot \boldsymbol{v}\,\mathrm{d}^3r = -\frac{\boldsymbol{v}}{24\pi c^2}\int E^2\,\mathrm{d}^3r\,.$$

Since $\boldsymbol{E}$ is the self-field produced by the charge distribution $\varrho(r)$, we can write

$$E^2 = -\boldsymbol{E}\cdot\nabla\varphi = \varphi\nabla\cdot\boldsymbol{E} - \nabla\cdot(\varphi\boldsymbol{E}) = 4\pi e\varrho\varphi - \nabla\cdot(\varphi\boldsymbol{E})\,,$$

where φ is the scalar potential of the self-field. Thus we get finally

$$\frac{1}{c^2}\int \overleftrightarrow{T}\cdot \boldsymbol{v}\,\mathrm{d}^3r = -\frac{e}{6c^2}\,\boldsymbol{v}\int \varrho\varphi\,\mathrm{d}^3r = -\frac{e^2}{6c^2}\,\boldsymbol{v}\int\frac{\varrho(r)\varrho(r')}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r\,\mathrm{d}^3r'\,,$$

as stated above. Hence, for a spherically symmetric charge distribution and by considering only linear terms in $\boldsymbol{v}$ and its time derivatives, the self-force becomes

$$\text{(8)}\qquad \boldsymbol{F}_{\text{self}} = -\frac{2e^2}{3c^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}\frac{\partial^n\dot{\boldsymbol{v}}}{\partial t^n}\int\mathrm{d}^3r\int\mathrm{d}^3r'|\boldsymbol{r}-\boldsymbol{r}'|^{n-1}\varrho(r)\varrho(r') + \frac{e^2}{6c^2}\,\dot{\boldsymbol{v}}\int\mathrm{d}^3r\int\mathrm{d}^3r'|\boldsymbol{r}-\boldsymbol{r}'|^{-1}\varrho(r)\varrho(r')\,.$$

From a more pragmatical point of view, we may consider eq. (8) just to be eq. (6), but with a correct electromagnetic-mass term with which the undesired $\frac{4}{3}$ factor disappears. The series in eq. (8) may be easily summed; in fact, writing for simplicity $R = |\boldsymbol{r}-\boldsymbol{r}'|$, we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}R^n\frac{\partial^n}{\partial t^n}\boldsymbol{a}(t) = \exp\left[-\frac{R}{c}\frac{\partial}{\partial t}\right]\boldsymbol{a}(t) = \boldsymbol{a}\left(t-\frac{R}{c}\right),$$

where $\boldsymbol{a}$ stands for acceleration:

$$\text{(9)}\qquad \boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t)\,.$$

Thus eq. (8) is equivalent to

$$\text{(10)}\qquad \boldsymbol{F}_{\text{self}} = -\frac{2e^2}{3c^2}\int\mathrm{d}^3r\int\mathrm{d}^3r'\frac{1}{R}\varrho(r)\varrho(r')\left[\boldsymbol{a}\left(t-\frac{R}{c}\right)-\frac{1}{4}\,\boldsymbol{a}(t)\right]$$

and the equation of motion for the particle, if we assume that the external force remains essentially unchanged within the dimensions of the charge, is

$$\text{(11)}\qquad \mu\boldsymbol{a} = \boldsymbol{F}_{\text{ext}} - \frac{2e^2}{3c^2}\int\mathrm{d}^3r\int\mathrm{d}^3r'\frac{\varrho(r)\varrho(r')}{R}\left[\boldsymbol{a}\left(t-\frac{R}{c}\right)-\frac{1}{4}\,\boldsymbol{a}(t)\right],$$

where μ is the mechanical (bare) mass. By adding to both sides the term $(e^2/2c^2)\int d^3r\int d^3r' R^{-1}\varrho(r)\varrho(r')$ and defining

$$m = \mu + \frac{e^2}{2c^2}\int d^3r\int d^3r' \frac{\varrho(r)\varrho(r')}{R}, \tag{12}$$

eq. (11) transforms into

$$m\boldsymbol{a} = \boldsymbol{F}_{\text{ext}} - \frac{2e^2}{3c^2}\int d^3r\int d^3r' \frac{\varrho(r)\varrho(r')}{R}\left[\boldsymbol{a}\left(t-\frac{R}{c}\right)-\boldsymbol{a}(t)\right]. \tag{13}$$

This is the desired equation of motion. In the limit of a point particle, it reduces to the Abraham-Lorentz equation, as a Taylor expansion of $\boldsymbol{a}(t-R/c)$ around t shows. Thus, in this limit, m becomes the dressed (classical) mass of the particle, as will be shown below. Equation (13) has been derived by KAUP [11], but using a much more cumbersome procedure.

Equation (13) shows that the self-force of an extended particle produces retarded effects on itself—as it should do since we have explicitly used retarded potentials in its derivation. This means that its present motion depends on all past accelerations and hence on the whole trajectory; thus the particle possesses memory.

Equation (13) may be somewhat simplified by writing it in terms of the form factor of the charge distribution,

$$\tilde{\varrho}(k) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \varrho(r)\exp[-i\boldsymbol{k}\cdot\boldsymbol{r}]\,d^3r. \tag{14}$$

Since we assume spherical symmetry for the charge distribution, all angular integrations can be explicitly performed and eq. (13) reduces to

$$m\boldsymbol{a}(t) = \boldsymbol{F}_{\text{ext}} - 16\pi^2 m\tau c^2\int_{-\infty}^{t} g\big(c(t-t')\big)[\boldsymbol{a}(t')-\boldsymbol{a}(t)]\,dt', \tag{15}$$

where the structure factor $g(r)$ is given by

$$g(r) = \int_0^\infty k|\tilde{\varrho}(k)|^2 \sin kr\,dk. \tag{16}$$

Now eq. (15) still contains an explicit mass correction with

$$\delta m = 16\pi^2 m\tau c\int_0^\infty g(r)\,dr. \tag{17}$$

Hence, introducing m_0 (to be carefully distinguished from μ, the bare mass, see eqs. (11) and (12)) given by

$$m_0 = m - \delta m\,, \tag{18}$$

we can recast eq. (15) in its simplest form

$$m_0\boldsymbol{a}(t) = \boldsymbol{F}_{\text{ext}} - m_0\eta\int\limits_{-\infty}^{t} g\big(c(t-t')\big)\,\boldsymbol{a}(t')\,\mathrm{d}t'\,, \tag{19}$$

where

$$\eta = 16\pi^2\tau c^2\frac{m}{m_0}\,. \tag{20}$$

It is possible to express δm in another form by combining eqs. (16) and (17) and using the formula (P stands for principal value)

$$\int\limits_0^\infty \sin kr\,\mathrm{d}r = \lim_{\sigma\to 0}\frac{k}{k^2+\sigma^2} = P\frac{1}{k}\,.$$

One obtains

$$\delta m = \lim_{\sigma\to 0} 16\pi^2 m\tau c\int\limits_0^\infty \frac{k^2}{k^2+\sigma^2}\,|\tilde{\varrho}(k)|^2\,\mathrm{d}k\,. \tag{21}$$

3. – General properties of the motion.

Here and in the following sections we shall assume that the external force depends only on time. We demonstrate that, in general, the acceleration $\boldsymbol{a}(t)$ has causal behaviour and investigate some properties of the motion. For this purpose a formal solution of the integro-differential equation (18) is convenient. We define

$$g_0(t) = H(t)\,g(t)\,, \tag{22}$$

where $H(t)$ is the Heaviside step function; then the integral in (18) may be extended, so that we have

$$m_0\boldsymbol{a}(t) = \boldsymbol{F}(t) - m_0\eta\int\limits_{-\infty}^{\infty} g_0\big(c(t-t')\big)\,\boldsymbol{a}(t')\,\mathrm{d}t'\,. \tag{23}$$

Fourier transforming this equation and solving for $\tilde{\boldsymbol{a}}$ yields

$$\tilde{\boldsymbol{a}}(\omega) = \frac{1}{m_0}\,\frac{1}{1+\sqrt{2\pi}\,\eta\tilde{g}_0(\omega)}\,\tilde{\boldsymbol{F}}(\omega) \equiv \tilde{G}(\omega)\,\tilde{\boldsymbol{F}}(\omega)\,. \tag{24}$$

Here the entire effect on the response of the structure of the particle is contained in $\tilde{G}(\omega)$, which is somewhat like a generalized inverse-mass operator

$$\tilde{G}(\omega) = \frac{1}{m_0} \cdot \frac{1}{1 + \sqrt{2\pi}\eta\tilde{g}_0(\omega)} . \tag{25}$$

The force factor $\tilde{\boldsymbol{F}}(\omega)$, on the other hand, is determined by the action of the surroundings on the particle. The Fourier inverse of eq. (24) is

$$\boldsymbol{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(t-t')\,\boldsymbol{F}(t')\,\mathrm{d}t' . \tag{26}$$

Here $G(t)$ plays the role of a response or transfer function, in the language of linear-response theory. The acceleration will show a causal behaviour if

$$G(t) = 0 \qquad \text{for } t < 0 , \tag{27}$$

since then eq. (26) predicts a retarded response:

$$\boldsymbol{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} G(t-t')\,\boldsymbol{F}(t')\,\mathrm{d}t' . \tag{28}$$

That the causality condition (27) is actually satisfied under very mild conditions has been shown previously by MONIZ and SHARP [12] and, more explicitly, by FRANÇA *et al.* [13]; we outline their argument in the appendix, for completeness' sake. The conditions referred to are that the form factor $\tilde{\varrho}(k)$ should have no poles for $\mathrm{Re}\, k \geqslant 0$ and that the mass correction δm of eq. (17) should be positive; the latter imposes significant conditions on the minimum radius of the extended charge, as discussed in the appendix. Here we shall consider only the causal case.

That then neither preaccelerations nor run-away solutions appear may be seen by considering a force that acts only for the finite interval $t_0 < t < t_1$. Now eq. (28) may be rewritten as

$$\begin{aligned} \boldsymbol{a}(t) &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} G(t')\boldsymbol{F}(t-t')\,\mathrm{d}t' = \\ &= \sqrt{2\pi}\, i \sum_n \mathrm{Res}\, \tilde{G}(z_n) \int_0^{\infty} \boldsymbol{F}(t-t') \exp[i\omega_n t'] \exp[-\sigma_n t']\,\mathrm{d}t' , \end{aligned} \tag{29}$$

where we have expressed $G(t')$ in terms of the poles $z_n = \omega_n + i\sigma_n$ of $\tilde{G}(z)$. If $t < t_0$, the argument of $\boldsymbol{F}$ is always less than t_0 and all contributions to the force vanish, and so there is no preacceleration. On the other hand, for $t > t_1$ the acceleration does not immediately drop to zero, there are appreciable memory effects, since the domain of $\boldsymbol{F}$ is completely contained in the interval of integration and, in general, the dynamics of the extended charged particle differs significantly from that of a Newtonian one, though (as will be seen in the next section) it can be essentially recovered in the limit $t \to \infty$. Now, as is shown in the appendix, the imaginary parts σ_n of the poles of $\tilde{G}(z)$ are strictly positive; hence, for a force that stops acting at t_1, the values of t' that contribute to the integral in (29) increase with increasing t, the acceleration decreases more and more nearly in exponential fashion and there are no run-away solutions.

4. – The Yukawa distribution.

A specific example is helpful at this point. In the preceding section we saw that the motion depends on the poles of $\tilde{G}(\omega)$; to simplify matters, we chose a case with only one pole pair, namely a Yukawa distribution for the charge:

$$\varrho(r) = \frac{\beta^2}{4\pi}\,\frac{\exp[-\beta r]}{r}\,. \tag{30}$$

For this $\varrho(r)$, we have

$$\tilde{\varrho}(k) = \frac{\beta^2}{(2\pi)^{\frac{3}{2}}}\,\frac{1}{\beta^2 + k^2}\,, \qquad g(r) = \frac{\beta^3}{32\pi^2}\, r \exp[-\beta r]$$

and

$$\tilde{g}_0(\omega) = \frac{\beta^3 c}{32\pi^2\sqrt{2\pi}}\,\frac{1}{(\beta c + i\omega)^2}\,. \tag{31}$$

It is the mathematical simplicity and fast convergence of $g(r)$, as well as the physical plausibility of $\varrho(r)$, that give to this distribution a special appeal as a model of the extended particle. Introducing eq. (31) into eq. (24), we get

$$\tilde{\boldsymbol{a}}(\omega) = \frac{\tilde{\boldsymbol{F}}(\omega)}{m_0}\cdot\frac{(\beta c + i\omega)^2}{(\beta c + i\omega)^2 + \omega_{\text{R}}^2}\,, \tag{32}$$

where

$$\omega_{\text{R}}^2 = \frac{\eta\beta^3 c}{32\pi^2} = \frac{1}{2}\,\tau c^3\beta^3\,\frac{m}{m_0} = \frac{e^2\beta^3}{3m_0}\,. \tag{33}$$

Equation (32) then yields acceleration, velocity and position vectors:

$$(34)\qquad \boldsymbol{a}(t)=\frac{1}{\sqrt{2\pi}m_0}\int_{-\infty}^{\infty}\frac{(\beta c+i\omega)^2}{(\beta c+i\omega)^2+\omega_{\mathrm{R}}^2}\tilde{\boldsymbol{F}}(\omega)\exp[i\omega t]\,\mathrm{d}\omega\,,$$

$$(35)\qquad \boldsymbol{v}(t)=\boldsymbol{v}_1-\frac{i}{\sqrt{2\pi}m_0}\int_{-\infty}^{\infty}\frac{(\beta c+i\omega)^2}{\omega[(\beta c+i\omega)^2+\omega_{\mathrm{R}}^2]}\tilde{\boldsymbol{F}}(\omega)\exp[i\omega t]\,\mathrm{d}\omega$$

and

$$(36)\qquad \boldsymbol{r}(t)=\boldsymbol{r}_1+\boldsymbol{v}_1 t+\frac{1}{\sqrt{2\pi}m_0}\int_{-\infty}^{\infty}\frac{(\beta c+i\omega)^2}{\omega^2[(\beta c+i\omega)^2+\omega_{\mathrm{R}}^2]}\tilde{\boldsymbol{F}}(\omega)\exp[i\omega t]\,\mathrm{d}\omega\,.$$

In addition to the kinematic poles at $\omega=0$ for $\boldsymbol{r}$ and $\boldsymbol{v}$, the poles of the integrands are located at

$$(37)\qquad \omega_{\pm}=i\sigma\pm\omega_{\mathrm{R}}$$

with

$$(38)\qquad \sigma=\beta c\,.$$

We evaluate the integrals by analytic continuation to the complex plane z. As before, for $t<0$ the integrals cancel out, but for $t\geqslant 0$ the contoui of integration lies in the upper half of the complex plane and the integrals do not vanish. Thus, for $t\geqslant 0$, one obtains

$$(39)\qquad \boldsymbol{a}(t)=\frac{\boldsymbol{F}(t)}{m_0}-\frac{\omega_{\mathrm{R}}}{m\sigma^2}(\sigma^2+\omega_{\mathrm{R}}^2)\int_{-\infty}^{t}\boldsymbol{F}(t')\exp[-\sigma(t-t')]\sin\omega_{\mathrm{R}}(t-t')\,\mathrm{d}t'\,,$$

$$(40)\qquad \boldsymbol{v}(t)=\boldsymbol{v}_1+\frac{1}{m}\int_{-\infty}^{t}\boldsymbol{F}(t')\,\mathrm{d}t'+ \\ +\frac{\omega_{\mathrm{R}}}{m\sigma^2}\int_{-\infty}^{t}\boldsymbol{F}(t')\exp[-\sigma(t-t')][\omega_{\mathrm{R}}\cos\omega_{\mathrm{R}}(t-t')+\sigma\sin\omega_{\mathrm{R}}(t-t')]\,\mathrm{d}t'\,,$$

$$(41)\qquad \boldsymbol{r}(t)=\boldsymbol{r}_1+\boldsymbol{v}_1 t+\frac{1}{m}\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}t''\boldsymbol{F}(t'')+ \\ +\frac{2\omega_{\mathrm{R}}^2}{m\sigma(\omega_{\mathrm{R}}^2+\sigma^2)}\int_{-\infty}^{t}\boldsymbol{F}(t')\,\mathrm{d}t'-\frac{\omega_{\mathrm{R}}}{m\sigma^2(\omega_{\mathrm{R}}^2+\sigma^2)}\int_{-\infty}^{t}\boldsymbol{F}(t')\exp[-\sigma(t-t')]\cdot \\ \cdot[(\sigma^2-\omega_{\mathrm{R}}^2)\sin\omega_{\mathrm{R}}(t-t')+2\sigma\omega_{\mathrm{R}}\cos\omega_{\mathrm{R}}(t-t')]\,\mathrm{d}t'\,.$$

In writing these equations we have taken into account that

$$\delta m = 16\pi^2 m\tau c\int_0^\infty g(r)\,\mathrm{d}r = \tfrac{1}{2}\,m\tau\sigma \tag{42}$$

and hence that

$$m = m_0 + \delta m = m_0\left(1 + \frac{\omega_R^2}{\sigma^2}\right). \tag{43}$$

We see that the solution is given in each case by the sum of Newtonian-type terms plus a transient term that oscillates with the characteristic frequency ω_R determined through (33) by the parameter β that measures the size of the particle. We shall come back to this point in the next section.

It is worth noting here that, by using (31) in eq. (A.3) of the appendix, the poles may be found as functions of β only; the imaginary part is given by (38) and the real part by

$$\omega_R^2 = \frac{\frac{1}{2}\tau c^3\beta^3}{1 - \frac{1}{2}\tau c\beta}\,. \tag{44}$$

Hence the poles remain in the upper half-plane only, while the «radius» β^{-1} satisfies

$$\beta^{-1} > \tfrac{1}{2}\tau c\,. \tag{45}$$

Thus the charge distribution must be larger than the classical radius τc for the particle to show causal behaviour; and the peculiarities of preacceleration and run-away solutions appear well before the point particle limit [13]. Similar results may be obtained for other charge distributions and are discussed elsewhere [12].

To get a better insight into these results, it seems worthwhile analysing them both in time and frequency domains.

A) To study the spectral properties of the asymptotic solution, we take

$$\boldsymbol{F}(t) = \boldsymbol{F}_0 H(t)\sin\omega t$$

and consider the case $t \gg \sigma^{-1}$; the results are

i) for $\omega \ll \omega_R$

$$\boldsymbol{a}(t) = \frac{\boldsymbol{F}(t)}{m};$$

ii) for $\omega \approx \omega_R$

$$\boldsymbol{a}(t) = \frac{\boldsymbol{F}(t)}{m}\left[1 + \frac{\omega_R^2}{\sigma^2}\frac{3\omega_R^2}{\sigma^2 + 4\omega_R^2}\right] - \frac{1}{m\sigma^3}\frac{(\sigma^2 + \omega_R^2)(\sigma^2 + 2\omega_R^2)}{\sigma^2 + 4\omega_R^2}\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t};$$

iii) for $\omega \gg \omega_R$

$$\boldsymbol{a}(t) = \frac{\boldsymbol{F}(t)}{m_0} - \frac{1}{m\sigma}\frac{\sigma^2 + \omega_R^2}{\sigma^2 + \omega^2}\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}t}.$$

In general, the response of the extended particle is far from being Newtonian, even in the asymptotic region. The classical behaviour is only obtained in the case $\omega \ll \omega_R$ for sufficiently large times.

B) To study the time response, let us take

$$\boldsymbol{F}(t) = \boldsymbol{F}_0\,\delta(t).$$

The solution is now

$$\boldsymbol{a}(t) = \frac{\boldsymbol{F}_0}{m_0}\delta(t) - \frac{\omega_R}{m\sigma^2}(\sigma^2 + \omega_R^2)\boldsymbol{F}_0 \exp[-\sigma t]\sin\omega_R t,$$

$$\boldsymbol{v}(t) = \boldsymbol{v}_1 + \frac{\boldsymbol{F}_0}{m}H(t) + \frac{\omega_R}{m\sigma^2}\boldsymbol{F}_0 H(t)\exp[-\sigma t][\omega_R\cos\omega_R t + \sigma\sin\omega_R t],$$

$$\boldsymbol{r}(t) = \boldsymbol{r}_1 + \left(\boldsymbol{v}_1 + \frac{\boldsymbol{F}_0}{m}\right)t + \\ + \frac{2\omega_R^2}{m\sigma^3}\boldsymbol{F}_0 H(t) - \frac{\omega_R}{m\sigma^3}\boldsymbol{F}_0 H(t)\exp[-\sigma t][\sigma\sin\omega_R t + 2\omega_R\cos\omega_R t].$$

Thus the acceleration produced by the impulsive force corresponds to the mass m_0, not to m, in agreement with case *A*). At time $t = 0$ there appears an impulsive acceleration and then an oscillatory negative transient (an undershoot) begins to develop. Both the frequency ω_R and the decay constant σ of this transient are determined by the size of the particle. The velocity has a more complicated behaviour; for $t \gg \sigma^{-1}$, $\boldsymbol{v}(t)$ has the classical form $\boldsymbol{v}_1 + \boldsymbol{F}_0/m$, which ascribes to the particle the « classical » mass m. However, for very short times, the velocity may be approximated by

$$\boldsymbol{v}(t) \approx \boldsymbol{v}_1 + \frac{\boldsymbol{F}_0}{m} + \frac{\omega_R^2}{m\sigma^2}\boldsymbol{F}_0 = \boldsymbol{v}_1 + \frac{\boldsymbol{F}_0}{m_0},$$

where we have used eq. (43); thus, if we measure the mass of the particle through its velocity immediately after the application of the impulsive force, we will get the value m_0, not m. The Newtonian part of $\boldsymbol{r}(t)$ is also characterized by the classical mass m for all $t \gg 0$; however, there is a non-Newtonian contribution to $\boldsymbol{r}(t)$ that never disappears, namely the term $(2\omega_R^2/m\sigma^3)\boldsymbol{F}_0$.

5. – Possible connection with quantum dynamics.

So far the « radius » of the particle $r_0 = \beta^{-1}$ has remained as a free parameter except for the lower bound (45). We may attempt to fix it by arguments like those used in quantum electrodynamics ([15]). It seems, however, more natural to determine r_0 by assigning a « reasonable » value to ω_R: there exists a general characteristic frequency of oscillation of the free electron, namely that associated with the zitterbewegung predicted by the Dirac theory. We, therefore, propose the identification

$$\omega_R = \frac{2mc^2}{\hbar} \,. \tag{46}$$

To investigate the radius predicted by this assumption, we substitute (46) in (44) and express the result in terms of the fine-structure constant $\alpha = e^2/\hbar c$ and the Compton wave-length $\lambda\!\!\!^-_C = \hbar/mc$. We get after simple algebraic manipulations

$$(\sqrt{\alpha}\,\lambda\!\!\!^-_C\beta)^3 + 4\alpha(\sqrt{\alpha}\,\lambda\!\!\!^-_C\beta) = 12\sqrt{\alpha}\,.$$

This equation has a single real root near $\sqrt{\alpha}\,\lambda\!\!\!^-_C\beta \simeq 1$; hence we may neglect the second term to get

$$r_0^{-1} = \beta = \frac{(12\sqrt{\alpha})^{\frac{1}{3}}}{\sqrt{\alpha}\,\lambda\!\!\!^-_C} \simeq \frac{1}{\sqrt{\alpha}\,\lambda\!\!\!^-_C}\,,$$

or

$$r_0^2 = r_c\lambda\!\!\!^-_C\,,$$

where $r_c = e^2/mc^2 = \alpha\lambda\!\!\!^-_C$ is the classical electron radius. Equation (46) implies that the electron radius is equal to the geometric mean of the classical radius and Compton wave-length, just the radius that quantum electrodynamics assigns to the electron due to the radiative corrections ([15,16]). This most

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([15]) N. N. Bogoliubov and S. V. Tyablikov: *Izv. Akad. Nauk Ukr. SSR*, **5**, 10 (1946).
([16]) The argument goes essentially as follows. The Lamb shift for the ground state of a harmonic oscillator is, in order of magnitude, $\alpha\hbar^2\omega^2/\pi mc^2$; if we ascribe this energy to vibrations of the oscillator with amplitude a, so that it equals $m\omega^2 a^2$, then a is to be interpreted as an effective radius of the electron. One gets $a^2 \approx \alpha\hbar^2/\pi m^2c^2 \sim \alpha\lambda\!\!\!^-_C{}^2$, which is the result referred to above.

gratifying result seems to justify the use of eq. (46). The picture that emerges is interesting, because the theory, in spite of its classical (nonquantal and nonrelativistic) nature, gives acceptable results.

The above hypothesis may be used to make some other estimates. From eq. (42) we get

$$\frac{\delta m}{m_0} \simeq \frac{\omega_R^2}{\sigma^2} = 4\alpha\,,$$

which shows that the mass renormalization is of order α. The period of oscillation will now be

$$T_R = \frac{2\pi}{\omega_R} = \frac{3\pi}{2\alpha}\tau \sim 650\tau\,,$$

that is almost three orders or magnitude larger than the radiation time τ. Finally, these results also show that the amplitude of the transient part of the velocity is rather small compared with the corresponding classical contribution. For example, from eq. (40) we see that this ratio is of order $\omega_R^2/\sigma^2 \simeq \delta m/m_0$ or 3 %. Moreover, the logarithmic decrement is σ/ω_R, so that this rather small oscillation is damped to 1/360 of its amplitude in a single period.

6. – Concluding remarks.

The preceding results show that taking into account the structure of the classical self-interacting particle is enough to solve all fundamental problems characteristic of the Abraham-Lorentz theory. More specifically, we have seen that the theory applies for all $\boldsymbol{F}(t)$, including the free particle; that, for physically acceptable charge distributions, all the mass parameters μ, m, m_0 and δm are finite and the annoying factor 4/3 relating μ and m does not appear; that the response is causal (retarded and finite), which implies freedom from preacceleration and run-away solutions. Thus all the discussions about the need of modification of our usual points of view in connection with causality [17] within the classical context are at least unnecessary: preacceleration is the price for a bad approximation, not a physical phenomenon.

Another interesting result of the theory is related to the properties of the mass parameters, which seem to be less simple than is normally assumed from our naive generalizations from Newtonian physics. In particular, we have seen that the Newtonian mass is the low-frequency, long-time mass, as measured by the velocity or the acceleration. A point of principle that must not

[17] See, *e.g.*, A. GRÜNBAUM and A. JANIS: *Am. J. Phys.*, **46**, 337 (1978).

pass unnoticed in connection with this discussion is the following. In classical dynamics, there is no conceptual problem in turning off or on the external force, but then the mass of the extended particle is undefined. However, when we go to deeper theories, like quantum electrodynamics for example, we recognize that the « free » particle is just a concept that has no physical counterpart, since all particles interact at least with the residual (stochastic) vacuum of each fundamental field. Therefore, the particle presents itself always dressed and no conceptual uncertainty appears.

The infinite memory shown by the extended electron endows its motion with specific and complex properties that make the dynamical problem far richer than its corresponding structureless approximation. These complexities may reveal themselves even more important for a confined particle due to the cumulative effects of the memory, which could produce essentially new results, unknown to the Newtonian theory. This is one of the reasons why we consider this theory important in connection with approaches such as stochastic electrodynamics ([18]), but we reserve the discussion of these problems for a forthcoming publication.

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APPENDIX

We have

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G}(\omega) \exp[+i\omega t] \, d\omega \,. \tag{A.1}$$

To evaluate this for $t > 0$, the contour in the complex plane is best closed by a semi-circle at infinity in the upper half-plane, while for $t < 0$ the lower half-plane is used. Thus condition (27) is equivalent to the condition that all the poles of $\tilde{G}(z)$ at which $z = \omega + i\sigma$ lie in the upper half-plane.

Using (16), (22) and the inverse transform to (A.1), we have

$$\tilde{g}_0(z) = \int_0^{\infty} dt \exp[-i\omega t] \exp[\sigma t] \int_0^{\infty} k |\tilde{\varrho}(k)|^2 \sin ckt \, dk \,. \tag{A.2}$$

([18]) A brief survey by T. H. BOYER is to be found in *Foundations of Radiation Theory and Quantum Electrodynamics*, edited by A. O. BARUT (New York, N. Y., 1980); P. CLAVERIE and S. DINER: *Int. J. Quantum Chem.*, **12**, *Suppl.* 1, 41 (1977); L. DE LA PEÑA and A. M. CETTO: *J. Math. Phys.* (*N. Y.*), **18**, 1612 (1977); **20**, 469 (1979).

For negative σ, the integration over t can be carried out, yielding

$$\tilde{g}_0(z) = \int_0^\infty \frac{k}{c^2k^2 - z^2} |\varrho(k)|^2 \mathrm{d}k \, .$$

If, in eq. (25), $m_0 > 0$, the poles of $\tilde{G}(z)$ are the solutions of

$$1 + \sqrt{2\pi}\eta\tilde{g}_0(z) = 0 \, . \tag{A.3}$$

Separating the real and imaginary parts, we have the equations

$$1 + \sqrt{2\pi}\eta \int_0^\infty \frac{k|\tilde{\varrho}(k)|^2(c^2k^2 - \omega^2 + \sigma^2)}{(c^2k^2 - \omega^2 + \sigma^2)^2 + \omega^2\sigma^2} \mathrm{d}k = 0 \tag{A.4}$$

and

$$\omega\sigma \int_0^\infty \frac{k|\tilde{\varrho}(k)|^2}{(c^2k^2 - \omega^2 + \sigma^2)^2 + \omega^2\sigma^2} \mathrm{d}k = 0 \, . \tag{A.5}$$

Since the integral in (A.5) is positive definite, ω must vanish for a solution to exist when $\sigma < 0$. But, for $\omega = 0$, the integral in (A.4) is also positive definite and no solution exists if $\eta \geqslant 0$. There are no poles in the lower half-plane. Nor are there any on the real axis if one assumes that $g(r)$ has support on a set of nonzero measures, for then (A.3) becomes

$$\tilde{g}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_0(t) \exp[-i\omega t] \mathrm{d}t = -\frac{1}{\sqrt{2\pi}\eta} \, ,$$

the only solution of which is $g_0(t) \sim \delta(t)$. But this is not physically plausible: as eq. (16) shows, an infinitely extended uniform charge would have this behaviour. Hence, if $\tilde{\varrho}(k)$ has no poles for $k \geqslant 0$, then neither $\tilde{G}(z)$ has poles on the real axis and all poles lie strictly in the upper half-plane; the only possible exception to this result demands the use of very peculiar charge distributions ([13,19]).

Equation (A.2) shows that, whenever $\omega + i\sigma$ is a pole, then so is $-\omega + i\sigma$, for this changes only the sign of the imaginary part of $\tilde{g}_0(z)$, which from (A.5) must vanish. Thus the poles occur in pairs in the upper half-plane, arranged symmetrically around the imaginary axis. Note that, in general, the origin does not represent a pole pair, though there are forms of $\varrho(r)$ for which it can be reached; thus the point particle is not necessarily the limiting case of the extended charge distribution.

([19]) These exceptional charge distributions that generate poles on the real axis ($\sigma = 0$) and hence correspond to stationary oscillations have been used as models to explain the atomic stability. See, *e.g.*, D. BOHM and M. WEINSTEIN: *Phys. Rev.*, **74**, 1789 (1948); G. H. GOEDECKE: *Phys. Rev. Sect. B*, **135**, 281 (1964).

The above demonstration heavily depends on the assumption $\eta \geqslant 0$. This condition characterizes what we may call the zone of defined causality; it can be expressed in a physically more transparent form as follows. According to eqs. (17) and (20), η is given by

$$\eta = 16\pi^2 \tau c^2 \left[1 - 16\pi^2 \tau c \int_0^\infty g(r)\, \mathrm{d}r\right];$$

in terms of the characteristic radius $\mathcal{R}$ of the charge distribution, defined by

$$\mathcal{R}^{-1} = 16\pi^2 \int_0^\infty g(r)\, \mathrm{d}r\,,$$

the condition $\eta \geqslant 0$ is thus expressed as

$$\mathcal{R} \geqslant \tau c = \tfrac{2}{3} r_c. \tag{A.6}$$

This is the minimum size that a charge distribution must possess to guarantee that its motion is causal. For the Yukawa distribution eq. (A.6) reduces to eq. (45) in the text.

● RIASSUNTO (*)

Si analizza il moto di una particella non relativistica estesa autointeragente. L'equazione di moto è integrodifferenziale e genera, diversamente dal caso puntiforme, un comportamento strettamente causale, cosí superando tutti gli svantaggi fondamentali della teoria di Abraham-Lorentz. Il moto è dotato di memoria, che genera effetti totalmente assenti nel caso senza struttura, come l'esistenza di caratterestiche oscillazioni smorzate, la frequenza e il numero delle quali sono determinati dalla struttura specifica.

(*) *Traduzione a cura della Redazione.*

Классическое движение протяженной заряженной частицы.

Резюме (*). — Анализируется движение нерелятивистской протяженной самовзаимодействующей частицы. Уравнение движения является интегродифференциальным и, в противоречии с точечно-подобным случаем, приводит к строго причинному поведению, тем самым устраняются все основные недостатки теории Абрагама-Лоренца. Движение обладает памятью, что приводит к возникновению эффектов, полностью отсутствующих в бесструктурном случае, таких как наличие характеристических затухающих осцилляций, частота и число которых определяется специальной структурой.

(*) *Переведено редакцией.*