Soliton Surfaces.

II. - Geometric Unification of Solvable Nonlinearities (*).

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Summary. – Two applications of the concept of soliton surfaces are discussed. Firstly, soliton surfaces can serve as a territory of unification of four types of solvable nonlinearities: 1) soliton, 2) strings, 3) spins and 4) chiral models. It is conjectured that models 2), 3) and 4) associated with a given soliton system are gauge equivalent to this soliton system. Secondly, an explicit construction of the soliton surface associated with a given soliton solution gives simultaneously the corresponding solutions to models 2), 3) and 4). Using the Hilbert-Riemann problem technique a construction of N-soliton surfaces is described. Examples including new soliton systems are given.

The concept of soliton surfaces was introduced in (1). These surfaces can be associated with a broad class of soliton systems. This association is a generalization of the well-known connection between pseudospherical surfaces in E^3 and the sine-Gordon equation (2-4). The Gauss-Mainardi-Codazzi (GMC) system of the theory of submanifolds of flat spaces (5) when applied to the soliton surfaces of a given soliton system is reducible to that soliton system. It proves a geometric nature of solitons.

Here we discuss two applications of soliton surfaces. Firstly, we show soliton surfaces are a proper territory of unification of four types of solvable nonlinearities:

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1) solitons, 2) strings, 3) spins and 4) chiral models. We conjecture that models 2), 3) and 4) geometrically unified with a given soliton system are also gauge-equivalent to that system. Secondly, as a result of the unification an explicit construction of a soliton surface associated with a given soliton solution gives simultaneously the corresponding solutions to models 2), 3) and 4) (geometric way to solve nonlinear models). Using the Hilbert-Riemann problem technique [6,7], we describe general N-soliton surfaces (N-surfaces). An algorithm to construct N-surfaces is a purely algebraic (algebra of projectors) and the passage from N-surface to $(N+1)$-surface is a geometric interpretation of Bäcklund transformations [1] in the spirit of old geometry [3,9]. Very recently this sample technique allowed one to find new vortex motions of high complexity [8]. As an illustration of these concepts we give examples including new soliton systems *e.g.* 3-field extension of the sine-Gordon equation and of the Lund-Regge-Pohlmeyer-Getmanov system [10] as well.

The conventions used in this paper are: $x = x^\mu = (x^1, x^2)$ and $\psi_{,\mu} = \partial\psi/\partial x^\mu$ etc. Consider a soliton system for the fields $\psi^A(x)$ $(A = 1, 2, ..., f)$ with the associated linear problem

$$\Phi_{,\mu} = g_\mu \Phi \,, \tag{1}$$

where

$$g_\mu(x, \zeta) = \gamma_\mu[\psi^A(x), \psi^A_{,\nu}(x), ...; \zeta]$$

belongs to a d-dimensional, real, semi-simple Lie algebra g of $(n\times n)$-matrices, provided ζ (spectral parameter) is real, while the wave function $\Phi = \Phi(x, \zeta)$ is a $(n\times n)$-matrix function with values in G (Lie group of the Lie algebra g). For a given soliton field ψ^A solving eq. (1) gives the corresponding wave function $\Phi(x, \zeta)$ and the ζ-family of soliton surfaces associated with ψ^A is given by [1]

$$g \ni r = r(x, \zeta) = \Phi^{-1}(x, \zeta)\, \Phi_{,\zeta}(x, \zeta) \,. \tag{2}$$

Soliton surfaces are embedded into g treated as a flat $R^d(++\,...\,-)$ space equipped with the Killing-Cartan scalar product [11] $x\cdot y$ $(x, y \in g)$. The independent variables x^1, x^2 become co-ordinates upon surfaces (2).

Soliton surfaces (2) define (and also are defined by) the metric and the second fundamental forms denoted by I and II^a $(a = 1, 2, ..., d-2)$, respectively [5],

$$\mathrm{I} = \mathrm{d}s^2 = g_{\mu\nu}(x, \zeta)\, \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{3}$$

$$\mathrm{II}^a = d^a_{\nu\mu}(x, \zeta)\, \mathrm{d}x^\mu \mathrm{d}x^\nu \qquad (a = 1, 2, ..., d-2) \,. \tag{4}$$

From (2) the metric can be easily calculated in terms of the matrices of the linear

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problem (1):

(5) $$g_{\mu\nu}(x, \zeta) = g_{\mu,\zeta}(x, \zeta)\cdot g_{\nu,\zeta}(x, \zeta)\,,$$

while, generally, the forms (4) admit more complicated formulae, see ([1]). In the further discussion we choose $g = su_{p,q}$ $(p + q = n)$ and assume the following structure of the linear problem (1):

(6a) $$g_1 = \zeta i h_1 + k_1[\psi^A(x), \psi^A_{,\nu}(x), \ldots]\,,$$

(6b) $$g_2 = \omega(\zeta) i h_2 + k_2[\psi^A(x), \psi^A_{,\nu}(x), \ldots; \zeta]\,,$$

where ih_m $(m = 1, 2)$ are constant and commuting elements of $su_{p,q}$ with $ih_m\cdot ih_m < 0$ $(m = 1, 2)$, $\omega \in R$ if $\zeta \in R$ and for $\psi^A = 0$ (identically) $k_m = 0$ $(m = 1, 2)$. In this case it is convenient to equip $su_{p,q}$ with a new scalar product: $(x, y) = x\cdot y/ih_1\cdot ih_1$. Of course, eq. (5) should be changed correspondingly.

The soliton surfaces (2) corresponding to a given soliton field ψ^A equipped with the co-ordinate system x^μ (independent variables) carry some geometric fields: 1) $g_{\mu\nu}(x, \zeta)$ and $d^a_{\mu\nu}(x, \zeta)$ $(a = 1, 2, \ldots, d-2)$, 2) co-ordinate curves $x^2 = \text{const}$, 3) tangent vector $r_{,1}(x, \zeta)$ and 4) normal vectors $n^a(x, \zeta)$ $(a = 1, 2, \ldots, d-2)$.

The idea of geometric unification of solvable nonlinearities consists in a proper identification of the above geometric fields as solutions to some physical nonlinear models. Table I summarizes this idea (ζ-dependence is omitted). We recall the Gauss-Weingarten (GW) equations describe the kinematics of a moving frame upon a submanifold of an affine space ([5]).

TABLE I. – *Geometric unification of solvable nonlinearities.*

	Geometric field	Physical meaning	Dynamics (kinematics for strings)
1	fundamental tensors $g_{\mu\nu}(x)$ $d^a_{\mu\nu}(x)$ $(a = 1, 2, \ldots, d-2)$	soliton fields (gauge equivalent to original soliton fields ψ^A)	GMC equations
2	co-ordinate curve $x^2 = \text{const}$ $r = r(x^1, x^2)$	string at a time x^2	gauge transformation from moving (on surface) frame to Frenet-Serret frame
3	Tangent vector $r_{,1}(x', x^2)$	Spin field	From GW equation
4	Normal vectors $n^a(x^1, x^2)$ $(a = 1, 2, \ldots, d-2)$	Chiral fields	From GW equation

Some explanations are in order. Firstly, we note that

(7) $$g_{11} = (g_{1,\zeta}, g_{1,\zeta}) = (ih_1, ih_1) = 1\,,$$

that is

(8) $$\mathrm{I} = ds^2 = (dx^1)^2 + 2g_{12}\,dx^1\,dx^2 + g_{22}(dx^2)^2\,.$$

The above-introduced string may be interpreted as a generalization of the Lamb-Hasimoto curves [12]. For details see [13]. From eq. (8) we have $ds^2|_{x^2=\text{const}} = (dx^1)^2$. Therefore, we can put $x^1 = s$ (arc length parameter along the string). Then if we identify x^2 as t (time), the soliton surface equation (2) reads $r = r(s, t)$ and we are in a position to state that the soliton surface $r = r(s, t)$ is swept out by its Lamb-Hasimoto curve. The identification $r_{,1}$ as a spin field S is reasonable since $(S, S) = (r_{,1}, r_{,1}) = g_{11} = 1$.

The following examples of the above unification scheme are presented in a rather concise way. All of them concern the su_2-ZS-AKNS linear problem [6,14] (except for the Liouville equation (12)). Soliton surfaces (for particular choices of ζ) are all embedded into $su_2 = E^3$. For each example the most interesting ingredients of the unification are listed (by the numbers 1, 2, 3 and 4 in the sense of table I).

A) 3-field extension of the since-Gordon equation

$$\left\{\begin{aligned} &\omega_{,12} = \sin\omega - \varphi_{,1}\psi_{,2}/\sin\omega\,, \\ &\varphi_{,12} = \omega_{,1}\psi_{,2}/\sin\omega\,, \\ &\psi_{,12} = \varphi_{,1}\omega_{,2}/\sin\omega\,. \end{aligned}\right. \tag{9}$$

This is a soliton system: its linear problem can be easily calculated using the standard geometric techniques [15].

*A*1)

$$\left\{\begin{aligned} &\mathrm{I} \;\;= (dx^1)^2 + 2\cos\omega\, dx^1\, dx^2 + (dx^2)^2\,, \\ &\mathrm{II} = \mathrm{II}^1 = \varphi_{,1}(dx^1)^2 - 2\sin\omega\, dx^1\, dx^2 + \psi_{,2}(dx^2)^2\,. \end{aligned}\right. \tag{10}$$

*A*2) For the sine-Gordon equation ($\varphi, \psi = \text{const}$) the Lamb-Hasimoto curve has a constant torsion (Beltrami-Enneper theorem [3,9,13]).

*A*3) O_3-invariant 2-spin model: $S = r_{,1}$, $T = r_{,2}$.

$$\left\{\begin{aligned} &S_{,2} = \;\;\; T\times S \qquad \text{(skew product of } E^3\text{-vectors)}\,, \\ &T_{,1} = -\,S\times T\,. \end{aligned}\right. \tag{11}$$

*A*4) For the sine-Gordon equation n (normal) solves the O_3-invariant relativistic σ-model equation [16].

B) Elliptic Liouville equation

$$\varphi_{,11} + \varphi_{,22} = -2\exp\varphi\,. \tag{12}$$

*B*1)

$$\left\{\begin{aligned} &\mathrm{I} \;\;= \exp[-\varphi][(dx^1)^2 + (dx^2)^2]\,, \\ &\mathrm{II} = (dx^1)^2 - (dx^2)^2\,. \end{aligned}\right. \tag{13}$$

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B3) O_3-invariant 2-spin model: $S = r_{,1}\exp[\varphi/2]$, $T = r_{,2}\exp[\varphi/2]$.

$$S_{,2} = S\times(T_{,2}\times T)\,, \qquad T_{,1} = T\times(S_{,1}\times_1)\,. \tag{14}$$

B4) The normal n solves the O_3-invariant Euclidean σ-model equation as an antiinstanton solution. We mention that in this case soliton surfaces are minimal with the total curvature (topological charge) negative.

C) Nonlinear Schrödinger equation

$$i\psi_{,2} + \psi_{,11} + \tfrac{1}{2}|\psi|^2\psi = 0\,. \tag{15}$$

C1)

$$\begin{cases} \mathrm{I} \;= (\mathrm{d}x^1)^2 + \varrho^2(\mathrm{d}x^2)^2\,, \\ \mathrm{II} = \varrho(\mathrm{d}x^1)^2 - 2\varrho\varphi_{,1}\mathrm{d}x^1\mathrm{d}x^2 + (\tfrac{1}{2}\varrho^3 - \varrho\varphi_{,2})(\mathrm{d}x^2)^2\,, \end{cases} \tag{16}$$

where $\psi = \varrho\exp[i\varphi]$.

C2) The motion of the vortex filament in the so-called localized induction approach [12,18] is given by

$$r_{,2} = r_{,1}\times r_{,11}\,. \tag{17}$$

It is worthwhile mentioning that the vortex filament at any instant of time continues to be a geodesic of its soliton surface [13].

C3) The tangent vector $r_{,1} = S$ solves the 1-dimensional continuous Heisenberg ferromagnet equation [19]

$$S_{,2} = S\times S_{,11}\,. \tag{18}$$

We conjecture that the models unified in the above-described way are all gauge-equivalent. It is suggested not only by particular examples, but also by the existence of the so-called Pohlmeyer transformations [1,16,20].

The geometric way mentioned at the beginning to find exact solutions to models 2) 3) and 4) consists in a construction of soliton surfaces (2). In the Hilbert-Riemann problem version of the inverse method [6,7] a general formula for a matrix wave function of N-soliton solution has been derived [7]. It is a product of $N+1$ matrices and N of them are built from projectors $P_k(x)$ ($i = 1, 2, \ldots, N$) which can be constructed in an inductive way. This result allows us to write down a general formula for N-soliton surfaces (N-surfaces) $r_N = r_N(x, \zeta)$ in the case of our interest ($su_{p,q}$ linear problem (6)).

For instance, the O-surface is a 2-dimensional plane, carrying Euclidean geometry, spanned by ih_m ($m = 1, 2$) matrices of the linear problem (6):

$$r_0 = x^1 ih_1 + \omega'(\zeta)x^2 ih_2\,. \tag{19}$$

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Generally, the N-surface is given by

$$r_N = \sum_{k=1}^{N} s_k + r_0 , \tag{20}$$

$$s_k = \frac{2 \operatorname{Im} \zeta_k}{|\zeta - \zeta_k|^2} \Phi_{k=1}^{-1} i \left(P_k - \frac{d_k}{n} \right) \Phi_{k-1} , \tag{21}$$

where ζ_k ($k = 1, 2, \dots, N$) are discrete and x^2-invariant eigenvalues of the scattering problem (6a) localized in the upper half-plane of the complex plane, Φ_k are SU_{pq} wave functions of the k-soliton solution and $d_k = \operatorname{Tr} P_k = \dim \operatorname{Im} P_k$.

It is interesting to point out that eq. (20) is a generalization of the Bianchi-Lie transformation of the classical differential geometry [1,3,9]. Originally, this transformation has been introduced as a surface-geometric analog of the Bäcklund transformation for the sine-Gordon equation. Like in the classical Bianchi-Lie transformation the vectors s_k are of constant (Euclidean) length

$$|s_k| = \frac{2 \operatorname{Im} \zeta_k}{|\zeta - \zeta_k|^2} \sqrt{d_k \left(1 - \frac{d_k}{n} \right) \Big/ \operatorname{Tr} h_1^2} .$$

Finally, we mention that formula (20) for $N = 2$ when applied to the non-linear Schrödinger equation gives a new vortex motion of a high complexity [8].

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