Discrete & Computational

Geometry

A Construction of Inflation Rules Based on n-Fold Symmetry*

K.-P. Nischke and L. Danzer[†]

Institut für Mathematik, Universität Dortmund, D-44221 Dortmund, Germany

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Summary. In analogy to the well-known tilings of the euclidean plane $\mathbb{E}^2$ by Penrose rhombs (or, to be more precise, to the equivalent tilings by Robinson triangles) we give a construction of an inflation rule based on the n-fold symmetry D_n for every n greater than 3 and not divisible by 3. For given n the inflation factor η can be any quotient $\mu_{n,k} := \sin(k\pi/n)/\sin(\pi/n)$ as well as any product $\prod_{k=2}^{n/2} \mu_{n,k}^{\alpha_k}$, where $\alpha_2, \alpha_3, \ldots, \in \mathbb{N} \cup \{0\}$. The construction is based on the system of n tangents of the well-known deltoid $\mathcal{D}$, which form angles with the ξ-axis of type $\nu\pi/n$. None of these tilings permits two linearly independent translations. We conjecture that they have no period at all. For some of them the Fourier transform contains a $\mathbb{Z}$-module of Dirac deltas.

1. The Construction

1.1. *The Deltoid $\mathcal{D}$ and Its Tangents*

If a unit-circle S^1 is rolled around inside a circle of radius 3, a point $x(\varphi)$ fixed on S^1 will move along a hypocycloid $\mathcal{D}$, which can be described in parametric form by

$$x(\varphi) := (\xi(\varphi), \eta(\varphi))^{\mathrm{T}}, \qquad 0 \le \varphi \le 2\pi,$$

where

$$\xi(\varphi) := 2\cos\varphi + \cos(2\varphi), \qquad \eta(\varphi) := 2\sin\varphi - \sin(2\varphi).$$
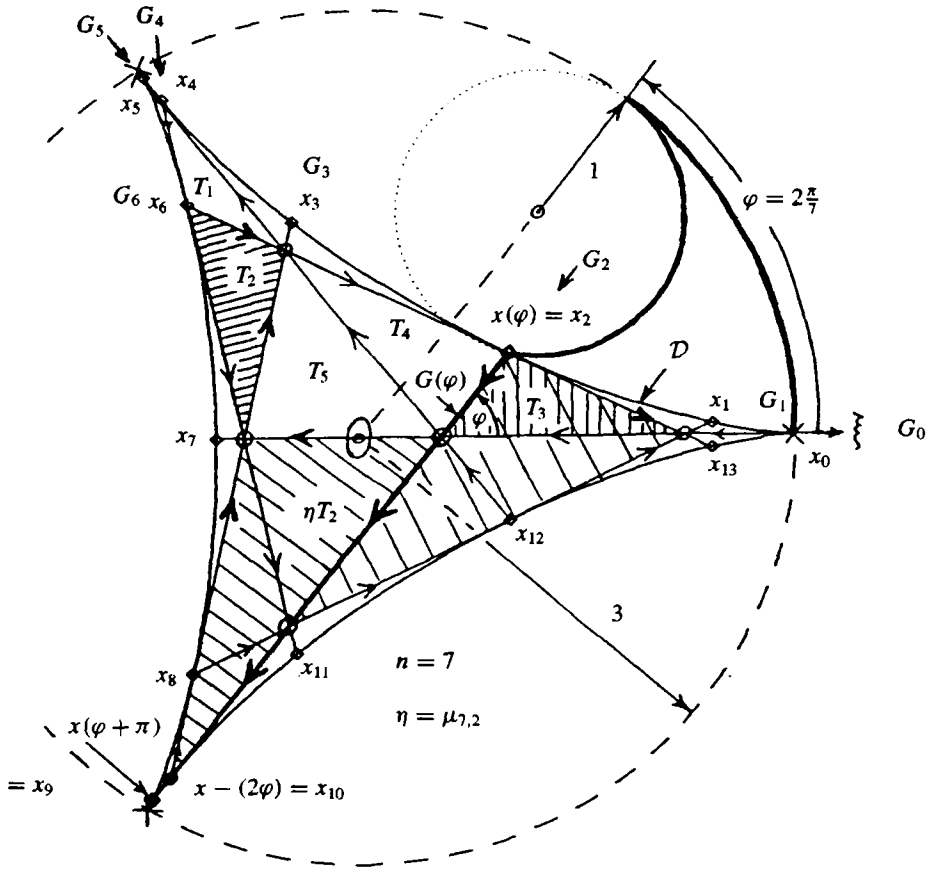

Fig. 1. The deltoid D.

D is a quartic as its implicit equation

$$(x^2 + 12x + 9 + y^2)^2 = 4(2x + 3)^3$$

shows (for the sequel see Fig. 1). We define the segment

$$G(\varphi) := \overline{x(\varphi); x(\varphi + \pi)}, \qquad 0 \le \varphi \le 2\pi.$$

Obviously $G(\varphi) = G(\varphi + \pi)$. We are mainly interested in the segments whose parameters φ are of the form $\nu\pi/n$. Provided n is odd, we mark $G(\nu\pi/n)$ with an arrow aiming from the even to the odd vertex. For brevity we write x_ν, G_ν, s_ν, and c_ν instead of $x(\nu\pi/n)$, $G(\nu\pi/n)$, $\sin(\nu\pi/n)$, and $\cos(\nu\pi/n)$. Finally we put $\mu_{n,k} := s_k/s_1$ for $1 \le k < n/2$.
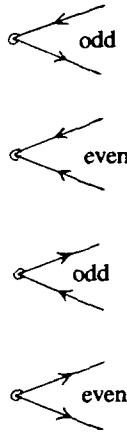
Fig. 2. The rule for the arrows.

1.2. The Family $\mathcal{F}_n$ of Prototiles

Throughout this paper—except Section 4—let $n = 2m + 1$ be an arbitrary but fixed odd natural number, $n \geq 5$ and $n \not\equiv 0 \pmod 3$. Whenever μ, ν, λ are natural numbers summing up to n there is a triangle $\triangle$ with angles $\mu\pi/n$, $\nu\pi/n$, $\lambda\pi/n$ and opposite sides of lengths $4s_1 s_\mu$, $4s_1 s_\nu$, $4s_1 s_\lambda$. We may put arrows on the sides of $\triangle$, and since n is odd we can do this in exactly two ways according to the rule shown in Fig. 2. If $\triangle$ is isosceles the angle at its apex is odd and even the two *arrowed* triangles are congruent. In this case we choose one of them as a prototile T. Otherwise μ, ν, λ are pairwise distinct and we receive from $\triangle$ two incongruent arrowed triangles T and T', and we take both of them as prototiles. The family of all these prototiles is denoted by $\mathcal{F}_n$. Its cardinality turns out to equal $(n - 1)(n - 2)/6$.

In Fig. 1 for $n = 7$ all five prototiles $T_1, T_2, \ldots, T_5$ are shown. T_2 and T_3 differ only in the arrows.

2. The Geometric Properties

The following properties, (1)–(11), are—in this order—all proved in Section 5. The arguments are rather straightforward and the calculations use only the well-known formulae of trigonometry and the identity

$$\mu_{n,k} \cdot \sin\left(\lambda\frac{\pi}{n}\right) = \sum_{\nu=0}^{k-1} \sin\left((\lambda + 1 - k + 2\nu)\frac{\pi}{n}\right). \qquad (*)$$

(1) $G(\varphi)$ touches $\mathcal{D}$ at $x(-2\varphi)$ (not really essential).
(2) $|G(\varphi)| = 4$ (length of the segment).
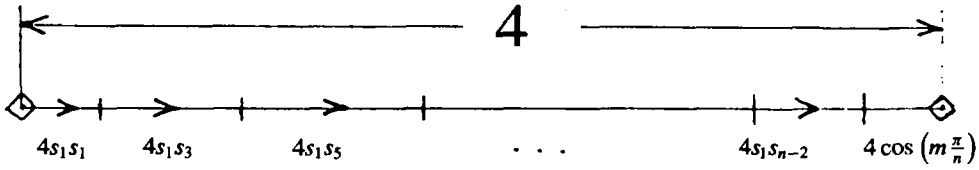(3) The angle between $G(0)$ and $G(\varphi)$ equals $\pi - \varphi$.

Fig. 3. The common dissection of the segments G_μ.

(4) If $0 \le \varphi < \psi < \chi < \pi$ and $\omega := \varphi + \psi + \chi$, then

$$|G(\varphi) \cap G(\chi) - G(\varphi) \cap G(\psi)| = 4|\sin \omega| \cdot \sin(\chi - \psi).$$

(5) $\varphi + \psi + \chi \equiv 0 \ (\mathrm{mod} \, \pi)$ implies $G(\varphi) \cap G(\psi) \cap G(\chi) \ne \emptyset$.

(6) If $0 \le \mu < \nu < \lambda < n$ $(\mu, \nu, \lambda \in N \cup \{0\})$ and $\sigma := \mu + \nu + \lambda$ then
$|G_\mu \cap G_\lambda - G_\mu \cap G_\nu| = 4|s_\sigma| s_{\lambda-\nu}$ and G_μ, G_λ are either concurrent (see (5))
or form a triangle $\Delta(\mu, \nu, \lambda)$ congruent to $\mu_{n,k} \cdot T$ for some $T \in \mathcal{F}_n$, where
$s_k = |s_\sigma| > 0$.

(7) The system $\mathcal{G} := \{G_\nu \mid \nu = 1, 2, \ldots, n\}$ makes up a triangular pattern $\mathcal{T}$ inside
$\mathcal{D}$. Every elementary triangle is of the type described under (6) with $\sigma \equiv \pm 1$
$(\mathrm{mod} \, n)$. Every interior vertex is shared by exactly six triangles.

(8) Every segment G_μ is cut by the other segments G_ν into pieces whose lengths
form—independent of μ—the sequence shown in Fig. 3 (see ($*$)). Obviously,
read from right to left, the points have distances $2c_1, 2c_2, \ldots, 2c_n = -2$ from
the center of $\mathcal{G}_\mu$.

(9) Given integers σ and α, β, γ with $\alpha + \beta + \gamma \equiv 0 \ (\mathrm{mod} \, n)$ the number of triangles
in $\mathcal{T}$ with angles α, β, γ and the given σ (see (6)) equals one, if α, β, γ are not
pairwise distinct (the triangle then is isosceles), and equals two otherwise, and
the two triangles then differ with respect to the arrows.

(10) If two triangles as described under (6) $(\sigma \not\equiv \pm 1 \ (\mathrm{mod} \, n))$ are congruent—
including the arrowing—they are dissected in the same way. (In Fig. 1 the
triangles corresponding to $(0, 2, 3)$ and to $(1, 2, 6)$ have $\sigma \equiv -2$ and $\sigma \equiv 2$,
respectively. They have congruent carriers, but are dissected in different ways.)

Definition 1. Given $n = 2m + 1$ and k, where $2 \le k \le m$, for every $T \in \mathcal{F}_n$ we define
$\mathrm{infl}_{n,k,+}(T)$ by dissecting $\mu_{n,k} T$ according to (9) and $\mathrm{infl}_{n,k,-}(T) := \mathrm{infl}_{n,k,+}(T')$, where
T' is congruent to T except for the arrows, which are reversed.

Hence for every isosceles T $\mathrm{infl}_{n,k,+}(T) = \mathrm{infl}_{n,k,-}(T)$. For example, in Fig. 1 the
triangle ηT_2 formed by G_0, G_2, G_3 carries $\mathrm{infl}_{7,2,+}(T_2) \simeq \mathrm{infl}_{7,2,-}(T_3)$.

(11) $\mathrm{infl}_{n,k,\varepsilon}(T)$ $(\varepsilon = +$ or $\varepsilon = -)$ is always a face-to-face patch of triangles, each
congruent to some member of $\mathcal{F}_n$; and even if several of these inflations are
applied one after the other the resulting patch is face-to-face.

Definition 2. *By* $S(\mathcal{F}_n; \text{infl}_{n,k,\varepsilon})$ $(\varepsilon = \pm)$ *we denote the species of all tilings* $\mathcal{P}$ *of the entire plane, where every patch* $\mathcal{A}$ *of* $\mathcal{P}$ *is congruent to some patch in some* $\text{infl}^j_{n,k,\varepsilon}(T)$ *with* $T \in \mathcal{F}_n$.

Originally infl is defined only for members of $\mathcal{F}$. However, using routine arguments and well-known techniques, unlimited iteration of the inflation finally leads to tilings of the entire plane, and these are exactly described by Definition 2. Thus we have arrived at:

Theorem 1. *Assume* $m \in \mathbb{N}$, $m \geq 2$, $n = 2m + 1$, $n \not\equiv 0 \pmod{3}$, $2 \leq k \leq m$, $\varepsilon \in \{+, -\}$. *Then the species* $S(\mathcal{F}_n, \text{infl}_{n,k,\varepsilon})$ *(see Definition 2) is not empty. It consists of face-to-face tilings of* $\mathbb{E}^2$ *by tiles being congruent to members of* $\mathcal{F}_n$; *and the arrows match. Every such tiling can be considered as a tiling of* patches, *each of which is of type* $\text{infl}_{n,k,\varepsilon}(T)$ *for some* $T \in \mathcal{F}_n$. *The inflation factor equals* $\mu_{n,k}$.

A protoset $\mathcal{F}$ is called *minimal* with respect to an inflation infl if, for no proper subset $\mathcal{G}$ of $\mathcal{F}$ and no j, every member T of $\mathcal{G}$ has as the jth inflation a patch, where every tile is congruent to a member of $\mathcal{G}$. In other words, no member of $\mathcal{F}$ is avoidable. Standard arguments of inflation theory lead to:

Corollary 1. *If* $\mathcal{F}_n$ *is minimal with respect to* $\text{infl}_{n,k,\varepsilon}$, *then the species* $S(\mathcal{F}_n, \text{infl}_{n,k,\varepsilon})$

— *consists of only one local isometry class,*
— *is repetitive,*
— *consists of* $2^{\aleph_0}$ *(uncountably many) congruence classes.*

Two tilings $\mathcal{P}$ and $\mathcal{Q}$ are said to be *locally isometric* if to every patch $\mathcal{A}$ contained in $\mathcal{P}$ there is an isometric patch $\mathcal{B}$ in $\mathcal{Q}$ and vice versa. A species S is said to be *repetitive* if to every radius r there is an R such that to *every* patch $\mathcal{A}$ of circumradius r, which occurs in some member $\mathcal{P}$ of S, a congruent copy can be found in *every* ball of radius R in *any* member $\mathcal{Q}$ of S.

Because of (11) not only can every inflation of type $\text{infl}_{n,k,\varepsilon}$ be iterated, but they can even be combined in any order. This leads to:

Corollary 2. *Let* m *and* n *be as in Theorem 1. Assume* $2 \leq k_\sigma \leq m$ *and* $\varepsilon_\sigma \in \{+, -\}$ *for* $\sigma = 1, 2, \ldots, s$. *Then the* s *inflations can be composed (i.e., can be applied one after the other). Thus the inflation*

$$\text{infl} := \text{infl}_{n,k_s,\varepsilon_s} \circ \cdots \circ \text{infl}_{n,k_2,\varepsilon_2} \circ \text{infl}_{n,k_1,\varepsilon_1}$$

defines a nonempty species $S(\mathcal{F}_n, \text{infl})$ *with inflation factor*

$$\eta = \mu_{n,k_s} \cdot \cdots \cdot \mu_{n,k_2} \cdot \mu_{n,k_1}.$$

3. Some Algebraic Properties

For a protoset $\mathcal{F} = \{T_1, T_2, \ldots, T_s\}$ and an inflation infl the inflation matrix $M := M(\mathcal{F}, \text{infl}) = (a_{i,j})$ is defined by

$$a_{i,j} := \#\{T \mid T \subset \text{infl}(T_j), T \simeq T_i\}, \qquad i, j = 1, 2, \ldots, s.$$

For instance,

$$M(\mathcal{F}_7, \text{infl}_{7,2,+}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix},$$

$$M(\mathcal{F}_7, \text{infl}_{7,2,-}) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

If η is the inflation factor and d is the dimension of the space considered, the row-vector $(v_1, v_2, \ldots, v_s)$, where $v_j := \text{vol}(T_j)$, obviously is an eigenvector to η^d, hence η^d is an eigenvalue of M. By our assumption on n and k:

(12) $\quad \eta^d = \eta^2 = \mu_{n,k}^2$ necessarily is an irrational algebraic number.

The column-vector $(f_1, f_2, \ldots, f_s)$, where f_i is the relative frequency of T_i, almost as obviously is also an eigenvector to η^d. Hence, if η^d is irrational, $f_1, f_2, \ldots, f_s$ cannot all be rational (since the $a_{i,j}$ are integers) and this implies that $S(\mathcal{F}, \text{infl})$ cannot contain any tiling, which is invariant under d linearly independent translations. So we deduce from (11):

Corollary 3. *If $\mathcal{P} \in S(\mathcal{F}_n, \text{infl}_{n,k,\varepsilon})$, $\mathcal{P}$ is not crystallographic (i.e., does not permit two linearly independent translations).*

Every tiling in $S(\mathcal{F}_n, \text{infl}_{n,k,\varepsilon})$ can be trivially *deflated*, that is to say (see Theorem 1), it can be considered as a face-to-face tiling of patches of type $\text{infl}(T_j)$. If this deflation is *unique* (literally, not only up to rigid motions), the tiling cannot be periodic at all. The deflations with respect to $\text{infl}_{7,2,\pm}$ and $\text{infl}_{7,3,\pm}$ are indeed all unique. So we get:

Corollary 4. *If $\mathcal{P} \in S(\mathcal{F}_7, \text{infl}_{7,k,\varepsilon})$ $(k = 2, 3; \varepsilon = +, -)$, $\mathcal{P}$ does not permit any translation. In other words, these four species are* aperiodic.

The Fourier transform of a tiling[1] defined by some inflation (see Definition 2) will show Dirac deltas if and only if the inflation factor η is a PV number; i.e., if $\eta > 1$, but $|\sigma_j(\eta)| < 1$ for all algebraic conjugates $\sigma_j(\eta)$ different from η (see [1]). Therefore, in physics inflations whose factors are PV numbers are of special interest.

Let K be the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta)$ (degree $\frac{1}{2}\varphi(n)$). Denote by $\sigma_1 := \mathrm{id}, \sigma_2, \ldots, \sigma_m$ the automorphisms of K with

$$\sigma_j(\mu_{n,k}) = (-1)^{(j+1)(k+1)} \frac{\mu_{n,jk}}{\mu_{n,j}}.$$

From algebraic number theory (see [2] and [3]) we know that if n is a prime, then the matrix

$$L := \left(\log(|\sigma_i(\mu_{n,j})|)\right) \qquad (2 \leq i, j \leq m)$$

is regular. So

(13) there is a PV number of the form

$$\eta = \mu_{n,2}^{i_2} \cdot \mu_{n,3}^{i_3} \cdot \cdots \cdot \mu_{n,m}^{i_m} \qquad (i_j \in \mathbb{Z}),$$

which is also a unit in the ring $\mathcal{O}_K$ of all algebraic integers of K.

Proof. The first statement comes from algebraic number theory. For the second statement we choose a real solution x of

$$Lx = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}.$$

Approximating x by a rational vector $x' \in \mathbb{Q}^{m-1}$ close enough, we still have $Lx' < 0$ (componentwise). Multiplication with a positive common denominator of the coefficients yields an integer vector y with $Ly < 0$. Now we simply take $i_2, i_3, \ldots, i_m$ as $y_1, y_2, \ldots, y_{m-1}$. □

Due to Minkowski's theorem on lattice points in a centrally symmetric convex body we also have

(14) a PV number $\eta \in \sum_{k=1}^{m} \mathbb{Z}\mu_k = \mathcal{O}_k$ with $0 < \eta \leq p^{(m-1)/2}$.

The following factors turn out to be PV numbers: $\mu_{5,2}$ ($= \tau$), $\mu_{7,3}$, $\mu_{11,3} \cdot \mu_{11,5}$, $\mu_{11,4} \cdot \mu_{11,5}$, $\mu_{13,5} \cdot \mu_{13,6}$. Hence the corresponding species of tilings (see Definition 2) based on 5-fold, 7-fold, 11-fold, and 13-fold rotational symmetries and defined by inflation have Fourier transforms with a $\mathbb{Z}$-module of Dirac deltas. Of course, the Fourier transform of any tiling $\mathcal{P} \in \mathbf{S}(\mathcal{F}_n, \mathrm{infl}_{n,k,\varepsilon})$ shows n-fold dihedral symmetry.

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[1] To be more precise: the Fourier transform of the distribution, which is the sum of the countably many Dirac deltas placed at the vertices of the tiling.

In contrast, for $n = 19$ we have

$$\left| \sigma_2(\mu_{19,k})^5 \cdot \sigma_3(\mu_{19,k})^4 \cdot \sigma_5(\mu_{19,k})^4 \right| > 1$$

for all $k \in \{2, 3, \ldots, 9\}$.[2] So there cannot be a PV number according to (13) with only nonnegative exponents i_j and therefore the inflation rules of Corollary 2 give no PV number as an inflation factor.

4. Examples

4.1. *The Case $n = 5$*

Here we have two tiles A, B. For the corresponding $\mathcal{T}$ see Fig. 4. The only possible inflation factor of the form $\mu_{5,k}$ is

$$\eta = \mu_{5,2} = \frac{1 + \sqrt{5}}{2} = \tau.$$

In this case there are two possibilities of defining an inflation rule: The first one reversing and the second one preserving the arrows as in Fig. 4. We obtain the Penrose tiling if we use a "mixture" of both: A is replaced by $\text{infl}_{2,+}(A)$ and B by $\text{infl}_{2,-}(B)$. For all three inflations we have the same inflation factor and the same substitution matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

$\mathcal{T}$
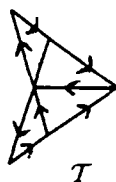
	Tiles	infl$_{2,-}$	infl$_{2,+}$
A			
B			

Fig. 4. Inflation rules for A, B; $n = 5$.

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[2] Private communication by Walter Parry, Department of Mathematics, Eastern Michigan University. 19 is the smallest such prime.
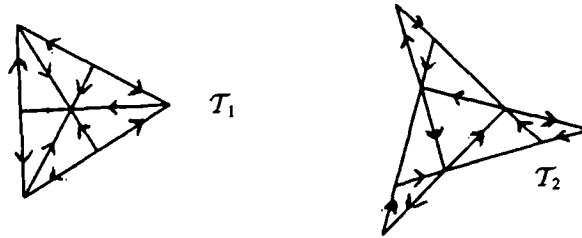
4.2. The Case $n = 6$

Fig. 5. The triangular patterns for $n = 6$.

Though $n = 6$ is not included in Theorem 1, a similar construction will work. Since $n \equiv 0 \pmod 3$ there are *two* triangular patterns T_1 and T_2, the second corresponding to $G(k\pi/6 + \pi/12)$ as shown in Fig. 5. First we consider the factor $\eta = \mu_{6,2} = \sqrt{3}$. There are (at least) two possibilities of how to define an inflation (see Fig. 6). We consider especially infl$_1$. In the resulting tiling we observe the patches P and Q shown in Fig. 7. Their second inflations are completely composed of patches of the same type. Hence we obtain an equivalent tiling by the tiles P' and Q' defined by their inflation, which is also given in Fig. 7. Figure 8 shows a part of the resulting global tiling.

For the inflation factor $\eta = \mu_{6,3} = 2$ we obtain an inflation for three tiles as given by Fig. 9(a). For the patches R and S (Fig. 9(b)) the same procedure works as for P and Q above (Fig. 10).
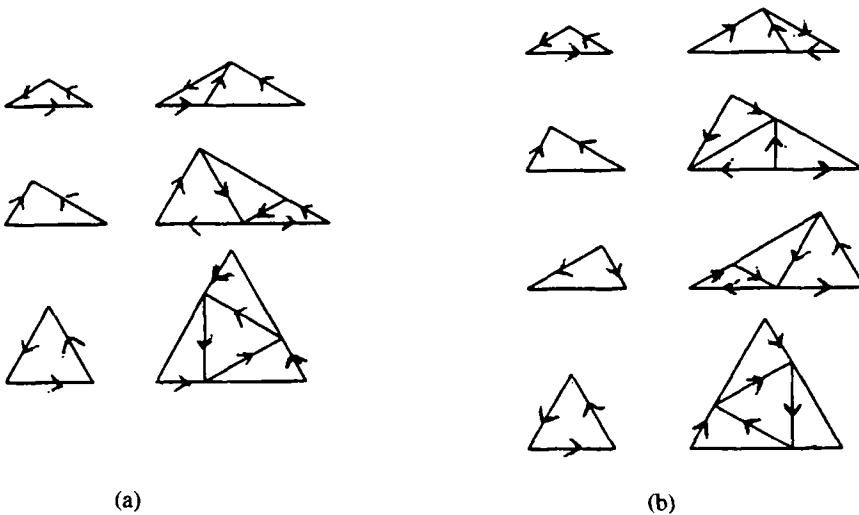

(a)

(b)

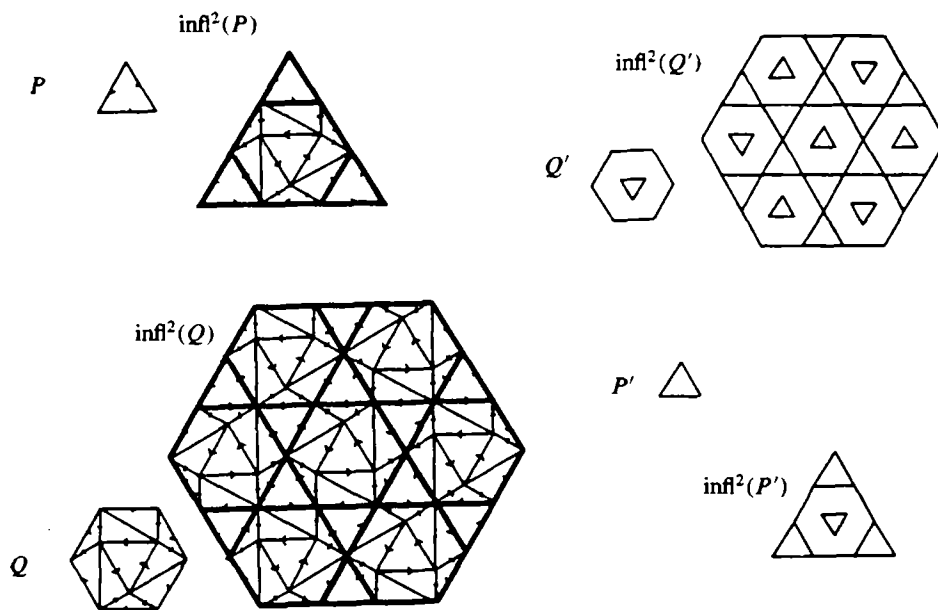**Fig. 6.** (a) infl$_1$ for $\eta = \mu_{6,2}$ and (b) infl$_2$ for $\eta = \mu_{6,2}$.

Fig. 7. The patches P, Q, P', Q' from infl_1 for $\mu_{6.2}$.
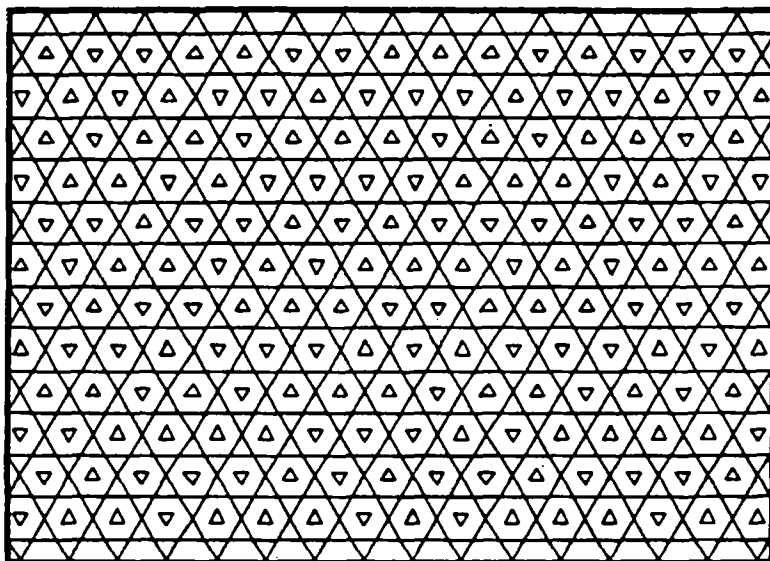
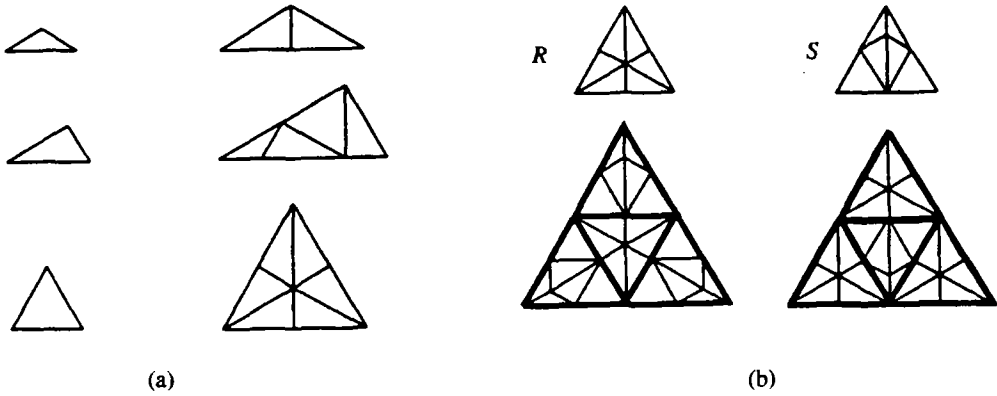**Fig. 8.** A larger patch from $\mathrm{S}(\{P', Q'\}, \mathrm{infl}_1)$.
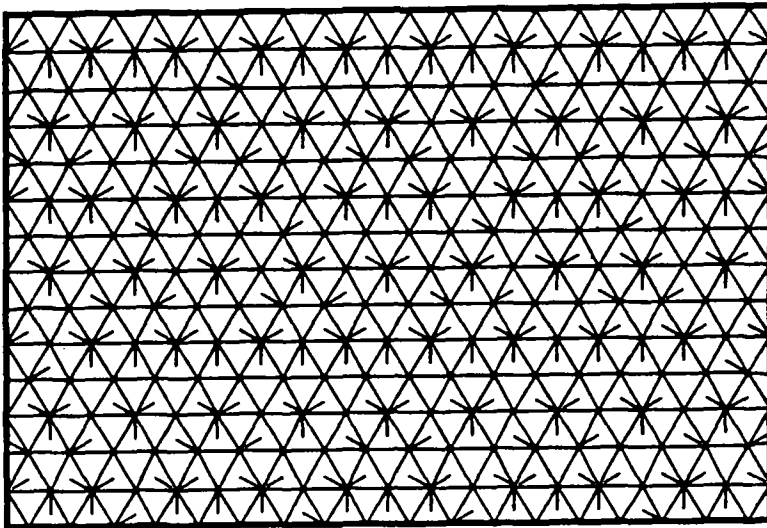
Fig. 9. An inflation for $\mu_{6,3}$.

Fig. 10. Part of a global tiling by R' and S'.

4.3. The Case $n = 7$

For $\eta = \mu_{7,2}$ see Fig. 1 and the inflation matrices given in Section 3. For $\eta = \mu_{7,3}$ (which is a PV number) we have

$$M(\mathcal{F}_7, \text{infl}_{7,3,+}) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 2 & 2 \end{pmatrix}$$

and

$$M(\mathcal{F}_7, \text{infl}_{7,3,-}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}.$$
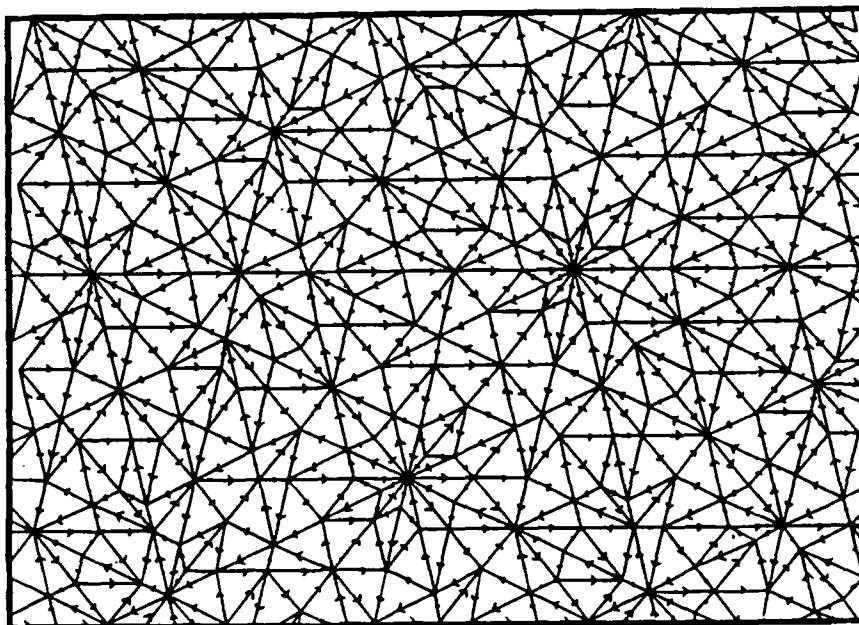

Fig. 11. A patch from $S(\mathcal{F}_7, \text{infl}_{7,3,+})$.

Finally we give an impression of the species $S(\mathcal{F}_7, \text{infl}_{7,3,+})$, which is—after the Penrose species—the first to be called quasiperiodic (see Fig. 11).

5. Proofs of Statements (1)–(11)

In this section every variable that refers to a multiple of π/n is considered as a residue class modulo n; especially, the index of our G_μ is taken modulo n.

Ad (1).   The tangent to $\mathcal{D}$ at $x(-2\varphi)$ is given by

(15) $\quad \xi \sin\varphi - \eta \cos\varphi = \sin(3\varphi)$.

This equation is satisfied by $x(\varphi)$ as well as by $x(-\varphi)$. Hence (15) is also the equation of $G(\varphi)$.

Ad (2).   Trivial trigonometry.

Ad (3). By (15) $G(\varphi)$ has slope $\mathrm{tg}(\varphi)$, while $G(0)$ is horizontal.

Ad (4).

(16) The point

$$p(\varphi, \chi) := \left(\begin{array}{c} 3 - 4(\sin^2 \varphi - \sin^2 \varphi \sin^2 \chi + \sin^2 \chi + \sin \varphi \sin \chi \cos \varphi \cos \chi) \\ -4 \sin \varphi \sin(\varphi + \chi) \sin \chi \end{array} \right)$$

satisfies (15) and is symmetric in φ and χ, hence lies in $G(\varphi) \cap G(\chi)$.

By the assumption of (4) $G(\varphi) \neq G(\chi)$, whence

$$\{p(\varphi, \chi)\} = G(\varphi) \cap G(\chi);$$

analogously

$$\{p(\varphi, \psi)\} = G(\varphi) \cap G(\psi).$$

Standard trigonometry yields

$$(p(\varphi, \chi) - p(\varphi, \psi))^2 = 16 \sin^2(\varphi + \psi + \chi) \sin^2(\chi - \varphi).$$

Obviously,

(17) $p(\varphi, \psi)$ is π-periodic in either variable.

Ad (5). This is an immediate consequence of (4).

Ad (6). The angle of the triangle $\triangle_{(\mu, \nu, \lambda)}$ at $G_\lambda \cap G_\nu$ equals $(\lambda - \nu)(\pi/n)$. The opposite side on G_μ has length $4|s_\sigma| s_{\lambda - \nu}$ (by (4)). The other angles are $(\nu - \mu)(\pi/n)$ and $(\mu - \lambda)(\pi/n) + \pi$, and the corresponding edge lengths now of course equal $4|s_\sigma| s_{\nu - \mu}$ and $4|s_\sigma| s_{n + \mu - \lambda}$.

Ad (7). Given $\mu \in I := \{0, 1, 2, \ldots, n - 1\}$, $\mu \neq 0$, choose ν from I, different from μ, -2μ, $-\frac{1}{2}\mu \pmod{n}$; then there is exactly one λ such that

$$\mu + \nu + \lambda \equiv 0 \pmod{n}$$

and μ, ν, λ are pairwise distinct. So, by (5), G_μ, G_ν, G_λ are three different segments meeting in $p_{\mu, \nu} = p_{\nu, \lambda} = p_{\lambda, \mu}$, where $p_{\mu, \nu}$ stands for $p(\mu\pi/n, \nu\pi/n)$, and in general we obtain $1/2(n - 3)$ such points on G_μ. The only two segments of $\mathcal{G}$ not yet used are $G_{-2\mu}$ and $G_{-(1/2)\mu}$. The former meets G_μ in its tangent point to $\mathcal{D}$; the tangent point of the latter is the endpoint of G_μ, whose index is even (the one where the arrows come from). For $\mu \equiv 0$, the two special points coincide and there are $1/2(n - 1)$ triple points.

Thus all points $p_{\mu, \nu}$, which are on only two of our segments, lie on $\mathcal{D}$ and we have proved the last statement of (7), of which the first is an immediate consequence.

Ad (8). Consider the triangle $\triangle(\mu, \nu, \lambda)$ in $\mathcal{T}$: either $p_{\mu, -\mu-\nu+1}$ or $p_{\mu, -\mu-\nu-1}$ is on the

half ray $\overrightarrow{p_{\mu,\nu}p_{\mu,\lambda}}$. Since by (6) $|p_{\mu,\nu} - p_{\mu,\lambda}| = 4|s_\sigma||s_{\lambda-\nu}|$ and $|\sigma| = 1$ by (7), the proof of (8) is completed. Clearly,

(18) adjacent elementary triangles in $\mathcal{T}$ have opposite signs of σ.

Ad (9). Because $n \not\equiv 0 \pmod 3$ we may assume $\gamma \neq \alpha, \beta$. Any triangle Δ in $\mathcal{T}$ with angles α, β, γ satisfies

$$\Delta = \begin{cases} \Delta(\mu, \mu + \alpha, \mu - \beta) \\ \text{or} \\ \Delta(\mu, \mu - \alpha, \mu + \beta). \end{cases}$$

So we have to solve the congruences $\mu + \mu \pm \alpha + \mu_+^- \beta \equiv \sigma \pmod n$. The unique solutions are

$$\mu \equiv \frac{\sigma \mp \alpha \pm \beta}{3}.$$

In case Δ is isosceles, i.e., $\alpha = \beta$, this is *one* solution. Otherwise we get *two* triangles with *different* arrowings. Since there are altogether $((n+1)(n-1))/12$ similarity classes of triangles (arrows neglected) and $(n-1)/2$ belong to the isosceles case, and since σ varies from 1 to $n - 1$, we arrive at

(19) $\mathcal{T}$ contains exactly

$$\frac{(n-1)^2}{2} + 2\frac{(n-5)(n-1)^2}{12} = \frac{n-2}{6} \cdot (n-1)^2$$

triangles and exactly

$$\frac{(n-2)(n-1)}{3} = \frac{2}{3}\binom{n-1}{2}$$

elementary triangles ($\sigma = \pm 1$).

Ad (10). By (9) there are exactly two congruent triangles with the same σ and the same arrowing. Since every triangle in $\mathcal{T}$ can be reflected in G_0, they have to be mirror images of each other and hence are dissected in the same way.

Ad (11). The first statement is included in (7), while the second is a consequence of (8) and

$$\text{(20)} \qquad \mu_k \cdot 4s_1 s_j = 4s_1(s_{j+1-k} + s_{j+3-k} + \cdots + s_{j+k-1}) \qquad (\text{see } (*))$$

$$= 4s_1(s_{n-j-1+k} + s_{n-j-3+k} + \cdots + s_{n-j-k+1}).$$

In case $j \equiv k \pmod 2$ the first sum represents—after canceling terms like $s_1 + s_{-1}$—a unique interval on every G_μ, in case $j \not\equiv k \pmod 2$ the second expression does.

6. Concluding Remarks and Open Questions

6.1. The whole construction can be extended to the cases

$$n \equiv 3, 6, 9 \pmod{12} \qquad \text{(with two different } \mathcal{G}\text{'s)}$$

and

$$n \equiv 2, 6, 10 \pmod{12} \qquad \text{(with lack of some arrows)},$$

but then becomes more complicated. The case $n = 6$ has been treated in Section 4.

6.2. For the construction of the system $\mathcal{G}$ of segments (see (7)) the deltoid $\mathcal{D}$ is not really necessary. One may instead begin with one segment satisfying (8) and then apply trigonometry in order to meet (3)–(6). A check of all possible combinations shows $\mathcal{G}$ to be unique (for (3)–(8)).

6.3. There are other inflation rules for triangles with angles of type $v\pi/n$, especially for $n = 7$. One of them (see Fig. 12) even permits a perfect l.m.r. (local matching rule). In fact the 29 vertex stars may serve as such. These tilings are exotic also in that the relative frequencies of the prototiles are *not* proportional to their areas.
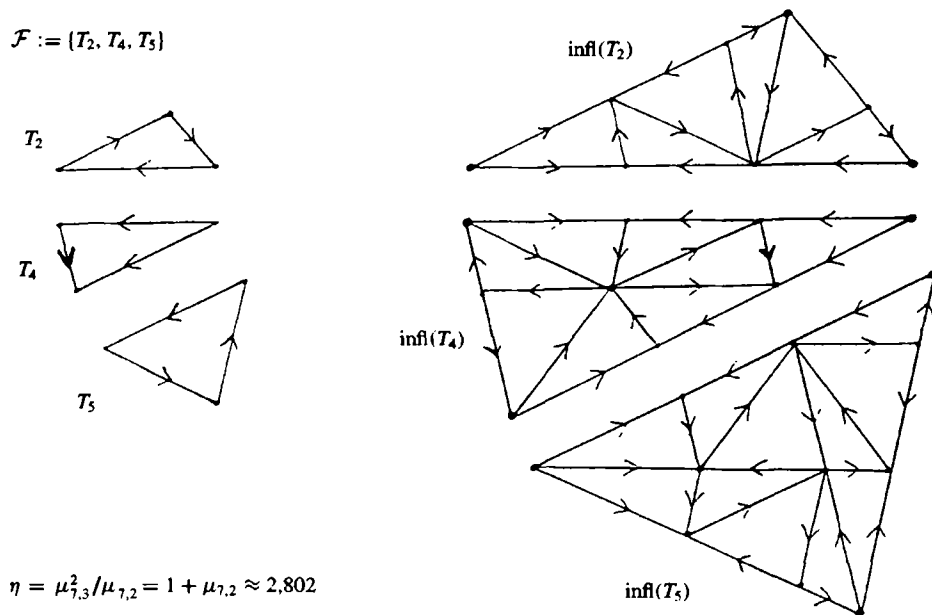

$$\mathcal{F} := \{T_2, T_4, T_5\}$$

$$\eta = \mu_{7,3}^2/\mu_{7,2} = 1 + \mu_{7,2} \approx 2{,}802$$

Fig. 12. The inflation for $n = 7$, which permits a local matching rule.

6.4. We do not know whether any one of the species given by Definition 2 does permit a l.m.r. In all cases where we have pursued this question it was easy to show that no l.m.r. can exist.

6.5. It is very easy to check whether $\mathcal{F}_n$ is minimal with respect to $\text{infl}_{n,k,\varepsilon}$ (see Corollary 1). In all cases for which we have done this (including $n = 5, 7, 11$ with all k and ε), $\mathcal{F}_n$ turned out to be minimal. However, we should not be surprised if some $\mathcal{F}_n$ are not minimal with respect to some $\mu_{n,k}$, especially if n is not a prime.

6.6. The inflation matrices $M(\mathcal{F}_n, \text{infl}_{n,k,+})$ seem to be symmetric, but we lack a proof for the general case. If M is symmetric, this implies that the relative frequencies of the prototiles are proportional to their volumes (areas). Since reversing arrows does not change areas this applies also to the case $\varepsilon = -$.

6.7. Whether a species S possesses a unique deflation is not so easily decided. For $n \geq 11$ it is an open question.

6.8. In the case $n = 5$ the most interesting species was a result of our constructions only indirectly (see Section 4.1). It may well be that also for other pairs (n, k) "blending" of $\text{infl}_{n,k,+}$ with $\text{infl}_{n,k,-}$ yields species with special properties.

6.9. The authors do not claim priority for statements (1)–(8); probably most of these facts were already known to Jacob Bernoulli.

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