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Estimation Procedures for a Family of Density Functions Representing Various Life-Testing Models

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Abstract: A family of density functions is considered which contains several life-testing models as specific cases. Uniformly minimum variance unbiased estimators are obtained for the positive and negative powers of the parameter, moments and reliability function. These general results provide the estimators for the specific models.

Key Words and Phrases: Family of densities, moments, reliability function, uniformly minimum variance unbiased estimators.

1 Introduction

A lot of work has been done in order to obtain the uniformly minimum variance unbiased estimators (UMVUE's) of the parameters, moments and reliability function associated with various life-testing models, like, exponential, gamma, Weibull, half-normal, Rayleigh, Erlang, generalized gamma, Maxwell and other distributions. For some citations, one may refer to Lehmann and Scheffè (1950, 1955), Pugh (1963), Basu (1964), Patil and Wani (1966, pp. 41–44), Sathe and Varde (1969), Zacks (1971, pp. 101–102), Gray, Watkins and Schucany (1973), Tyagi and Bhattacharya (1989), and others.

The purpose of the present note is many-fold. In section 2, we consider a family of probability density functions (p.d.f.'s), which covers many life-testing models as specific cases. In section 3, we obtain the UMVUE's of the positive and negative powers of the parameter involved with the model and utilize them in order to obtain the UMVUE's of the moments. We also provide the UMVUE of the reliability function.

2 The Set-Up of the Model and the Problems

Let us consider the family of p.d.f.'s

$$f(x;\theta,a,b,c)=\frac{cx^{ac-1}}{\theta^{ba}\Gamma_a}\exp(-x^c/\theta^b)\ ; \quad (x,\theta,a,b,c>0)\ , \tag{2.1}$$

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where θ is assumed to be unknown and a, b and c are known. We note the following:

(i) For $a = b = c = 1$, (2.1) gives the p.d.f. of the one-parameter exponential distribution [see Johnson and Kotz (1970, p. 207)].
(ii) For $b = c = 1$, (2.1) becomes the p.d.f. of the gamma distribution and taking 'a' as positive integer, (2.1) turns out to be the p.d.f. of an Erlang distribution [see Johnson and Kotz (1970, p. 166)].
(iii) For $b = c$, (2.1) gives the p.d.f. of the generalized gamma distribution [see Johnson and Kotz (1970, p. 197)].
(iv) For $a = 1, b = c$, (2.1) represents the p.d.f. of a Weibull distribution [see Johnson and Kotz (1970, p. 250)].
(v) For $a = 1/2,\ b = c = 2$, (2.1) is the p.d.f. of a half-normal distribution [see Davis (1952)].
(vi) For $a = b = 1,\ c = 2$, (2.1) turns out to be the p.d.f. of the Rayleigh distribution [see Sinha (1986, p. 200)].
(vii) For $a = \alpha/2,\ b = 1,\ c = 2$, (2.1) becomes the p.d.f. of the chi-distribution [see Patel, Kapadia and Owen (1976, p. 173)].
(viii) For $a = 3/2,\ b = 1,\ c = 2$, (2.1) gives the p.d.f. of the Maxwell's failure distribution [see Tyagi and Bhattacharya (1989)].

From (2.1), it can be seen that the r^{th} moment about origin is

$$\mu_r' = E(X^r)$$

$$= \frac{\Gamma(a + r/c)}{\Gamma(a)} \theta^{br/c} , \quad (r > -ac) , \tag{2.2}$$

and the reliability function $R(t)$ for a specified mission time t (> 0) is

$$R(t) = P(X > t)$$

$$= \frac{\gamma(a, t^c/\theta^b)}{\Gamma(a)} , \tag{2.3}$$

where $\Gamma(a)$ and $\gamma(a, y)$ are, respectively, the gamma and incomplete gamma functions, defined by

$$\Gamma(a) = \int_o^\infty x^{a-1} \exp(-x) dx , \tag{2.4}$$

and

$$\gamma(a, y) = \int_y^{\infty} x^{a-1}\exp(-x)dx \ , \quad (a, y > 0) \ . \tag{2.5}$$

3 UMVUE's of the Moments and Reliability Function

Given a random sample $X_1, \ldots, X_n$, in what follows, we denote by the statistic $T = \sum_{i=1}^{n} X_i^c$. The following theorem provides the UMVUE's of various powers of the parameter θ associated with the model (2.1) and the moments.

Theorem 1: For r (> 0), the UMVUE's of θ^r, θ^{-r} and μ'_r are, respectively $\hat{\theta}^r$, $\hat{\theta}^{-r}$ and $\hat{\mu}'_r$, where

$$\hat{\theta}^r = \frac{\Gamma(na)}{\Gamma(na + r/b)} T^{r/b} \ , \tag{3.1}$$

$$\hat{\theta}^{-r} = \frac{\Gamma(na)}{\Gamma(na - r/b)} T^{-r/b} \ , \quad (r < nab) \tag{3.2}$$

and

$$\hat{\mu}'_r = \frac{\Gamma(na)\Gamma(a + r/c)}{\Gamma(na + r/c)\Gamma(a)} T^{r/c} \ . \tag{3.3}$$

Proof: From (2.1), the likelihood of observing $\underline{X} = (X_1, \ldots, X_n)$ is

$$L(\theta; \underline{X}) = g(\theta, T)h(\underline{X}) \ , \tag{3.4}$$

where

$$g(\theta, T) = \bar{\theta}^{nba}\exp(-T/\theta^b)$$

and

$$h(\underline{X}) = \left\{\frac{c}{\Gamma(a)}\right\}^n \left\{\prod_{i=1}^n X_i\right\}^{ac-1} .$$

Applying Fisher-Neyman factorization theorem, it follows from (3.4) that T is a sufficient statistic for the family of distributions given by the p.d.f. (2.1). Denoting by $Y = X^c$, it is easy to check that the p.d.f. of Y is

$$g(y;\theta,a,b) = \frac{y^{a-1}}{\theta^{ba}\Gamma(a)} \exp(-y/\theta^b) \; ; (y,\theta,a,b > 0) , \tag{3.5}$$

i.e. Y has the gamma distribution $G(y;\theta^{-b},a)$. From the additive property of the gamma distribution, it follows that T has the $G(t;\theta^{-b},na)$ distribution, the p.d.f. of which can be obtained from (3.5) on replacing y by t and 'a' by na. By a general result on the exponential family [see Zacks (1971, p. 69)], T is complete. Thus we conclude that T is a complete sufficient statistic for the family of distributions presented by the p.d.f. (2.1). From Theorem 1.2 of Lehmann (1983, p. 80) [see also Bahadur (1957)], for any function $g(\theta)$, if we can find an unbiased estimator, which is a function of T, then it is a UMVUE of $g(\theta)$. From the distribution of T, it is easy to check that $\hat{\theta}^r$, $\hat{\theta}^{-r}$ and $\hat{\mu}'_r$, defined at (3.1), (3.2) and (3.3), respectively, are unbiased for θ^r, θ^{-r} and μ'_r, respectively, thus completing the proof of the theorem.

Corollary 1: The UMVUE of the $r^{th}(r \geq 2)$ moment about mean $\mu_r = E(X - \mu'_1)^r$ is given by

$$\hat{\mu}_r = \left\{\frac{\Gamma(na)}{\Gamma(na + r/c)\Gamma(a)}\right\} \left[\sum_{i=0}^r \binom{r}{i}(-1)^i \Gamma\left(a + \frac{r-i}{c}\right) \left\{\frac{\Gamma(a+1/c)}{\Gamma(a)}\right\}^i\right] T^{r/c} .$$

Proof: We have the recurrence relation

$$\mu_r = \sum_{i=0}^r \binom{r}{i}(-1)^i \mu'_{r-i}\mu_1'^i ,$$

which on using (2.2) gives that

$$\mu_r = \sum_{i=0}^r \binom{r}{i}(-1)^i \left\{\frac{\Gamma(a+(r-i)/c)}{\Gamma(a)}\right\} \left\{\frac{\Gamma(a+1/c)}{\Gamma(a)}\right\}^i \theta^{br/c} . \tag{3.6}$$

The result now follows from (3.6) utilizing (3.3).

In the next theorem, we obtain the UMVUE of the reliability function. We first prove a lemma.

Lemma 1: Let $Y_1 = X_1^c$. The random variables (r.v.'s) $V_1 = Y_1/T$ and T are independent and the ratio V_1 has the p.d.f.

$$g(v_1) = \frac{1}{B(a,(n-1)a)} v_1^{a-1}(1-v_1)^{(n-1)a-1} \qquad (o \le v_1 \le 1) .$$

Proof: Since $Y_1 \stackrel{d}{=} G(y_1; \theta^{-b}, a)$ and $T \stackrel{d}{=} G(t; \theta^{-b}, na)$, the lemma follows from the well-known results for gamma distribution [see Johnson and Kotz (1970, p. 182)].

Now we prove the main theorem.

Theorem 2: The UMVUE of the reliability function $R(t)$ is

$$\hat{R}(t) = \begin{cases} 1 - I_{t^c/T}(a,(n-1)a) & \text{if} \quad T > t^c \\ 0 & \text{if} \quad T \le t^c , \end{cases}$$

where $I_z(p,q)$ is the incomplete beta function ratio, given by

$$I_z(p,q) = \frac{1}{B(p,q)} \int_o^z w^{p-1}(1-w)^{q-1} dw \quad (0 < z < 1; p, q > 0) .$$

Proof: We denote by $F_{V_1}(v_1)$, the cumulative distribution function of V_1. Let us consider the statistic

$$S = \begin{cases} 1 & \text{if} \quad X_1 > t \\ 0 & \text{otherwise} \end{cases}$$

based on a single observation X_1. Obviously, S is unbiased for $R(t)$ and is complete and sufficient. We can write

$$S = \begin{cases} 1 & \text{if} \quad V_1 > t^c/T \\ 0 & \text{otherwise} . \end{cases}$$

Now Rao-Blackwellization [see Lehmann (1983, p. 81)] gives

$$E(S|T) = \bar{F}_{V_1}(t^c/T) \ ,$$

where $\bar{F}_{V_1}(v_1) = 1 - F_{V_1}(v_1)$. The theorem now follows utilizing Lemma 1.

Remarks: The UMVUE's for the powers of the parameter, moments and reliability function associated with the life-testing models considered in section 1 can be obtained from theorems 1 and 2, merely substituting the values of a, b and c.

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