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Gaussian States in de Sitter Spacetime and the Evolution of Semiclassical Density Perturbations.

1. Homogeneous Mode

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Abstract. The evolution of Gaussian quantum states in the de Sitter phase of the early universe is investigated. The potential is approximated by that of an inverted oscillator. We study the origin and magnitude of the density perturbations with special emphasis on the nature of the semi-classical limits.

Key words: cosmology—early universe—density perturbations

1. Introduction

A Friedmann universe with power law expansion for the scale factor [$S(t) = t^n$, $n< 1$] fails to explain the origin of galaxies on two major counts. Firstly, it does not have any natural seeds for the origin of density inhomogeneities. Secondly the scales on which the inhomogeneities exist today would all have originated outside the ‘physical horizon’ (which is $(\dot{S}/S)^{-1}$) in the early universe; it is difficult to imagine physical processes which can give rise to such coherence.

An inflationary model can solve both these problems: The quantum fluctuations of the scalar field which drive inflation can provide the seeds for density perturbations. The second difficulty is circumvented because, during the phase of exponential expansion the Hubble radius remains constant, but the proper wavelengths grow exponentially. Thus the galactic scales can originate from inside the horizon at the early epochs.

Given any model for inflation it is therefore possible to compute the spectrum and amplitude of the density perturbations. Such calculations have been done by several people (Guth & Pi 1982; Starobinski 1982; Hawking 1982; Bardeen, Steinhardt & Turner 1983) with the following result: Inflation leads to a (desirable) ‘scale-invariant’ spectrum; but generically the amplitude of perturbation is too large (by a factor 10^5–10^6). This amplitude can be brought down only if the inflationary potential is fine-tuned in a very unnatural way. This makes inflation aesthetically unappealing.

With the aim of re-analyzing the origin of large semiclassical density perturbation, in the inflationary scenario, we consider a toy-model. There are two main features in our toy-model. Firstly we consider only the homogeneous mode of the scalar field. Strictly speaking, a homogeneous mode cannot produce a density inhomogeneity. This however is not a serious drawback since the density inhomogeneity in any case is expected to have a weak k-dependence. Further, the results of the toy-model are in confirmity with those obtained by a complete analysis by taking into account the inhomogeneous modes also. The second feature of our toy-model is the simple form for the potential which we have used

$$V(\phi) = V_0 - \tfrac{1}{2}\omega^2\phi^2. \tag{1}$$

The significance of this potential is the following. In the standard inflationary potential,

$$V(\phi) = V_0 - \tfrac{1}{2}\omega^2\phi^2 + \lambda_0\phi^4. \tag{2}$$

The minimum of the potential occurs at

$$\phi_f = \frac{\omega}{2\sqrt{\lambda_0}}. \tag{3}$$

Assuming $V_{\min}=0$, we get

$$V_0 = \left(\frac{\omega^2}{4\sqrt{\lambda_0}}\right)^2 \tag{4}$$

Hence $V(\phi)$ can be expressed as

$$V(\phi) = \lambda_0\,\phi_f^4\left[1 - \left(\frac{\phi}{\phi_f}\right)^2\right]^2. \tag{5}$$

Inflation proceeds as long as $V(\phi)$ is a constant, *i.e.* $\phi^2 \ll \phi_f^2$. In this limit $V(\phi)$ can be approximated by the expression in (1).

The potential of type given in Equation (1) has been previously used by Guth and Pi (1985). However they had used the complete scalar field (homogeneous and inhomogeneous modes). Since we are using only the homogeneous mode, the analysis of the problem is greatly simplified.

The main issue of concern here is the method by which *classical* density perturbations are computed from *quantum* mechanical operators. Let us briefly review the conventional approach (as proposed in references Guth & Pi 1982; Starobinski 1982; Hawking 1982; Bardeen, Steinhardt & Turner 1983), and—what we believe to be—its unsatisfactory features.

It is natural that if inflation occurs at GUT scale or earlier, the driving scalar field should be described by a quantum field theory. A self-consistent treatment would then require that the spacetime metric be also quantized. Not having such a theory, one is compelled to describe the system by semiclassical equations which treat gravity classically and matter quantum mechanically. Such a semiclassical description of gravity has a long history, and—in a way—formed part of the subject 'quantum field theory in curved spacetime'. It is usually believed—at least in the days before the invention of inflation—that the *source for semiclassical gravity is the expectation value of* T_{ik}. According to this view-point semiclassical gravity is described by the equations

$$G_{ik} = 8\pi\langle\psi|T_{ik}|\psi\rangle, \tag{6}$$

$$i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle, \tag{7}$$

where $|\psi\rangle$ denotes the quantum state of the field .$\hat{H}$ is the Hamiltonian governing the evolution of ϕ and G_{ik} stands for ($R_{ik} - \frac{1}{2} g_{ik} R$). Here (6) is the semiclassical Einstein's equation and (7) is the Schrödinger picture evolution equation for the quantum state of the field ϕ.

This viewpoint, however, leads to difficulties in the inflationary scenario. It is usual to assume that the quantum field driving the inflation is in the vacuum state in the de Sitter spacetime; but the expectation value $\langle 0|T_{ik}(x, t)|0\rangle$ is homogeneous (*i.e.*, independent of **x**) because of the translational invariance of the vacuum state $|0\rangle$. Thus, we will never get an **x**-dependent ($\delta\rho/\rho$) out of this prescription. We must abandon the rule that $\langle 0|T_{ik}|0\rangle$ is the source for semiclassical gravity.

Once we abandon it, we are at a loss to select another unique 'source'. (The proper approach will be to start with the Wheeler-DeWitt equation in quantum gravity and consider its semiclassical limit. Unfortunately, there are several subtilities involved in this approach *cf.* Hartle 1987; Halliwell 1987; Padmanabhan 1989; Padmanabhan & Singh 1989). We need to proceed in a somewhat intuitive manner. The conventional view has been the following (Brandenberger 1985).

We define a classical field ϕ_{cl} (**x**, t) as consisting of a homogeneous part and a perturbation

$$\phi_{cl}(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t). \tag{8}$$

Since ϕ_0 (t) cannot be defined as $\langle 0|\phi(x, t)|0\rangle$ (which vanishes), it is defined as the (regularized) rms value:

$$\phi_0(t) \equiv [\langle 0|\phi^2(\mathbf{x}, t)|0\rangle]^{1/2}. \tag{9}$$

Defining $\delta\phi$ (x, t) is trickier. We first define the power spectrum of the scalar field by

$$P(\mathbf{k}, t) = \int d^3\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{x}} \langle 0|\phi(\mathbf{x}+\mathbf{y}, t)\phi(\mathbf{y}, t)|0\rangle \tag{10}$$

and construct $\delta\phi$ (x, t) as the Fourier transform of $\sqrt{P(k,t)}$. In this manner, $\delta\phi(\chi, t)$ is made to carry information about the inhomogeneities. This classical field $\phi_{cl}(x, t)_{\,l}$ is then used to construct a classical energy-momentum tensor T_{ik} (which is no longer homogeneous). From this T^{cl}_{ik} a nonzero density perturbation can be obtained. This, essentially, is the conventional approach.

It is clear that the information regarding the spatial dependence can be smuggled in only through the expectation values of the two-point functions. This procedure of defining a classical field through a correlation function is somewhat *ad hoc* and arbitrary. The necessity of such a round-about method of constructing ϕ_{c1} arose because the scalar field was assumed to be in a quantum state which is translationally invariant (a vacuum state in the conventional approach).

A second issue of importance is the mode of transition from the quantum to the classical limit. Although in literature attempts have been made to study this transition, the schemes followed conventionally have several serious drawbacks. The most detailed discussion available today is probably the analysis by Guth and Pi: They assume the field ϕ to be in a vacuum state and study the classical limit using a classical distribution function, $F(x, p, t)$ which is constructed from the vacuum state. It is claimed that at late times, F peaks around the classical trajectory. In other words, for late times the distribution function F behaves like

$$F(x, p, t) = |\psi(\mathbf{x}, t)|^2 \delta(p - p_{cl}) \tag{11}$$

where p_{cl} is the classical momentum (Guth & Pi 1985).

To study the classical limit, the proper way is to start with the wavefunction and construct the Wigner function using this. We can construct the Wigner function explicitly and study its behaviour at late times. In particular we will check if the Wigner function peaks around the classical trajectory (Wigner 1932; Landau & Lifshitz 1985, Section 6). This has been discussed in Appendix 1.

As has been shown in Appendix 1, for late times, F behaves as

$$F = \frac{1}{\pi}\left(1 - \frac{x^2}{2\sigma^2} + \frac{x^4}{8\sigma^4} + \ldots\right) e^{-2\sigma^2 (p - p_{cl})^2} \tag{12}$$

where σ is the spread in the position x. As $t \to \infty$, $\sigma \to \infty$ and F tend to a vanishingly small uniform value for all x. Thus the analysis shows that F does not peak around any classical trajectory.

The above discussion emphasizes the need to develop an alternative approach to the study of the classical limit. In the conventional approach, we cannot interpret the expectation value of the field operator $\langle \phi \rangle$ as ϕ_{cl} because the field is assumed to be in a vacuum state. (The vacuum state is the Gaussian state with a zero mean value.)

We know from quantum mechanics that coherent state is the nearest approximation to a classical, state. Since we cannot define coherent states in an expanding background in a strict sense, we may use general Gaussian states. Since these quantum states need not be the ones with zero mean value, the classical value of ϕ can now be identified with the expectation value $\langle \phi \rangle$ in these states. We explore this approach in this paper.

Yet another motivation for the use of Gaussian states arises from the finite temperature effects. If the field is in equilibrium with the sourrounding radiation, then it is not in a pure state but in a mixed state. Its quantum description necessitates the use of density matrix. However, the effects of finite temperature can be mimicked to a certain extent by a Gaussian state whose spread has the appropriate dependence on the temperature of the surrounding radiation. (For a discussion on density matrix in quantum mechanics refer to Landau & Lifshitz 1985, Section 14). Thus the Gaussian state is a very natural choice for the field to be in.

The rigorous method to approach the problem is to start from the full field operator $\hat{\phi}(x, t)$ and decompose it into the Fourier components $\bar{\phi}_k(t)$. The full analysis would involve the use of all these Fourier modes. (Seshadri & Padmanabhan (1989); hereafter referred to as Paper 2). The toy-model which we use in this paper makes use of only the homogeneous mode of the field operator. This simplification allows us to see the various constraints and assumptions involved in a clear way. This is a major advantage of the toy-model. Further, the use of only the homogeneous mode is not a major drawback since the perturbations in density, in any case, is expected to give only a weak k-dependence. Strictly speaking, however, the use of merely the homogeneous mode cannot produce spatial inhomogeneity in density. Thus our method cannot be justified within the framework of the toy-model.

However, a complete analysis gives us the result which is in agreement with the results arrived at in this toy-model. Hence this model is a very helpful guide to the analysis of the origin of large density perturbation.

The plan of the paper is as follows: Gaussian states in an exponentially expanding background are considered in Section 2. We use these results to compute density perturbation in Section 3. Section 4 deals with the discussion of the main features of the result.

2. Gaussian states in exponentially expanding background

We will assume a spatially flat universe with the line element

$$ds^2 = dt^2 - S^2(t)\,(dx^2 + dy^2 + dz^2). \tag{13}$$

In this spacetime we are given a scalar field with the action

$$\mathscr{A} = \int \left(\frac{1}{2}\phi^i \phi_i - V(\phi) \right) S^3 dt\, d^3x. \tag{14}$$

We will consider only the homogeneous mode of the scalar field.

For a homogeneous scalar field, the space derivatives of ϕ vanish and the action in (14) reduces to

$$\mathscr{A} = \int \left(\frac{1}{2}\Omega S^3 \dot{\phi}^2 - \Omega S^3 V(\phi) \right) dt \tag{15}$$

where Ω is the comoving normalization volume. Since Ω appears as an overall multiplicative factor the exact value of Ω does not affect our final results. For a closed universe it is convenient to take $\Omega = 2\pi^2$. For the sake of uniformity we will set Ω to this value. (This is just a choice of normalization volume.) We rewrite the action in a slightly different form using the transformation of the time coordinate

$$T = \int^t \frac{dt}{S^3}. \tag{16}$$

In terms of T, we get

$$\mathscr{A} = 2\pi^2 \int \left(\frac{1}{2}\left(\frac{d\phi}{dT}\right)^2 - \frac{1}{2} V S^6 \right) dT. \tag{17}$$

This action is similar to the action for a nonrelativistic particle with a time-dependent potential energy $(2\pi^2)S^6\, V$. ϕ is like the coordinate of the particle and T is like the usual flat-spacetime coordinate.

By inspection we can write the Schrödinger equation as

$$i\frac{\partial}{\partial T}\psi = -\frac{1}{4\pi^2}\frac{\partial^2 \psi}{\partial \phi^2} + 2\pi^2 (S^6 V)\,\psi. \tag{18}$$

We will now assume ψ to be of the form

$$\psi = A(t) e^{-B(\phi - f)^2}, \tag{19}$$

(where A, B and f are functions, only of time) and V is a quadratic function $V = a\phi^2 + b\phi + c$.

In order to determine A, B, and f in Equation (19), we substitute the expression for ψ in the Schrödinger Equation (18) and compare coefficients of ϕ^2, ϕ^1 and ϕ^0 on either side of the equation separately. The results we obtain are

$$i\dot{B}=\frac{B^2}{\pi^2 S^3}-2\pi^2 S^3 a, \tag{20}$$

$$i\dot{f}=2\pi^2 S^3\frac{f}{B}+2\pi^2 S^3\frac{b}{2B}, \tag{21}$$

$$\frac{i}{A}\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{B}{2\pi^2 S^3}+2\pi^2 S^3(af^2+bf+c), \tag{22}$$

where overhead dot denotes differentiation with respect to t.

Using the ansatz for ψ in Equation (19) we can compute $|\psi|^2$ to be

$$|\psi|^2=N(t)\exp\left[-\frac{(\phi-\bar{\phi}(t))^2}{2\sigma^2_{(t)}}\right], \tag{23}$$

where

$$\bar{\phi}=\frac{Bf+B^*f^*}{B+B^*}, \tag{24}$$

$$\sigma^2=\frac{1}{2(B+B^*)}, \tag{25}$$

and N is the normalization factor. $\bar{\phi}$ satisfies the classical equations of motion

$$\ddot{\bar{\phi}}+3\dot{\bar{\phi}}\frac{\dot{S}}{S}+\left.\frac{\partial V}{\partial\phi}\right|_{\phi=\bar{\phi}}=0. \tag{26}$$

To solve for B we substitute

$$B=-i\pi^2 S^3\frac{\dot{Q}}{Q} \tag{27}$$

in Equation (20). This gives the equation for Q as

$$\ddot{Q}+3\dot{Q}\frac{\dot{S}}{S}+2aQ=0. \tag{28}$$

It is worth noting that $\bar{\phi}$ is affected both by the linear as well as the quadratic terms in V, while Q is affected only by the quadratic part. We will be using the solutions to the above equations in the context of inflationary scenario in the next section.

Inflation occurs when the constant energy density V_0 drives the expansion. For a homogeneous and isotropic universe whose energy density is dominated by a constant V_0, the Einstein's equations are

$$\frac{\dot{S}^2+k}{S^2}=\frac{8\pi G}{3}V_0 \tag{29}$$

where $K=-1$, 0 or +1. The solutions to the above equation are

$$S=\begin{cases} H^{-1}\cosh(Ht) & K=+1 \quad (30)\\ H^{-1}e^{Ht} & K=0 \quad (31)\\ H^{-1}\sinh(Ht) & K=-1 \quad (32)\end{cases}$$

where

$$H^2 = \frac{8\pi G}{3} V_0. \tag{33}$$

We will be only interested in the asymptotic limit of these solutions, for $Ht \gg 1$. Asymptotically all three kinds of solutions behave similarly. For $Ht \gg 1$, 1, the scale factor may be taken to be

$$S \simeq \frac{1}{H} e^{Ht} \tag{34}$$

in all the three cases. The exponential growth of $S(t)$ makes $\dot{S}/S$ a constant (= H).

We will now consider the solutions to (26) and (28) for the inverted oscillator potential when $S(t)$ is given by (34).

For such a potential, the mean value, $\bar{\phi}$ evolves according to the classical equations of motion:

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} - \omega^2 \bar{\phi} = 0. \tag{35}$$

This equation has the solution

$$\bar{\phi} = e^{-\frac{3}{2}Ht}\left(\phi_i \cosh(\lambda t) + \left(\frac{v_i}{\lambda} + \frac{3}{2}\frac{H}{\lambda}\phi_i\right)\sinh(\lambda t)\right) \tag{36}$$

Where ϕ_I denote the values of $\bar{\phi}$ and $\dot{\bar{\phi}}$ at $t = 0$ and

$$\lambda^2 = \omega^2 + (\tfrac{3}{2}H)^2. \tag{37}$$

To compute the spread σ, we first have to evaluate Q in Equation (28). For the present case it turns out to be

$$Q = e^{-\frac{3}{2}Ht}[A\cosh(\lambda t) + \sinh(\lambda t)]. \tag{38}$$

Substituting this, in Equation (27), B can be evaluated. For the computation of σ we require only the real part of B. We have,

$$Re(B) = B_0\left(\frac{\pi^2}{B_0 H^2}\right)\frac{\lambda}{H}\left\{\frac{-2A_i/(1+|A|^2)}{[|A|^2 - 1 + 2A_R\sinh(2\lambda t)]/[1+|A|^2] + \cosh(2\lambda t)}\right\} \tag{39}$$

where $A = A_R + iA_i$ and

$$A_R = \frac{3}{2}\left(\frac{\lambda}{H}\right)\left(\frac{\pi^2}{B_0 H^2}\right)^2\left[1 + \left(\frac{3}{2}\frac{\pi^2}{B_0 H^2}\right)^2\right]^{-1}, \tag{40}$$

$$A_i = -\frac{\lambda}{H}\left(\frac{\pi^2}{B_0 H^2}\right)\left[1 + \left(\frac{3}{2}\frac{\pi^2}{B_0 H^2}\right)^2\right]^{-1}. \tag{41}$$

The spread σ can be computed to be

$$\sigma^2 = \frac{1}{2(2\pi^2)S^3\lambda}\frac{1+|A|^2}{(-2A_i)}\left[\cosh(2\lambda t) + \sinh(2\lambda t)\left(\frac{2A_R}{1+|A|^2}\right) + \frac{|A|^2 - 1}{|A|^2 + 1}\right]. \tag{42}$$

In terms of the initial spread σ_0 this can be written as

$$\sigma^2 = \sigma_0^2 e^{-3Ht}\left[\cosh(2\lambda t)\frac{1+|A|^2}{2|A|^2} + \sinh(2\lambda t)\frac{A_R}{|A|^2} + \frac{|A|^2 - 1}{2|A|^2}\right]. \tag{43}$$

The above results have been derived for the case of the exponentially expanding background. We will see how $\bar{\phi}$ and σ evolve in this case and compare it with the case of flat space-time.

The late-time behaviour of $\bar{\phi}$ turns out to be (from Equation 36)

$$\bar{\phi}=\frac{1}{2}\left[\phi_i\left(1+\frac{3}{2}\frac{H}{\lambda}\right)+\frac{V_i}{\lambda}\right]\mathrm{e}^{(\lambda-\frac{3}{2}H)t} \tag{44}$$

where

$$\lambda^2=\omega^2+(\tfrac{3}{2}H)^2 \tag{45}$$

and that of σ to be

$$\sigma\sim\exp\{[(\omega^2/H^2+\tfrac{9}{4})^{1/2}-\tfrac{3}{2}]\,Ht\}. \tag{46}$$

The flat spacetime case is achieved by using $H\to 0$ limit and we get

$$\bar{\phi}=\frac{1}{2}\left(\phi_i+\frac{V_i}{\omega}\right)e^{\omega t}, \tag{47}$$

$$\sigma\sim e^{\omega t}. \tag{48}$$

Both $\bar{\phi}$ and σ grow exponentially with time in the case of exponentially expanding background as well as that of flat spacetime. However, in the case of the expanding background these quantities grow at a slower rate than in the case of flat spacetime. This is because of the fact that the background expansion provides a damping factor in their evolution.

3. Density perturbations with Gaussian states

Large density inhomogeneities have always been a problem in developing a workable inflationary scenario. With the aim of probing the origin of such large inhomogeneities we will consider a toy-model. In a complete theory we have to include the homogeneous as well as the inhomogeneous parts of the 'inflation field'. In our toy-model we will neglect the inhomogeneous part. This reduces the field theory problem into a quantum mechanical one.

Earlier we have argued why it is more natural to assume the field to be in a Gaussian state. We will therefore assume that the wave-function for the homogeneous mode is given by equation (19). The quantum spread σ as well as the mean value $\bar{\phi}$ have been computed in the last section. We will estimate the density inhomogeneity $\delta\rho/\rho$ at the epoch of entering the horizon by the expression

$$\left.\frac{\delta\rho}{\rho}\right|_{t_f}=H\left.\frac{\sigma}{\dot{\bar{\phi}}}\right|_{t=t_1}\mathcal{O}(1) \tag{49}$$

where t_1 is the time when the relevant scales freeze out of the horizon.

The spread σ is the spread of the wave-packet. Thus $\sigma/\dot{\bar{\phi}}$ is a measure of the time interval between the leading and the lagging edge of the wave packet. The right-hand side is a dimensionless number related to the quantum spread of the wave-packet. So we expect that the expression (49) will give the correct order of magnitude for the density perturbations.

We consider a Gaussian wave packet as given in Equation 19. In the last section we have derived the expressions for $\bar{\phi}$ and the spread of the Gaussian wave-packet for such a potential. At later times the expression for $\bar{\phi}$ is given by Equation (44). At late times the spread evolves as

$$\sigma^2 = \sigma_0^2 \frac{1 + |A|^2 + 2A_R}{4|A|^2} e^{(2\lambda - 3H)t} \tag{50}$$

where

$$A = A_R + iA_i,$$
$$A_R = \frac{3}{2}\left(\frac{\lambda}{H}\right)\left(\frac{\sigma_0}{H/2\pi}\right)^4 \left(1 + \frac{9}{4}\left(\frac{\sigma_0}{H/2\pi}\right)^4\right)^{-1},$$
$$A_i = -\frac{\lambda}{H}\left(\frac{\sigma_0}{H/2\pi)}\right)^2 \left[1 + \frac{9}{4}\left(\frac{\sigma_0}{H/2\pi}\right)^4\right]^{-1},$$

and

$$\lambda^2 = \omega^2 + (\tfrac{3}{2}H)^2.$$

Density inhomogeneity is computed using the expression,

$$\frac{\delta\rho}{\rho} = \mathcal{O}(1) H \frac{\sigma}{\dot{\bar{\phi}}}\bigg|_{t=t_1}. \tag{51}$$

We see that both $\dot{\bar{\phi}}$ and σ have the same time dependence at late times. Hence $\delta\rho/\rho$ is time-independent for late times.

It is clear that the density perturbation will depend on the initial spread, σ_0, of the wave-packet. The value of σ_0 cannot be arbitrary but is constrained in a de Sitter spacetime. A de Sitter spacetime is assumed to have an intrinsic temperature ($= H/2\pi$). The initial spread is assumed to be greater than or of the order of this temperature:

$$\sigma_0 \gtrsim \frac{H}{2\pi}. \tag{52}$$

This point has been discussed in an earlier paper of ours (Padmanabhan & Seshadri (1986). This gives

$$\frac{\delta\rho}{\rho} = \frac{H}{\phi_i\left(1 + \frac{3}{2}\frac{H}{\lambda}\right) + \frac{V_i}{\lambda}\left(2\frac{\lambda}{H} - 3\right)\pi} \frac{1}{} \frac{(1 + A_R^2 + A_i^2 + 2A_R)^{1/2}}{\sqrt{2}(A_R^2 + A_i^2)^{1/2}} \tag{53}$$

where

$$A_i = -\frac{4}{13}\frac{\lambda}{H} \tag{54}$$

and

$$A_R = \frac{6}{13}\frac{\lambda}{H}. \tag{55}$$

There is, however, one more constraint. In order to solve the problems of Standard cosmology, the Universe should undergo sufficient exponential expansion. Since we assume that the expectation value of T_0^0 is the source term for Einstein's equations, we require that the constant term V_0 should dominate $\langle T^0_0 \rangle$ for a duration of at least Δt (55-60)H^{-1} . This imposes further constraints on the parameters of the potential; we will now discuss these constraints.

The action for the homogeneous scalar field in this potential is given by

$$\mathscr{A} = \int (\tfrac{1}{2}\dot{\phi}^2 - V_0 + \tfrac{1}{2}\omega^2\phi^2)\, S^3 2\pi^2 \,\mathrm{d}t. \tag{56}$$

The momentum p_ϕ conjugate to the field ϕ is given by

$$p_\phi = 2\pi^2 S^3 \dot{\phi}. \tag{57}$$

Using this the energy density T^0_0 is given by

$$T^0_0 = \frac{1}{2}\left(\frac{p_\phi}{2\pi^2 S^3}\right)^2 + V_0 - \frac{1}{2}\omega^2\phi^2. \tag{58}$$

In our analysis, the expectation value of T^0_0 is the source for the semiclassical Einstein's equation

$$\frac{S^2}{S^2} = \frac{8\pi G}{3}\langle T^0_0 \rangle \tag{59}$$

where

$$\langle T^0_0 \rangle = V_0 + \frac{1}{2}\frac{1}{(2\pi^2 S^3)^2}\langle p^2_\phi \rangle - \frac{1}{2}\omega^2 \langle \phi^2 \rangle. \tag{60}$$

For an exponential expansion of the universe, we require that the expectation value of T^0_0 should be dominated by the constant term V_0. For sufficient inflation we require that inflation should last at least for a time (55-60) H^{-1}. So the expectation value of T^0_0 should be dominated by V_0 for at least this much time The scale factor grows exponentially for a constant value for < T^0_0 >:

$$S = \frac{1}{H}e^{Ht}. \tag{61}$$

where

$$H^2 = \frac{8\pi G}{3}V_0. \tag{62}$$

From the condition that inflation should last for at least 55 *e*-folding times we get,

$$\frac{3}{\pi}\left(\frac{E_p}{H}\right)^2 \geqslant \frac{1}{16}e^{55(2p-3)}(2p-3)\left(2+\frac{3}{p}\right)^2 \left\{\frac{3\pi^2}{2}\left[1+\left(\frac{2}{2p+3}\right)^2\right] + \left(\frac{\phi_i}{H}\right)^2 (2p+3)\right\} \tag{63}$$

where $\rho = \lambda/H$ and E_p is the Planck energy scale ($\simeq 10^{19}$ Gev). (Here we have used the condition that the initial spread σ_0 is equal to the de Sitter temperature, $H/2\pi$.)

We can now study the magnitude of density inhomogeneities produced in inflationary models when there is sufficient inflation.

In the standard GUT-scale inflation $V_0 = (10^{14}\ \text{Gev})^4$. Using the Equation (62) we have

$$H^2 = \frac{8\pi G}{3}V_0 \tag{64}$$

which gives $H \simeq 2\times 10_9$ Gev. Using these in Equation (63) we can put bounds on the value of $p(= \lambda/H)$. From the definition of p t follows that $p \geqslant 1.5$. ($\omega = 0$ implies $p = 1.5$. This corresponds to $2p - 3 = 0$.) Using the value of H in the inequality (63), we first of all see that e n for $p = 2$ the inequality cannot be satisfied. Hence p cannot be much different from unity. Assuming $\phi_i \simeq H$, we simplify the condition for sufficient inflation as

$$2\log\left(\frac{E_p}{H}\right) > 22(2p-3) + \log(2p-3). \tag{65}$$

We can put bounds on $(2p - 3)$ depending upon the order to which we want to satisfy this inequality. The left-hand side is about 20. If we demand that the right-hand side should be less by about 18, we get $(2p - 3) \leqslant 0.1$. A lower value of $(2p - 3)$ will improve the inequality.

Since in the region of interest, p is of order unity ($p \simeq 1.5$) the expression for density perturbation in Equation (53) reduces to

$$\frac{\delta\rho}{\rho} \simeq \frac{\mathcal{O}(1)}{\pi}\frac{H}{\phi_i}\frac{1}{2p-3}. \tag{66}$$

For $(2p - 3) \leq 0.1$, and $\phi_i \simeq H$,

$$(\delta\rho/\rho) \simeq \mathcal{O}(1) \times 10 \tag{67}$$

As is usually the case with density perturbation in inflation, the value of $\delta\rho/\rho$ is too large to be acceptable.

4. Discussion of the result

We have seen in the last section that the magnitude of density perturbations is very large. Similar toy-models using a constant potential and a potential with a constant slope, had also given large density inhomogeneity (Padmanabhan (1985); Padmanabhan & Seshadri (1986)). Thus merely changing the form of the potential has not helped to resolve the problem.

Production of density inhomogeneity in inflation using the inverted oscillator potential has been studied in literature before (Guth & Pi 1985). We are interested in small values for $(2p - 3)$. In this limit p is of order unity. Then the result obtained by Guth and Pi reduces to

$$\frac{\delta\rho}{\rho} \simeq \frac{H}{\phi_c}\frac{1}{(2p-3)}k^{-\frac{1}{2}(2p-3)}. \tag{68}$$

(In the paper by Guth and Pi, they have used a random variable, ϕ_α. However we can replace this random parameter by a nonrandom variable ϕ_c using a transformation in the time coordinate. The above result has been obtained after this transformation. For details refer to Appendix 2). Here ϕ_c is a constant and k's are the physical wave modes. For small values of $(2p - 3)$ we can express $\delta p/\rho$ as

$$\frac{\delta\rho}{\rho} \simeq (H/\phi_C)\frac{1}{(2p-3)}\left(1 - \frac{1}{2}(2p-3)\ln k\right). \tag{69}$$

For small $(2p - 3)$ we clearly have

$$\frac{\delta\rho}{\rho} \simeq (H/\phi_C)\frac{1}{2p-3}. \tag{70}$$

Clearly this result matches with the result obtained by us in equation (67).

Before concluding this paper we would like to discuss once again the simplifying assumption in this paper.

We have worked with a toy-model to compute density perturbations. The toy-model involves two simplifying assumptions: (i) the potential is of a very simple form, and (ii) we have considered only the homogeneous mode of the scalar field. We will now discuss the validity of these assumptions.

The inverted oscillator potential which we have used has the distinct advantage that we can do an exact analysis of the evolution of the Gaussian wave-packet. This is because the potential we have used does not include powers of ϕ higher than quadratic. This feature does not exist for a realistic inflationary potential. A realistic inflationary potential is too complicated to yield such a simple analysis as the one in this paper. The details of the result, however, are reasonably insensitive to the exact details of the potential. Thus the result obtained using our simple potential is expected to be very close to the realistic case, while, at the same time allowing us to do an exact analysis. Thus the simplification of the form of the potential is certainly a very valid and useful assumption.

The next question concerns the use of only the homogeneous mode of the scalar field to compute density perturbation. As is clear from Equation (49), the nonzero quantum spread in the wave-packet is responsible for the generation of inhomogeneities. Intuitively one would expect that this is certainly plausible. Although intuitively appealing, this procedure needs explanation. The trouble which arises is that if one uses only the homogeneous mode, it is not clear how a spatial dependence in density (which is what we need to compute density perturbation) arises. This is because we have not introduced any spatial dependence at any stage. Thus, strictly speaking, the computation of density perturbation in our analysis cannot be *rigorously justified* within the frame work of the toy-model. We are, however, able to compute density perturbation using the above formalism because the density perturbation has only a weak k-dependence. Thus the absence of spatial information in our model is not a serious drawback.

In order to justify the above toy-model a complete analysis (which takes into account, the contribution from the inhomogeneous modes also) needs to be done. Paper 2 deals with this detailed analysis.

Appendix 1

In the conventional approach the classical limit of the evolution of the scalar field is analyzed using a classical distribution function. It is also usually claimed (see *e.g.* Guth & Pi, 1985) that the distribution function peaks around the classical trajectory of the field, at late times.

In this appendix we will construct a classical distribution function and study its late-time behaviour. In order to do this we will consider the quantum mechanics of a particle of a mass *m* in an inverted oscillator potential

$$V(x) = -\tfrac{1}{2}m\omega^2 x^2. \tag{A1.1}$$

In classical statistical mechanics, one uses a probability distribution function $P(x, p, t)$ where p is the momentum of the particle. The probability of finding a particle in the position interval x and $x + dx$ and momentum interval p and $p + dp$ is $P(x, p, t)\,dx\,dp$ at time t; P obeys the Louville's equation:

$$\frac{\partial P}{\partial t} + \frac{p}{m}\frac{\partial P}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial P}{\partial p} = 0 \tag{A1.2}$$

where V is the potential experienced by the particle. The expectation value of position is given by

$$\langle x \rangle_c = \int P(x, p, t) \times \mathrm{d} \times \mathrm{d}p \tag{A1.3}$$

and the momentum expectation value by

$$\langle p \rangle_c = \int P(x, p, t) p \mathrm{d} \times \mathrm{d}p. \tag{A1.4}$$

In the classical statistical mechanics, one could define this probability distribution because, one can measure the position and the momentum simultaneously to arbitrary accuracy.

In quantum mechanics, however, a simultaneous measurement of position and momentum is not possible. Hence we cannot, strictly speaking, define a probability distribution function in quantum mechanics. This being the situation one can at best see if we can construct some function of x, p and t from the quantum state of the particle $\psi(x, t)$, which mimics the classical distribution function as much as possible. One such function is the Wigner function (Wigner 1932; Landau & Lifshitz 1985, Section 6). We will first define the Wigner function and mention some of the properties.

Given a wavefunction $\psi(x, t)$, the Wigner function is defined as

$$F(x, p, t) = \int_{-\infty}^{+\infty} \mathrm{d}u\, \psi^*(x - \tfrac{1}{2}u\hbar) e^{ipu} \psi(x + \tfrac{1}{2}u\hbar). \tag{A1.5}$$

The dynamics of the quantum-mechanical particle is described by a wavefunction $\psi(x, t)$ which is governed by the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi. \tag{A1.6}$$

If ψ obeys the above equation then the Wigner function satisfies the following equation

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial x} - \frac{\partial V}{\partial x}\frac{\partial F}{\partial p} = \sum_{l=1}^{\infty}\left(-\frac{i}{2}\hbar\right)^{2l}\frac{\partial^{2l+1} V}{\partial x^{2l+1}}\frac{\partial^{2l+1} F}{\partial p^{2l+1}}. \tag{A1.7}$$

Two points are worth noting in the above equation: (i) The right-hand side involves powers of h greater than or equal to two. For a general potential, the right-hand side can be ignored if we neglect second and higher powers in h. (ii) The right-hand side involves third and higher derivatives of the potential. If the potential is (at least

Approximately) quadratic, then the right-hand side vanishes (This is the case we are interested in.) In either of these cases the equation governing F is

$$\frac{\partial F}{\partial t}+\frac{p}{m}\frac{\partial F}{\partial x}-\frac{\partial V}{\partial x}\frac{\partial F}{\partial P}=0 \tag{A1.8}$$

which is identical to the Louville equation (A1.2).

Further, it can be shown that the Wigner function has the following properties:

$$\int_{-\infty}^{+\infty} \mathrm{d}p F(x, p, t)=|\psi(x, t)|^2, \tag{A1.9}$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x F(x, p, t)=|\tilde{\psi}(p, t)|^2, \tag{A1.10}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}x\mathrm{d}p F(x, p, t)(f(x, t)+g(p, t))=\langle f(x, t)\rangle+\left\langle g\left(-i\hbar\frac{\partial}{\partial x}, t\right)\right\rangle. \tag{A1.11}$$

Here $\tilde{\psi}(p, t)$ is the Fourier transform of $\psi(x, t)$ and represents the probability amplitude to find a particle with momentum p at time t. < > implies the expectations value in the given quantum state.

There is, however, a problem in interpreting F as a probability distribution function. Unlike the classical probability distribution, P, F may take negative values. It has been further argued by wigner (1932) that no expression, F, exists which satisfies (A1.11) and is everywhere positive. Of all the possible expressions for F, (A1.5) was chosen by Wigner, because it is the simplest. However, when one uses this function, one should ensure the F does not take negative values for the case it is being used. For the inverted oscillator case, F is always positive.

We next study the classical limit using this distribution function. A particle in an inverted oscillator potential is described by a wavefunction which is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}-\frac{1}{2}m\omega^2x^2\psi. \tag{A1.12}$$

From now on we will choose units for which $h = 1$.

Consider a Gaussian wave-packet of the form.

$$\psi=Ae^{-Bx^2} \tag{A1.13}$$

where A and B are functions only of time. The mean value of x in the above state is zero. Once A and B are given, the wavefunction is completely specified. Substituting ψ in the Schrödinger equation we get the expression for B as

$$B=\frac{1}{2}m\omega\frac{-2\alpha/(1+\alpha^2)-i\sinh(2\omega t)}{(\alpha^2-1)/(\alpha^2+1)+\cosh(2\omega t)}. \tag{A1.14}$$

where $\alpha = -m\omega/(2B_0)$. (B_0 is the initial value of B. We have imposed the condition that B_0 is the rel. This does not lead to any loss of generally as it just fixes the phase of the wave-packet.) The probability density $|\psi|^2$ for finding the particle between x *and* $x+\mathrm{d}x$ at time t is given by

$$|\psi|^2=N(t)e^{-x^2/(2\sigma^2)} \tag{A1.15}$$

where N is a normalization factor and σ is related to B by

$$\sigma^2 = \frac{1}{2(B+B^*)}. \tag{A1.16}$$

Using equations (Al.14) and (Al.16) we get for late times

$$\sigma = \frac{1}{2\sqrt{B_0}}\left[1+\left(\frac{2B_0}{m\omega}\right)^2\right]^{1/2} e^{\omega t}. \tag{A1.17}$$

The Wigner function for our case turns out to be

$$F = \left[\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2}\right)\right]\left[\sigma\sqrt{\frac{2}{\pi}}\exp(-2\sigma^2(p-\bar{p})^2)\right] \tag{A1.18}$$

Where

$$\bar{p} = -2x\mathrm{Im}(B). \tag{1.19}$$

We see that p is proportional to x at late times. This is precisely the behaviour for the momentum of a classical particle in the potential $V = -(l/2)m\omega^2x^2$. (For $t \rightarrow \infty$, $x_{\mathrm{cl}}(t) \sim e^{\omega t}$, $P_{\mathrm{cl}}(t) \sim e^{\omega t} \sim x$). Thus it is tempting to write (Al.18) as

$$F = |\psi|^2\left(\frac{1}{2\pi\Delta^2(t)}\right)^{1/2}\exp\left[-\frac{(p-p_{\mathrm{cl}}(x))^2}{2\Delta^2_{(t)}}\right] \tag{A1.20}$$

where $\Delta^2 = (1/4\sigma^2)$ and $|\psi|^2$ is the first factor in (A1.18). Since Δ goes as σ^{-1}, from Equation (Al.17) it is clear that $\Delta \rightarrow \infty$ as $t \rightarrow \infty$, so that the Gaussian in $(p - p_{cl})$ in equation (A1.18) becomes more and more close to a δ-function. This seems to suggest that F is sharply peaked around p_{cl}. Due to this, in the past, some authors (for *e.g* Halliwell 1987) have claimed that for large times the distribution function goes to a delta function.

This, however, is not (rigorously) true as can be easily seen from the full form of (A1.18). The limit in which the second factor becomes a δ-function, the first factor vanishes. Since $\sigma \rightarrow \infty$ as $t \rightarrow \infty$. we can expand F as

$$F = \frac{1}{\pi}\left(1-\frac{x^2}{2\sigma^2}+\frac{x^4}{8\sigma^4}+\ \dots\ \right)e^{-2\sigma^2(p-\bar{p})^2} \tag{A1.21}$$

at fixed, finite x, p. It is clear from this that strictly speaking, F goes to zero as $\sigma \rightarrow \infty$. (It does not become a δ-function). Thus one has to be careful in the interpretation of F as a classical distribution.

Appendix 2

In this appendix we give some sort of details of how one can arrive at Equation (68).

In the limit of small values for $2p - 3$, p is of order unity. In this limit the results obtained by Guth and Pi become

$$\frac{\delta\rho}{\rho} \simeq (H/\phi_\alpha)\frac{1}{(2p-3)}k'^{-\frac{1}{2}(2p-3)}. \tag{A2.l}$$

Here k' is not the physical wavenumber and ϕ_α is a random variable. This randomness is removed by transforming to a new time coordinate. This new time coordinate t' is defined as

$$t' = t + \frac{2}{(2p-3)H} \ln(\phi_\alpha/\phi_C) \tag{A2.2}$$

where ϕ_c is not a random variable. Using this transformation it can be shown that

$$\phi_C k^{\frac{1}{2}(2p-3)} = \phi_\alpha k'^{\frac{1}{2}(2p-3)}. \tag{A2.3}$$

k is now the physical wave mode. Using Equations (A2.1) and (A2.3) we get

$$\frac{\delta\rho}{\rho} \simeq \frac{H}{\phi_c} \frac{1}{(2p-3)} k^{-\frac{1}{2}(2p-3)}. \tag{A2.4}$$

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