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On the Damping of the Bending Waves in Saturn's Ring

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Abstract. We study numerically the motion of a single particle in the bending wave of finite thickness in Saturn's ring. We include the forcing due to the planet, a moon, the coriolis force and the self gravity of the ring. In particular, we compute the variation of the velocity arising due to the variation of the amplitude and the phase of the epicyclic motion across the local vertical height of the ring. We suggest that the dissipation of energy due to the collision of ring particles in this shear layer damps out the bending wave of Saturn's ring at the 5:3 vertical resonance of Mimas within a distance of 150 km from the site of its launching as is observed in Voyager data.

Key words: planetary rings, Saturn

Recently, through the analysis of the Voyager data it is discovered that the particulate Saturn's rings show varieties of the collective effects such as the spiral density waves and the spiral bending waves (Smith *et al.* 1981; Cuzzi, Lissauer & Shu 1981; Lane *et al.* 1982; Holberg, Forrester & Lissauer 1982; Shu, Cuzzi & Lissauer 1983; Shu 1984; Gresh *et al.* 1986). Spiral bending waves in Saturn's ring launched at the inner vertical resonance propagate radially inwards and the spiral density waves launched at the inner horizontal resonance propagate radially outwards (Shu 1984). In the case of the 5:3 vertical resonance of Mimas, the wave amplitude at the launch is about 1 km and within a distance of about 150 km from the site of the launching, the wave is almost completely damped.

In this paper, we provide an explanation for the rapid attenuation of the bending wave. We show that there is a significant variation of the radial component of the velocity across the vertical height of the ring. The energy of such a shearing motion is dissipated through collisions. Since the collisional frequency is about twice per orbit, the particle picture may be more suitable. Furthermore, the frequencies of the different forces acting on the particle being incommensurate, the particle trajectories in the phase space is expected to be ergodic, and therefore the time averaged behaviour of a single particle should be similar to the instantaneous behaviour of the whole disc. The analysis of the damping length is done in two parts: in the first part we study the motion of a single particle in the warped self-gravitating ring of finite thickness by numerically integrating the equations of motion of the particle. This provides a

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reasonable estimate for the vertical shear component σ_{xz}. In the second part we estimate the kinematic viscosity coefficient v_k by using the amplitude of the epicyclic motion. Using this we calculate the damping length of the bending wave. Although, the work is applied to the problem of Saturn's ring, as far as the mathematical part goes, it is completely general, and therefore can have much broader application. Details will be discussed elsewhere (Chakrabarti 1989).

We choose a right-handed Cartesian coordinate system (X, Y, Z) at a radial distance r from the centre of the planet (See Fig. 1). The X axis points radially outwards, the Y axis points toward the azimuthal direction, and the Z axis points vertically upwards, normal to the equatorial plane of the planet. The frame is rotating around the planet with the local Keplerian velocity $\Omega(r)$. The frame is also oscillating vertically with an amplitude ε and frequency ω, the frequency of perturbation due to the moon as seen from the rotating frame. In general, the frequency n at which the perturbation is launched at the inner vertical resonance as seen from the inertial frame is given by (Shu, Cuzzi & Lissauer 1983),

$$n = m\Omega_{\rm M} \pm \eta\mu_{\rm M} \pm p\kappa_{\rm M}. \tag{1}$$

Here, $\Omega_{\rm M}$ denotes the angular velocity of the moon, $\mu_{\rm M}$ denotes the vertical frequency, $\kappa_{\rm M}$ denotes the epicyclic frequency and m, η and p are integers. At the vertical resonance radius $r_{\rm v}$, one has (Brahic 1977),

$$n - m\Omega(r_{\rm v}) = \pm\mu(r_{\rm v}). \tag{2}$$

Assuming the planet to be spherically symmetric, $\Omega(r) = \mu(r) = k(r)$. In the case of the 5:3 vertical resonance of Mimas, for example, $m = 4$, and $n_{\rm v} = 3\Omega(r_{\rm v}) = 5\Omega_{\rm M}$.The

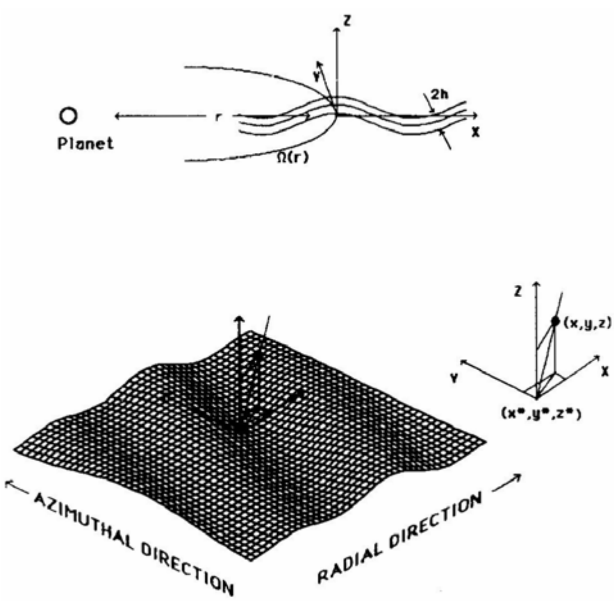

Figure 1. Geometry and coordinates of the problem are schematically shown. Rotating frame (X, Y, Z) also oscillates vertically with frequency ω of the perturbation due to the moon. Equation of the midplane is $z^* = \varepsilon \cos (k_x x^* + K_y y^* \omega t)$ · (x^*, y^*, z^*) is the coordinate on the midplane where the local normal drawn from the particle at (x, y, z) intersects.

perturbation frequency ω as seen from the rotating frame is given by, $\omega = n - 4\Omega(r)$, which is exactly equal to $\Omega(r_V)$ at the vertical resonance. Since the bending wave is launched inwards, the Keplerian frequency $\Omega(r)$ at the radius r that we are considering, is somewhat higher than $\Omega(r_V)$. We let k_χ and k_y denote the radial and the azimuthal wave numbers of the perturbation. The midplane of the warped ring is assumed to be of the form,

$$z^* = \varepsilon \cos (k_x x^* + k_y y^* - \omega t). \tag{3}$$

To the zeroth order, this represents the first Fourier component of the bending wave form (see, *e.g.* Shu 1984). By a star (*) symbol we denote the coordinates on the midplane of the ring. Denoting the phase of the wave by ϕ^* ($= k_x x^* + k_y y^* - \omega t$) and that of the particle by ϕ ($= k_x x + k_y y - \omega t$) the equations of motion are given by (Chakrabarti 1989),

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} + 3\Omega^2 x - \nu^2 x_1 - (\Omega^2 - \omega^2)\varepsilon^2\kappa_x \cos\phi \sin\phi^*/(1 + \varepsilon^2\kappa^2 \sin^2\phi^*)^{1/2}, \tag{4a}$$

$$\frac{d^2y}{dt^2} = 2\Omega\frac{dx}{dt} - \nu^2 y_1 - (\Omega^2 - \omega^2)\varepsilon^2\kappa_y \cos\phi \sin\phi^*/(1 + \varepsilon^2\kappa^2 \sin^2\phi^*)^{1/2}, \tag{4b}$$

and,

$$\frac{d^2z}{dt^2} = -\Omega^2 z - \nu^2 z_1 + (\Omega^2 - \omega^2)\varepsilon \cos\phi/(1 + \varepsilon^2\kappa^2 \sin^2\phi^*)^{1/2}. \tag{4c}$$

Here x_1, y_1, and z_1 are the projection of the local normal to the ring drawn from the particle position on our local frame of reference. Also, $k^2 = k_x^2 + k_y^2$. The first two terms of Equation (4a) and the first term of Equations (4b) and (4c) are due to the distant planet and the coriolis force. The terms proportional to v^2 are due to the self-gravity effect assumed to be arising due to the flat ring of uniform density ρ, so that $v^2 = 4pG\rho$. In the absence of the self-gravity and forcing due to the moon, the solution is given by,

$$y - y_0 = -v_g(t - t_0) - 2A \cos[(t - t_0)\Omega + \psi], \tag{5a}$$

$$x = -\frac{2v_g}{3\Omega} + A \sin[(t - t_0)\Omega + \psi], \tag{5b}$$

and

$$z = h \cos(\Omega t + \zeta). \tag{5c}$$

Here, A is the amplitude of the epicyclic motion, ψ and ζ are arbitrary phases, v_g is the speed of the guiding centre, t_0 is the origin of time for the problem, h is the half thickness of the ring.

Numerical solution of the above equations (4a–c) are obtained by using fourth order Runge-Kutta method. The dynamical timescale in the problem is of the order of 5.0×10^4 s. We take the time step of the integration dt = 1s, and perform the integration up to about 100 orbits. The simulation is carried out for h = 10 metres. The vertical frequencies in this case is $\mu = 1.948 \times 10^{-4}$. Eight different initial phases ($\psi = n\pi/4$, $0 \leqslant n \quad 7$) are used for averaging purposes. Since the local vertical motion is very nearly sinusoidal, ζ of Equation (5c) is always chosen to be 0 so that the particle starts from the midplane in every simulation. Other parameters include: ε = 1 km, $k_x = 5.0 \times 10^{-6}$, $k_y = 3.077 \times 10^{-10}$ cm^{-1}, σ = 50 g cm^{-2} appropriate for the 5:3 vertical resonance of Mimas. To obtain the variation of the velocity and the shear in the ring, the configuration space at each phase ϕ^* of the wave in the ring is divided into 729 bins with 9 equal bins along each of the x_1, y_1 and z_1 directions. Such divisions are made at eight phases of the wave centred around $\phi^* = n\pi/4, 0 \leqslant n \leqslant 7$. In each of the 729 zones at each ϕ^*, the peculiar velocity components are calculated from the number of times the particle visited that particular zone and the velocity it carried during each such visit. Such processes are carried out for each of the eight initial intrinsic phases (ψ) of the particle, over which the final average of the velocity is computed assuming each ψ is equally probable. Analytic functions are fit through average velocities as a function of the vertical zone co-ordinates at each phase of the wave, upon differentiating which,

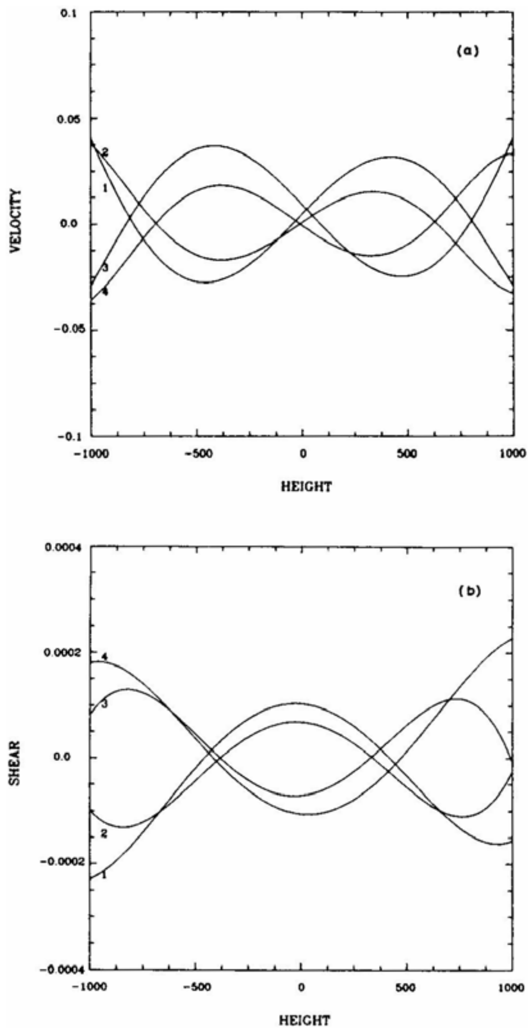

Figure 2. Variation of (a) $\langle v_x \rangle$ and (b) shear component $\partial\langle v_x \rangle/\partial z_1$ as a function of local height as obtained from numerical solution at phases: (1) $\phi^* = 0$, (2) $\phi^* = \pi/4$, (3) $\phi^* = 3\pi/4$, and (4) $\phi^* = \pi$. Half thickness of the ring = 10 m.

the shears are obtained. The analytic functions are found to be very smoothly fitting through the numerical data indicating that random fluctuations are virtually absent The 'shear' in our particle-dynamical model retains the micro-physical definition and is understood to be arising due to the variation of the particle velocities when they are colliding at the same instant of time (*i.e.*, at same phase ϕ^*). The results of our simulation are shown in Fig. 2. Fig. 2a illustrates $\langle v_x \rangle$ in cm s^{-1} as a function of the local height of the ring, and Fig. 2b shows the corresponding values of $\partial\langle v_x \rangle/\partial z_1$ in s^{-1} (the value of $\partial\langle v_x \rangle/\partial x_1$ is very small in our analysis.). The four curves denoted by the numbers 1, 2, 3, and 4 in each of the diagrams are for $\phi^* = 0, \pi/4, 3\pi/4$ and π respectively. The average shear is about 7×10^{-5} s^{-1} (Fig. 2a).

One can do an order of magnitude estimate for the damping length of the bending wave assuming this to be the only source of shear. The energy density of the wave is given by, $\frac{1}{2}\varepsilon^2\Omega^2$ which is about 85 erg g^{-1}. This energy is transferred to the particles through infrequent collisions (about twice per orbit). In the absence of a proper model, we assume that the coefficient of the kinematic viscosity v_k is roughly given by (Brahic 1977; Goldreich & Tremaine 1978, 1982),

$$v_k = \Omega\tau \max(R, R_{\text{epi}}), \tag{6}$$

where R is the size of the largest particles (about 5 m for 'A' ring of Saturn) and R_{epi} is the amplitude of the epicyclic motion which is similar to the half thickness of the ring. The coefficient v_k calculated in this manner probably provides the upper limit. With the optical depth $\tau \simeq 0.5$, one obtains $v_k = \Omega\tau h^2 \approx 64 \text{cm}^2 \text{ s}^{-1}$ for $h = 10$ m. The rate of dissipation of energy density is $v_k(\partial\langle v_x \rangle/\partial z_1)^2$. Substituting the values mentioned above, one obtains the time period for dissipation as 2.6×10^8 seconds. The group velocity is $c_g = -\pi G\sigma/(\omega - 4\Omega) \simeq 0.026$ cm s^{-1}. This provides the damping length $\lambda_d \approx 70$ km—roughly half of the observed length of about 150 km. It is observed that for simulation with higher half thickness shear remains very similar but v_k goes up reducing the damping length even further. We therefore conclude that, considering uncertainties in v_k, the total thickness of the disc is no more than 20 metres and probably closer to 15 metres is consistent with the other theoretical considerations that ring could be only tens of metres thick (Goldreich & Tremaine 1982).

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